MASLOV INDEX

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Contents

Drawing from [1].

1. Outline

Motivation. We want to put a grading on the chain complex generated by the critical points of the action functional, that is, the periodic orbits of $H_t : M \to \mathbb{R}$. However, we can't naively do the Morse homology thing (grading via the dimension of the negative eigenspace of the Hessian of the Morse function) "because the loop space $\mathcal{L}M$ is infinite-dimensional."

I put our catchphrase in quotes because even though $\mathcal{L}M$ is infinite-dimensional, for some actions the negative eigenspaces of the Hessian of a "Morse function" are still finite dimensional (e.g. if we're looking for closed geodesics, the functional is $E(x) = \int_0^1 |\dot{x}(t)| \, dt$). Let's try to make sense of the "dimension of the negative eigenspaces of $\mathcal{A}$."

$\nabla\mathcal{A}(x) = J_{x(t)}(\dot{x}(t)) + \nabla_g H_t(x(t))$. The negative eigenspace of the Hessian of $\mathcal{A}$ is the span of directions we can push off of x – sections of x^*TM – for which $\nabla\mathcal{A}(x)$ applied to that direction, changing as x

changes in that direction, goes down. That is, $s \in \Gamma(x^*TM)$ for which $\nabla\mathcal{A}(x+s)(s) < 0$, where I should probably be being more careful about exponentiating s and what "+" means. But essentially we want

$$\nabla\mathcal{A}(x+s)(s) = J_{x(t)+s(t)}(\dot{x}(t)+\dot{s}(t)) + \nabla_g H(t)(x(t)+s(t)) < J_{x(t)}(\dot{x}(t)) + \nabla_g H_t(x(t))$$

componentwise. When we restrict to thinking about only the decrease in $\nabla_g H_t$, even though H_t has only a finite-dimensional negative eigenspace, anything in the space of smooth functions from $[0,1]$ to this eigenspace will make $\nabla_g H(t)(x(t)+s(t)) < \nabla_g H_t(x(t))$. Since J is pointwise an isometry, there should be some length of $\dot{s}(t)$ to set to make the effect from J not too much to cancel the effect from $\nabla_g H$ (I think H_t's nondegeneracy makes this possible – it changes "like squaring" in the size of s).

Definition of the Maslov Index. Trivialize x^*TM (somehow), and choose a symplectic basis for $T_{x(0)}M$, providing a family of symplectic bases for $T_{x(t)}M$ by mapping the basis vectors forward via the trivialization. Let φ^t denote the time t flow of H and consider $A(t) := A_x(t) = d\varphi^t_{x(0)}$. In the symplectic basis for $T_{x(0)}M$ and $T_{x(t)}M$, $A(t) \in Sp(2n)$. $A(0)$ is the identity and the nondegeneracy of H_t forces $\det(A(1) - I) \neq 0$.

Construct $\rho : Sp(2n) \to S^1$ an isomorphism on π_1. For any $\gamma : [0,1] \to Sp(2n)$, choose a lift $\alpha : [0,1] \to \mathbb{R}$ of $\rho \circ \gamma$, and define

$$\Delta(\gamma) = \frac{\alpha(0) - \alpha(1)}{\pi}$$

Fix $W^\pm \in Sp(2n)^\pm := \{B \in Sp(2n) \mid \det(B - I) \geq 0\}$ or $\{< 0\}$, respectively. We will specifically use

$$W^+ = -I_{2n} \text{ and } W^- = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus -I_{2n-2}$$

For any $B \in Sp(2n)^* := \{B \in Sp(2n) \mid \det(B - I) \neq 0\}$, let γ_B denote a path from B to $W^\pm$ in $Sp(2n)^*$ (that is, if $B \in Sp(2n)^+$, choose a path to W^+).

Let $\psi : [0,1] \to Sp(2n)$ with $\psi(0) = I$ and $\det(\psi(1) - I) \neq 0$. Define the **Maslov index of** ψ as

$$\mu(\psi) := \Delta(\psi) + \Delta(\gamma_{\psi(1)})$$

the number of clockwise half-turns which $\rho \circ (\psi \cdot \gamma_{\psi(1)})$ makes around $Sp(2n)$.

Finally, define the **Maslov index of** x as

$$\mu(x) := \mu(A(t))$$

Questions Demanding Answers.

- What properties of the topology of M allow us to obtain a well-defined path in $Sp(2n)$ from an orbit of H?
- What is ρ and what properties of ρ do we need? e.g.
  - ρ must be an isomorphism on π_1

  - ρ must be independent of the trivialization of x^*TM

- What properties of $Sp(2n)^*$ do we need? e.g.
  - can we characterize its connected components as $Sp(2n)^{\pm}$, and how?
  - how can we obtain a well-defined path between $A(1)$ and $W\pm$?
  - why do we even need $W^{\pm}$?
- Does the Maslov index make sense as a grading
  - on the Hamiltonian Floer chain complex?
  - as a generalization of the Morse grading?
  - for which the grading difference between pairs of orbits is compatible with the Fredholm index of the linearized Floer equation on the space of cylinders between those orbits?

2. The Path of an Orbit

Of course we can trivialize x^*TM if we consider x as a map from $[0,1]$ to M and thence consider the derivative of the flow of H_t along x in the basis given by the trivialization. But how can we guarantee a trivialization as a map from S^1 to M? (Specifically, we need $A(1)$ to be in the same coordinate system as $A(0)$.) Hint: we will use the contractibility of x. However, we then need to show independence of the contraction for the path to be well-defined.

Obtaining A. Since x is contractible, there is some continuous u extending x to a disk, that is, $x = u|_{\partial D}$, because

Fact 1. *For every continuous $u : D^k \to M$, we get a trivialization of u^*TM as a symplectic fiber bundle. All its trivializations are homotopic.*

Remark. Trivialize the bundle by the contractibility of D^k. Push the symplectic structure around via the sections (that is, pull the tangent vectors back to one point at which the form is defined). Two trivializations differ only by a map $D^k \to Sp(2n)$, and $Sp(2n)$ is path connected: □

Fact 2. *$Sp(2n)$ retracts onto $U(n)$. Via this retraction it is path connected and its fundamental group is $\mathbb{Z}$.*

Remark. Use the polar decomposition of $GL(2n;\mathbb{R})$ restricted to $Sp(2n)$ and show that the fibers in $Sp(2n)$ over elements of $U(n)$ are contractible. The determinant realizes the isomorphism from $\pi_1(U(n))$ to $\mathbb{Z}$. □

Using the trivialization of u^*TM, obtain a trivialization of x^*TM as a map from S^1. In this trivialization, $A : [0,1] \to Sp(2n)$ is defined, has $A(0) = I$, and has $A(1) \in Sp(2n)^*$.

Independence of u. We need independence of the choice of u.

Let v be another extension of x to a disk. Let $w : S^2 \to M$ be given by u on one hemisphere and v on the other (they agree on the equator since they both agree with x). By our assumption that the first Chern class of (M, ω) pairs trivially with $\pi_2(M)$ (technically with its image under the Hurewicz map), we can trivialize w^*TM.

Since trivializations of u (respectively, v) are homotopic, they are in particular homotopic to the trivialization given via inclusion as the restriction of w to one hemisphere. These agree on the equator. Therefore the trivializations given by restricting the trivializations of u and v to their boundaries are homotopic.

Autonomous Case. When H is not time-dependent, $A(t) = \exp(tJ_0\mathrm{Hess}_x)$ (nondegenerate so long as Hess_x is nonsingular with all eigenvalues less than 2π). You can show this by showing that the derivative of the flow of X_H at x is $J_0\mathrm{Hess}_x$ times the flow again.

3. The Map $\rho : Sp(2n) \to S^1$

Theorem 1. *For every $n \in \mathbb{Z}_{>0}$ there exists a continuous map $\rho : Sp(2n) \to S^1$ such that*

(1) *naturality:* $\rho(TBT^{-1}) = \rho(B)$

(2) *product:* $\rho \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A)\rho(B)$

(3) *determinant: we have $U(n) \subset Sp(2n)$, and $\rho|_{U(n)} = \det_{\mathbb{C}}$*

(4) *normalization: if spec$(B) \subset \mathbb{R}$ then $\rho(B) = (-1)^{\frac{m_0}{2}}$, where m_0 is the total multiplicity of B's negative real eigenvalues*

(5) $\rho(A^T) = \rho\left(A^{-1}\right) = \overline{\rho(A)}$

Remark. We like (1) because it shows that ρ is independent of a choice of coordinates.

(3) induces an isomorphism on π_1, since the determinant provides an isomorphism between $\pi_1(U(n))$ and $\mathbb{Z}$, and **Fact 2** asserts that circles in $Sp(2n)$ are uniquely represented by homotopic circles in $U(n)$ (their images under the retraction).

We need (4) many times to prove that μ makes sense as an index for a homology.

(2) and (5) are useful to prove lemmas we will not prove today. $\square$

Example: $Sp(2)$**.** We have $Sp(2) = SL(2;\mathbb{R})$ (show by setting $B^T J_0 B = J_0$). The matrices with one as an eigenvalue are the 2×2 matrices with determinant one and trace two. The subsets of $Sp(2)$ with trace greater than or less than two are each connected:

$$\mathrm{tr}B > 2 \Rightarrow B \text{ has eigenvalues } \lambda \neq \lambda^{-1} > 0 \text{ (use the quadratic formula)}$$

$$\mathrm{tr}B < 2 \Rightarrow B \text{ has eigenvalues } \lambda, \lambda^{-1} \text{ such that}$$

$$\lambda^2 - (a+d)\lambda + 1 = 0 \text{ and } \frac{1}{\lambda^2} - (a+d)\frac{1}{\lambda} + 1 \text{ (since } \det B = 1 \text{ or by dividing by } \lambda^2)$$

$$\Rightarrow \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2}$$

$$a + d < -2 \Rightarrow -(a+d) > \sqrt{(a+d)^2-4} > 0 \Rightarrow \lambda \neq \lambda^{-1} < 0$$

$$a+d = -2 \Rightarrow \text{ Jordan normal form is either } -I \text{ or } \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \text{ the matrices similar to } \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

$$-2 < a+d < 2 \Rightarrow (a+d)^2 - 4 < 0 \Rightarrow \lambda \neq \lambda^{-1} \in \mathbb{C}\backslash\mathbb{R}$$

characterizing these sets by their (connected sets of) eigenvalues, and similarity classes of matrices are connected because GL^+ is connected.

Note that while $\rho|_{U(n)}$ is the complex determinant (take the determinant as if the matrix were in $GL(n;\mathbb{C})$ rather than in $GL(2n;\mathbb{R})$), $\rho(US)$ is not necessarily $\det_{\mathbb{C}}(U)$ for US the polar decomposition of some symplectic matrix:

$$U = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \text{ and } S = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

Since $\mathrm{tr}(US) > 2$, $\rho(US) = 1$ by (4) in **Theorem 1**, yet $\det_{\mathbb{C}}(U) = e^{i\frac{\pi}{3}}$.

The above example motivates the following proposition, due to Conley and Zehnder:

Proposition 1. *The open set $Sp(2n)^*$ has two connected components, $Sp(2n)^{\pm}$, each having trivial π_1.*

The proof follows from the lemma

Lemma 1. *For $B \in Sp(2n)^*$ there exists a path in $Sp(2n)^*$ connecting B to a matrix C whose eigenvalues are all distinct with exactly zero ($B \in Sp(2n)^+$) or two ($B \in Sp(2n)^-$) real positive eigenvalues.*

Given this lemma, we can specifically connect matrices B in $Sp(2n)^*$ to the matrices

$$-I \in Sp(2n)^+ \text{ for } B \in Sp(2n)^+ \text{ or } \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus -I \text{ for } B \in Sp(2n)^-$$

First connect B to C. Choose a symplectic basis $\{X_\lambda\}$ of eigenvectors of C which is invariant under complex conjugation (put together some stuff from §5.6 to get this). For eigenvalues $\lambda \notin \mathbb{R}_+$ of C choose a path $\lambda(s)$ connecting λ to -1 while avoiding 1, where the path associated to $\overline{\lambda}$ is $\overline{\lambda(s)}$, and the path chosen for $\frac{1}{\lambda}$ is $\frac{1}{\lambda(s)}$. Now let X_λ be the eigenvector with eigenvalue λ, and define a path from C by

$$C(s) \cdot X_\lambda = \begin{cases} \lambda(s) X_\lambda & \text{if } \lambda \notin \mathbb{R}_+ \\ \lambda X_\lambda & \text{if } \lambda \in \mathbb{R}_+ \end{cases}$$

Then $C(s) \in Sp(2n)^*$ with $C(0) = C$ and

$$C(1) \cdot X_\lambda = \begin{cases} -X_\lambda & \text{if } \lambda \notin \mathbb{R}_+ \\ \lambda X_\lambda & \text{if } \lambda \in \mathbb{R}_+ \end{cases}$$

If C has no positive real eigenvalues then $C(1) = -I$ and if C has precisely two positive real eigenvalues then in the basis $\{X_\lambda\}$,

$$C(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \oplus -I$$

which we need to show can be connected to $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus -I$ in $Sp(2n)^-$.

We need to change bases to obtain a matrix of the same form in the canonical basis, then adjust the eigenvalues. The second part is easy since all matrices of this form are symplectic, so we just need to choose paths between $\left\{\lambda, \frac{1}{\lambda}\right\}$ and $\left\{2, \frac{1}{2}\right\}$ which don't cross 1.

The first part is possible by conjugating by a path of unitary matrices interpolating between the identity and the actual change-of-basis matrix (which will be unitary since $C(1)$ is symmetric). Since all elements in a similarity class have the same eigenvalues, conjugation by any matrix in this path will keep $U^{-1}C(1)U \in Sp(2n)^-$.

Finally, to show the triviality of the fundamental groups, use the lemma

Lemma 2. *There exist continuous $\tilde{\rho}_\pm : Sp(2n)^\pm \to \mathbb{R}$ such that the diagrams commute:*

$$\begin{tikzcd} & \mathbb{R} \arrow[dr, "\exp"] & \\ Sp(2n)^\pm \arrow[ur, "\tilde{\rho}_\pm"] \arrow[r, "\iota"'] & Sp(2n) \arrow[r, "\rho"'] & S^1 \end{tikzcd}$$

Its proof relies on the details of the construction of ρ.

On π_1 we get

$$\begin{tikzcd} & 0 \arrow[dr, "\exp_*"] & \\ \pi_1(Sp(2n)^\pm) \arrow[ur, "(\tilde{\rho}_\pm)_*"] \arrow[r, "\iota_*"'] & \pi_1(Sp(2n)) \arrow[r, "\rho_*"'] & \mathbb{Z} \end{tikzcd}$$

Given the deduction from **Theorem 1** that ρ_* is an isomorphism on fundamental groups, we have

$$\begin{aligned}\rho_* \circ \iota_*(\pi_1(Sp(2n)^{\pm})) &= \exp_* \circ (\tilde{\rho}_{\pm})_*(\pi_1(Sp(2n)^{\pm})) \\ \iota_*(\pi_1(Sp(2n)^{\pm})) &= \rho_*^{-1} \exp_*(0) \\ &= 0\end{aligned}$$

5. Properties of the Maslov Index

Which Allow Us to Use it for Grading.

Proposition 2. (1) *μ takes values in $\mathbb{Z}$*

(2) *ψ_0 and ψ_1 are homotopic relative to $Sp(2n)^*$ if and only if $\mu(\psi_0) = \mu(\psi_1)$*

(3) *the sign of* $\det(\psi(1) - I) = (-1)^{\mu(\psi)-n}$

Proof. For (1), $\psi \cdot \gamma_{\psi(1)}$ starts at $\psi(0) = I$ and ends at $W^{\pm}$. I maps under ρ to 1 and both $W^{\pm}$ map under ρ to $\pm 1 \in S^1$, so must lift to integer multiples of π in $\mathbb{R}$.

We have answered the question about the relevance of $W^{\pm}$ – they provide endpoints whose lifts give precisely the count of half-turns, not up to homotopy in $(0, \pi)$.

For (2), because $\pi_1(Sp(2n)^{\pm}) = 0$, ψ_i are homotpic relative to $Sp(2n)^*$ if and only if their extensions $\psi_i \cdot \gamma_{\psi_i(1)}$ are homotopic relative to endpoints. But since ρ_* is an isomorphism on π_1, this is if and only if the number of half-turns they make are the same, which is if and only if their lifts in $\mathbb{R}$ have the same endpoints (since they both start at the same point), which is if and only if $\mu(\psi_0) = \mu(\psi_1)$.

For (3) we have

$$\begin{aligned}\det(\psi(1) - I) > 0 &\Leftrightarrow \psi(1) \in Sp(2n)^+ \\ &\Leftrightarrow \text{ the endpoint of } \gamma_{\psi(1)} \text{ is at } W^+ \\ &\Leftrightarrow \mu(\psi) - n = \frac{\psi \cdot \gamma_{\psi(1)}(0) - \psi \cdot \gamma_{\psi(1)}(1)}{\pi} - n \\ &\qquad = \frac{\text{lift of } 1 - \text{ lift of } (-1)^n - n\pi}{\pi} \\ &\qquad \in \begin{cases} \text{odd} - \text{odd} & \text{if } n \in 2\mathbb{Z} \\ \text{even} - \text{even} & \text{if } n \in 2\mathbb{Z} + 1 \end{cases}\end{aligned}$$

and similarly for $\det(\psi(1) - I) < 0$. □

Which Make Us Feel Confident it's Meaningful. Firstly, $\mu(x)$ relates to the index of x as a critical point of an autonomous Hamiltonian as follows:

Proposition 3. *If S is an invertible symmetric matrix with norm $< 2\pi$, then*

$$\mu(\exp(tJ_0 S)) = \#\{\textit{negative eigenvalues of } S\} - n$$

Secondly, we'll soon want the Fredholm index of the linearized $\overline{\partial}_J$ operator when applied to the curves between two orbits to agree with the Maslov index difference of those orbits. This works as follows.

The Maslov index is a "perfect" homotopy invariant. So we'll define paths of symplectic matrices realizing the desired Maslov indices, which are therefore homotopic to the true paths obtained from the orbits.

Next we'll use a correspondence ($\exp t$) between paths of symplectic matrices and symmetric matrices to calculate the Fredholm index of the linearized $\overline{\partial}_J$ operator with perturbation term corresponding to the paths realizing the Maslov indices. The Fredholm index will equal the difference in the Maslov indices in this case.

Since the Fredholm index is invariant under small perturbations, and the perturbation to get to the actual linearized $\overline{\partial}_J$ operator is small enough (since it arises from a homotopy on the paths of symplectic matrices), we preserve the equality.

6. What We Have Not Done

We haven't talked about how to obtain ρ (**Theorem 1**) or proved all lemmas on the way to understanding the topology of $Sp(2n)^*$ (**Lemma 1**, **Lemma 2**).

We do not understand how to associate a path of symplectic matrices to a critical point of an autonomous Hamiltonian, nor how to understand the association between the Maslov index and the Morse index in the autonomous case (**Proposition 3**).

Finally, we do not understand the correspondence between paths of symplectic matrices and symmetric matrices, nor do we know how to choose a symmetric matrix so as to obtain a path of symplectic matrices with prescribed Maslov index.

My suggestion: leave **Theorem 1**, and the **Lemmas** for independent linear algebrization. Leave the autonomous case as a (comparatively straightforward) exercise. Postpone the discussion of prescribing the Maslov index until it becomes necessary when we're trying to reconcile the grading given by the Maslov index with the definition of the differential via the Fredholm index of the linearized $\overline{\partial}_J$ operator.

However, the existence and properties of ρ and the topology of $Sp(2n)^*$ are what this whole operation depends on, so I can understand wanting to understand **Theorem 1** and the **Lemmas**.

References

[1] Audin, Michéle and Mihai Damian, trans. Reinie Erné. *Morse Theory and Floer Homology*, Springer-Verlag, London, 2014.