4-MANIFOLDS: CLASSIFICATION AND EXAMPLES

1. Outline

Throughout, "4-manifold" will be used to mean closed, oriented, simply-connected 4-manifold. Hopefully I will remember to append "smooth" wherever necessary. I will use PD for both the Poincaré duality map and its inverse.

Classification of topological 4-manifolds

- **intersection forms**, their algebraic classification, the effects of topological operations
- **simple examples**: $\mathbb{CP}^2$, $\bar{\mathbb{CP}}^2$, sphere bundles over spheres
- **Freedman's theorem**, sketch of existence part of its proof
- **harder example**: E_8 manifold and fake 4-balls

Restrictions for smooth 4-manifolds

- **Rohklin's and Donaldson's restrictions**, 11/8 conjecture and progress towards it
- **exotic** $\mathbb{R}^4$**s**, constructions, multitudes

Sources detailed in the bibliography (cited without reference).

2. Intersection Forms

M a closed, oriented 4 manifold has a symmetric bilinear **intersection form** Q_M on $H^2(M;\mathbb{Z})$ given by $(\alpha,\beta) \mapsto \alpha \cup \beta([M])$. Under the de Rham isomorphism $Q_M(\alpha,\beta) = \int_M \alpha \wedge \beta$. Since Q_M is a map to $\mathbb{Z}$, which has no torsion, it must vanish on torsion elements of $H^2(M;\mathbb{Z})$, therefore can use Q_M on $H^2_{dR}(M;\mathbb{R})$ for computations (however the fact that Q_M is a $\mathbb{Z}$-bilinear form will be important soon).

The name comes from the interpretation of $Q_M(\alpha,\beta)$ as the **intersection number** $PD(\alpha) \cdot PD(\beta)$, determined as follows: count the intersections with signs determined by whether or not the isomorphism on tangent spaces is orientation preserving or reversing. $PD(\alpha) \pitchfork PD(\beta)$ because transversality can always be achieved by a homotopy small enough to preserve embeddedness. In order to understand Q_M in this way it is necessary to show that elements of $H_2(M;\mathbb{Z})$ can be represented by compact embedded surfaces:

Proof.

$$H^2(M;\mathbb{Z}) \leftrightarrow \{\text{complex line bundles on } M\}$$

where the correspondence is given by the first Chern class. However, the zero set of a generic (transverse to zero section) section is Poincaré dual to c_1, and this zero set is an embedded submanifold. (As a closed subset of compact M it is compact.) $\square$

We need to show that $Q_M(\alpha, \beta) = \int_M \alpha \wedge \beta = PD(\alpha) \cdot PD(\beta)$.

Proof.

$$Q_M(\alpha, \beta) = \int_M \alpha \wedge \beta = PD(\beta)(\alpha) = \int_{PD(\beta)} \alpha$$

At such an intersection of surfaces Σ_1 and Σ_2 it is possible to choose local coordinates $\{x_1, x_2, y_1, y_2\}$ so that $\Sigma_1 = \{y_1 = y_2 = 0\}$ and $\Sigma_2 = \{x_1 = x_2 = 0\}$. $PD(\Sigma_1)$ can be represented near each intersection with any Σ_2 as $f(x_1, x_2)dy_1dy_2$ where f has total integral one and the support of f is contained in a neighborhood around $(0, 0, 0, 0)$ small enough to not interact with the supports of the other bumps at the other intersections.

Given this choice for $\Sigma_1 = PD(\alpha)$, $\Sigma_2 = PD(\beta)$, $Q_M(\alpha, \beta) = PD(\alpha) \cdot PD(\beta)$, as desired.

Why is it possible to make this choice of $PD(\Sigma)$? Because M is compact, its homology is finitely generated, so it is possible to represent all generators of $H_2(M; \mathbb{Z})$ as embedded surfaces in such a way that all pairwise intersections are distinct and transverse, and there are finitely many of them. So we can reduce to parities intersections. Then $PD(\Sigma_1)$ is determined by the way it acts on $\omega \in H^2(M; \mathbb{Z})$:

$$\int_{\Sigma_1} \omega = \int_M PD(\Sigma_1) \wedge \omega = \int_{PD(\omega)} PD(\Sigma_1)$$

The middle term is nonzero precisely when it is possible to choose coordinates $\{x_1, x_2, y_1, y_2\}$ so that $PD(\Sigma_1) \sim dy_1dy_2$ and $\omega \sim dx_1dx_2$. The left hand side is nonzero precisely when it is possible to choose coordinates $\{x'_1, x'_2, y'_1, y'_2\}$ so that $\omega \sim dx'_1dx'_2$ and the projection of $T_p\Sigma_1$ onto the span of $\partial_{x'_1}$ and $\partial_{x'_2}$ is surjective for all p contributing to the left hand integral. So we have $x_i = x'_i$. Analyzing the right hand term gives $y_i = y'_i$.

The fact that the fs can be chosen to have total integral one comes because else the middle term would get the total integrals from both forms while the left and right hand sides get the integrals from one. $\square$

Q_M is unimodular (determinant ± 1): $Q_M(\alpha, \cdot) = PD(\alpha)$, and so Q_M is invertible as a map $H^2(M; \mathbb{Z}) \to H_2(M; \mathbb{Z})$. However, if Q_M is invertible then the matrix expressing it as a bilinear form is invertible. But Q_M is a bilinear form on Z-modules, and so it must be invertible over Z; the only elements of Z which have inverses in Z are ± 1.

2.1. **Algebraic Classification.**

The **rank** of Q_M is the rank of $H^2(M; \mathbb{Z})$ mod torsion, that is, the dimension of $H^2(M; \mathbb{R})$, that is, b_2.

Diagonalizing Q_M over $\mathbb{R}$ gives a set of positive and negative eigenvalues; let $b_2^\pm$ be the number of positive/negative eigenvalues. The **signature** of Q_M is $\sigma_M = b_2^+ - b_2^-$.

If for all nonzero $\alpha \in H^2$, $Q_M(\alpha, \alpha) > 0$ then Q_M is **positive definite**. If for all nonzero $\alpha \in H^2$, $Q_M(\alpha, \alpha) < 0$ then Q_M is **negative definite**. Else it is **indefinite**.

If for all $\alpha \in H^2$ $Q_M(\alpha, \alpha)$ is even then Q_M is **even**. Else it is **odd**.

Indefinite symmetric bilinear unimodular forms are classified by rank, signature, and parity. Precisely, if Q is indefinite odd, it is equivalent over $\mathbb{Z}$ to $(1)^{\oplus m} \oplus (-1)^{\oplus n}$. If Q is indefinite even, it is equivalent over $\mathbb{Z}$ to $(\pm E_8)^{\oplus m} \oplus H^{\oplus n}$ with $n > 0$, where E_8 is the Cartan matrix of the Lie algebra E_8 (a definite form) and $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

There are too many definite symmetric bilinear unimodular forms to classify. This is kind of strange, and I don't have a good feeling as to why, since it seems like direct summing anything indefinite to a definite form would give a new indefinite form. There must somehow be "room" in basis changes over $\mathbb{Z}$ to simplify the form.

2.2. **Topological Operations.**

$Q_{M\#N} = Q_M \oplus Q_N$ since connect sum involves only 4-cells/handles and H_2/H^2 is influenced only by $<$ 4-cells/handles. Also $\sigma_{M\#N} = \sigma_M + \sigma_N$.

$Q_{\overline{M}} = -Q_M$ since positive intersections become negative in $\overline{M}$. Also $\sigma_{\overline{M}} = -\sigma_M$.

If M^4 is the boundary of an oriented 5-manifold W^5 then $\sigma_M = 0$. Conversely, if M is smooth and has $\sigma_M = 0$ then there is some smooth oriented W^5 with $M = \partial W$. (There is a bit of a discrepancy between the requirements of smoothness on page 123 of Scorpan cited here.)

2.3. **Simple Examples.**

$\mathbb{CP}^2$: $H_2(\mathbb{CP}^2) = \mathbb{Z}\langle[\mathbb{CP}^1]\rangle$, and two projective lines meet at a point. So $Q_{\mathbb{CP}^2} = (1)$.

$\overline{\mathbb{CP}}^2$: Its second homology is also generated by the projective line, two of which also meet at a point. But their orientations will be the same as the orientations computed for $\mathbb{CP}^2$, whereas the orientation of $\overline{\mathbb{CP}}^2$ is flipped. So So $Q_{\overline{\mathbb{CP}}^2} = (-1)$.

Line bundles over spheres are classified by their first Chern class, which takes values in $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Compactifying each fiber by adding a point at infinity gives an S^2 bundle over S^2. Take as generators of the homology the zero section and a fiber. Let L be a sphere bundle obtained in this way. The intersection form of its total space must be

$$\begin{pmatrix} c_1(L) & 1 \\ 1 & 0 \end{pmatrix}$$

However, an S^2 bundle over S^2 is defined by a choice of gluing over the equator of the base (that is, a map $S^1 \to SO(3)$ which tells you how to identify the fibers from the south with the fibers from the north), and $\pi_1(SO(3)) = \mathbb{Z}/2$. So there are only two such bundles up to homeomorphism, those for which $c_1(L)$ is even and those for which it is odd. However, the form is even in the trivial case and odd in the nontrivial case (corresponding to c_1 even and odd), so there are precisely two such bundles.

We'll denote them by $S^2 \times S^2$ and $S^2 \tilde{\times} S^2$. Note that $Q_{S^2 \times S^2} = H$ and $Q_{S^2 \tilde{\times} S^2} = Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2}$. Due to the classification theorem below, this is enough to show that $S^2 \tilde{\times} S^2$ is homeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$. It is also possible to show this directly.

3. Freedman's Theorem

If M^4 is simply-connected, then

$$H_1(M;\mathbb{Z}) = H^3(M;\mathbb{Z}) = 0 \text{ and } H_3(M;\mathbb{Z}) = H^1(M;\mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) = 0$$

so all homological information is contained in $H_2(M;\mathbb{Z}) = H^2(M;\mathbb{Z})$. In fact, all *topological* information is contained in $H_2 = H^2$ as well, as determined by the intersection form:

Theorem 1. *For each symmetric bilinear unimodular form Q over $\mathbb{Z}$ there exists a closed oriented simply-connected topological 4-manifold with Q as its intersection form. If Q is even there is precisely one; if Q is odd there are precisely two, at least one of which is nonsmoothable.*

I will only mention the existence part of the proof.

Proof. Outline: given Q, plumb disk bundles over spheres to get a manifold with intersection form Q; kill the π_1; the boundary will be a homology sphere; glue in a fake 4-ball to make the resulting manifold closed. These manifolds are all non-homeomorphic since the intersection form is a homeomorphism invariant (since H^2 is). If M has odd intersection form then one can characterize $*M$ by $(*M)\#\mathbb{CP}^2 \cong M\#(*\mathbb{CP}^2)$, and $*M \ncong M$ since one can show (but we will not do so here) that $*\mathbb{CP}^2 \ncong \mathbb{CP}^2$. Using the Kirby-Siebenmann invariant one can show that $*M$ is not smoothable.

The plumbing: Let a_{ij} denote the entries of Q. Denote b_2 copies of S^2 by $S_1, \ldots, S_{b_2}$. Build a D^2 bundle with Euler class a_{kk} over S^k; call its total space M_k. Next, plumb M_i and M_j a_{ij} times, where if $a_{ij} < 0$ do it in such a way that S_i and S_j intersect negatively. Plumbing is identifying a $D^2 \times D^2$ in M_i with a $D^2 \times D^2$ in M_j by crossing the factors.

Killing the π_1: Represent the generators of $\pi_1(M)$ by circles disjointly embedded in ∂M. Attach a 2-handle $D^2 \times D^2$ by sending the boundary of the first disk to the generator. This will kill π_1 but not touch H_2.

Boundary is a homology sphere: Diagram chasing. We have (for j the quotient map $C_n(M;\mathbb{Z}) \to C_n(M, \partial M;\mathbb{Z})$) (replacing $H^2(M;\mathbb{Z})$ on the lower right with $\mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z})$ with the universal coefficient theorem)

$$\begin{tikzcd} H_2(M;\mathbb{Z}) \arrow[r, "j^*"] \arrow[d, "Q"'] & H_2(M,\partial M;\mathbb{Z}) \arrow[d, "PD"] \\ \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \arrow[r, leftrightarrow] & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \end{tikzcd}$$

which commutes (takes an argument), so Q is injective/surjective precisely when j^* is. Using the long exact sequence on homology gives

$$0 \to H_2(\partial M;\mathbb{Z}) \xrightarrow{i^*} H_2(M;\mathbb{Z}) \xrightarrow{j^*} H_2(M,\partial M;\mathbb{Z}) \xrightarrow{\partial} H_1(\partial M;\mathbb{Z}) \to 0$$

therefore the bijectivity of Q forces both $H_2(\partial M;\mathbb{Z})$ and $H_1(\partial M;\mathbb{Z})$ to be zero.

$*\mathbb{CP}^2$: Attach a $D^2 \times D^2$ to the boundary S^3 of a D^4 by sending the boundary of the first D^2 to some knot in S^3. With respect to any oriented Seifert surface of the knot, choose the $S^1 \times D^2$ to twist once around the knot. If we choose the unknot, it can be shown that the boundary of the resulting manifold is an S^3, and capping off with another D^4 gives $\mathbb{CP}^2$. However, if we choose the trefoil, it is possible to show that the resulting boundary is the Poincaré homology sphere, which can be capped by a fake D^4, giving $*\mathbb{CP}^2$. $*\mathbb{CP}^2$ has the same intersection form as $\mathbb{CP}^2$: the only nontrivial closed surface comes from the $D^2 \times D^2$ attachment, taking the Seifert surface together with the handle's $D^2 \times \{0\}$.

The Kirby-Siebenmann invariant is $\mathbb{Z}/2$-valued and is 1 when the manifold is not smoothable. It is also additive under #. Finally, $ks(\mathbb{CP}^2) = 0$ and $ks(*\mathbb{CP}^2) = 1$ (I am not claiming to compute this, only stating it). □

3.1. **Homology S^3s and Fake D^4s.**

Theorem 2. *Every Y^3 with the integral homology of S^3 bounds a contractible 4-manifold Δ^4.*

In particular, note that Y is not necessarily contractible on its own, when the homotopy is not allowed to pass through Δ.

One constructs Δ by stacking end-to-end infinitely many copies of a homotopy copy of $S^3 \times [0,1]$ obtained from $Y \times [0,1]$ by killing π_1 and H_2, then compactifying by adding a point at one end. This is the difficult part of the existence half of Freedman's theorem, since killing the H_2 is pretty hard.

3.2. **Harder Example: the E_8 Manifold.**

The topological 4-manifold realizing the intersection form E_8 will complete our topological 4-manifold building blocks, and accounts for nonsmoothability. One obtains M_{E_8} via Freedman's plumbing construction, then capping off the resulting homology sphere with the fake 4-ball from Freedman's theorem on homology spheres. It turns out that the homology sphere is Poincaré's original homology sphere.

We have $E_8 \oplus (-E_8) = 8H$ because they are both indefinite and have the same rank (16), signature (0), and parity (even); this helps us reduce intersection forms which could be a sum of positive and negative copies of E_8 to a sum of $\pm E_8$s and Hs, with E_8s' signs consistent.

4. Restrictions on Smooth Manifolds

4.1. **Due to Rohklin.**

Theorem 3. *If M is smooth and simply-connected with even intersection form then 16 divides σ_M.*

There hypotheses can be weakened, giving an understanding of the role of $w_2(M)$, but that is not important for us here.

Note that M_{E_8} is not smoothable.

4.2. **Due to Donaldson.**

Theorem 4. *The only definite forms arising as the intersection form of a smooth 4-manifold are I_m and $-I_m$.*

Since indefinite forms are classified, we now have all homeomorphism types of smooth 4-manifolds: $(\mathbb{CP}^2)^{\#m}\#(\overline{\mathbb{CP}}^2)^{\#n}$ realizing the sums of (1) and (-1) in the odd case, and $(\pm M_{E_8})^{\#m}\#(S^2 \times S^2)^{\#n}$ realizing the sums of $\pm E_8$ and H in the even case.

The even forms are further restricted by Rohklin's Theorem: the terms with intersection form E_8 (M_{E_8}) must occur in pairs, as must those with intersection form $-E_8$ ($\overline{M}_{E_8}$). Moreover, n must be nonzero in the even case lest the form be definite.

However, not all of the even homeomorphism types are smoothable. (The odd ones are all fine since $\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2$ are both smoothable; however, M_{E_8} is not by Rohklin's Theorem.)

4.3. **Realization Constraints and Nonuniqueness of Smooth Structures.**

In an attempt to further refine the possible even intersection forms, we have the following conjecture:

Conjecture 1. *If M^4 is smooth with Q_M even then*

$$b_2 \geq \frac{11}{8}|\sigma_M|$$

This is equivalent to $n \geq 3|m|$:

Proof. The signature of E_8 is 8 (I have not shown this; it comes from elementary row operations) and the signature of H is 0. So $|\sigma_M| = 16|m| + 0n$ and $b_2 = 16|m| + 2n$. So we get

$$b_2 \geq \frac{11}{8}|\sigma_M| \Leftrightarrow 16|m| + 2n \geq \frac{11}{8} \cdot 16|m| \Leftrightarrow 8|m| + n \geq 11|m|$$

$\square$

Using Seiberg-Witten monopoles (in contrast to Donaldson's instantons), Furuta proved

Theorem 5. *If M^4 is smooth with Q_M even then*

$$b_2 \geq \frac{10}{8}|\sigma_M| + 2$$

or equivalently, $n \geq 2|m| + 1$.

The one hand was constraining the homeomorphism type of a smooth manifold; the other hand is investigating the smooth types of a (closed, orientable, simply-connected) topological manifold. For example, Friedman and Morgan have proved

Theorem 6. *The topological manifolds corresponding to $2n(-E_8) \oplus (4n-1)H$ with $n \geq 1$ and $(2k-1)(1) \oplus N(-1)$ with $k \geq 2$, $N \geq 10k-1$ each carry infinitely many nondiffeomorphic smooth structures.*

5. Exotic $\mathbb{R}^4$s

Here is the other hand for the simplest open, orientable, simply-connected topological manifold. Exotic $\mathbb{R}^4$s appear when the topological and smooth classifications clash.

There is something called a **small exotic** $\mathbb{R}^4$; we will concern ourselves here with **large exotic** $\mathbb{R}^4$**s**.

5.1. Constructions.

Let $E = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ (this is smooth). Its intersection form is $(1) \oplus 9(-1)$, therefore E is homeomorphic to $\overline{M}_{E_8} \# \overline{\mathbb{CP}}^2 \# \mathbb{CP}^2$. Let's try to decompose $E = N \# \mathbb{CP}^2$ smoothly.

Let $\alpha \in H_2(E;\mathbb{Z})$ correspond to the (1) in Q_E. Assume it can be represented by a smoothly embedded S^2. Because $Q_E(\alpha,\alpha) = 1$, a neighborhood of α can be viewed as a disk bundle with Euler number 1, just like a neighborhood of $\mathbb{CP}^1$ in $\mathbb{CP}^2$. This neighborhood of $\mathbb{CP}^1$ has boundary S^3, so we could cut out the neighborhood of α and glue in a 4-ball instead, obtaining N.

However, $Q_N = -E_8 \oplus (-1)$, a definite form and so not allowed under Donaldson's restrictions. So no such class α can exist. However, one can show that α may be represented by a topologically embedded S^2, and we can follow the above constructions obtaining only a topologically embedded 4-ball in $\mathbb{CP}^2$. It has a smooth structure from its embedding in $\mathbb{CP}^2$, but this cannot be $\mathbb{R}^4$'s. In $\mathbb{R}^4$ every compact set can be surrounded by a smoothly embedded S^3 (compact implies bounded, which is defined as "surroundable by a smoothly embedded S^3), but if this were the case for our 4-ball then one could pull that S^3 back to E and get a forbidden disk bundle.

5.2. **How Many?**

There are uncountably many. We will see that there are at least countably many: cutting out neighborhoods in two exotic $\mathbb{R}^4$s of a smoothly embedded $(0, \infty)$ (going towards infinity) and gluing the resulting $\mathbb{R}^4$s together along their boundary $\mathbb{R}^3$ gives a new exotic $\mathbb{R}^4$. It is a theorem of Gompf that if we take n copies of, for example, the exotic $\mathbb{R}^4$ above and sum them in this way, they will never be diffeomorphic.

References

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