Handles

def: $0 \le k \le n$

an <u>n-dim'l k-handle</u> h is $\mathbb{D}^k \times \mathbb{D}^{n-k}$ attached to the boundary of an n-manifold X along $\partial \mathbb{D}^k \times \mathbb{D}^{n-k}$ by an embedding $\varphi : \partial \mathbb{D}^k \times \mathbb{D}^{n-k} \hookrightarrow \partial X$. notation $X \cup_{\varphi} h$

(smoothing corners is canonical...)

note: homotopy equivalent to attaching a k-cell

terminology: <u>core</u> $\mathbb{D}^k \times 0$, <u>cocore</u> $0 \times \mathbb{D}^{n-k}$, attaching <u>map</u> φ

<u>attaching region</u> $\partial \mathbb{D}^k \times \mathbb{D}^{n-k} \leftrightarrow \varphi(\partial \mathbb{D}^k \times \mathbb{D}^{n-k})$

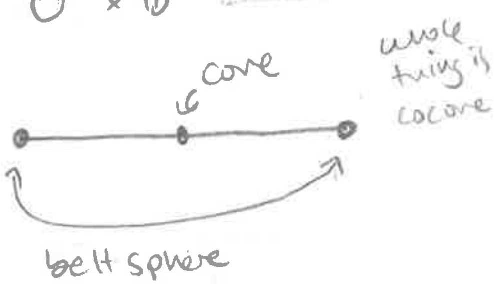
<u>attaching sphere</u> $\partial \mathbb{D}^k \times 0 \leftrightarrow \varphi($ it $)$ (<u>descending</u>)

<u>belt sphere</u> $0 \times \partial \mathbb{D}^{n-k}$ (<u>ascending</u>)

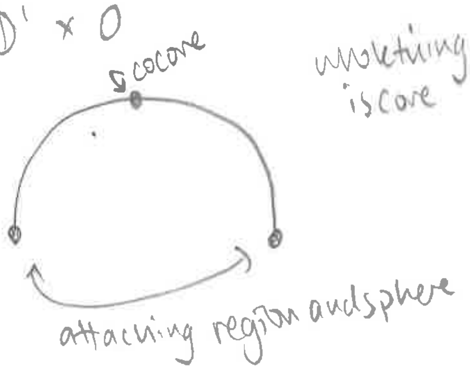
<u>index</u> of the handle is k (compare: Morse index)

1-dim'l...

0-handle: $0 \times \mathbb{D}^1$

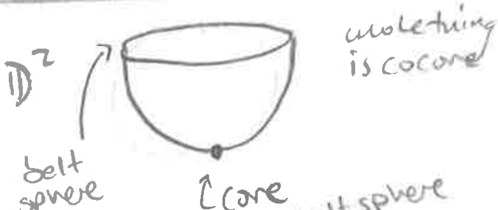


1-handle: $\mathbb{D}^1 \times 0$

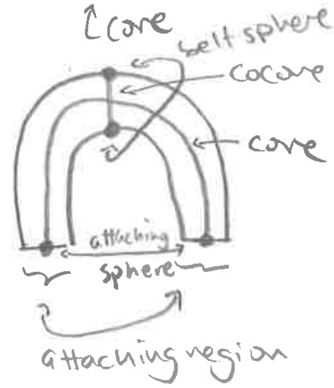


2-dim'l...

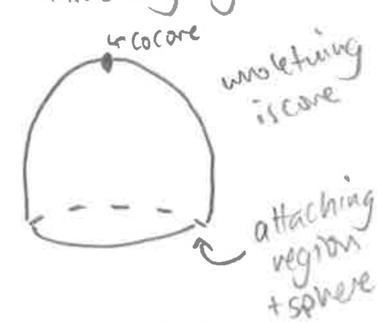
0-handle: $0 \times \mathbb{D}^2$



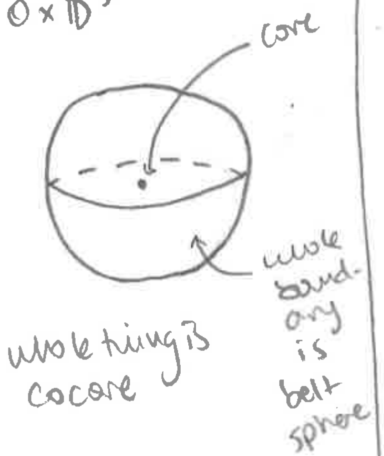
1-handle: $\mathbb{D}^1 \times \mathbb{D}^1$



2-handle: $\mathbb{D}^2 \times 0$

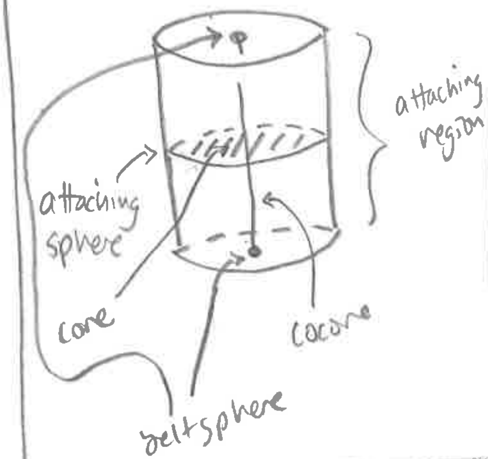


3-dim'l...
0-handle:
$0 \times \mathbb{D}^3$

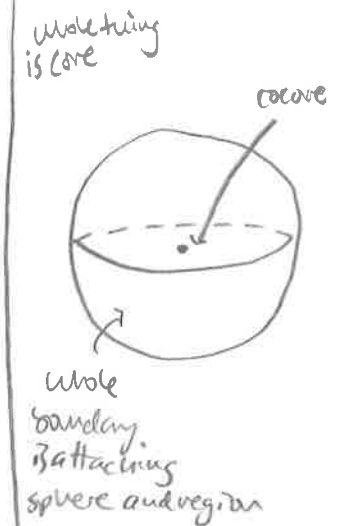


1-handle: $\mathbb{D}^1 \times \mathbb{D}^2$

2-handle: $\mathbb{D}^2 \times \mathbb{D}^1$



3-handle: $\mathbb{D}^3 \times 0$

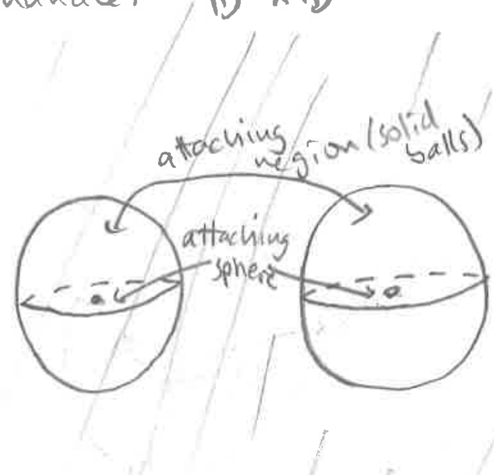


4-dim'l...
0-handle: $0 \times \mathbb{D}^4$

attaching region + sphere are empty

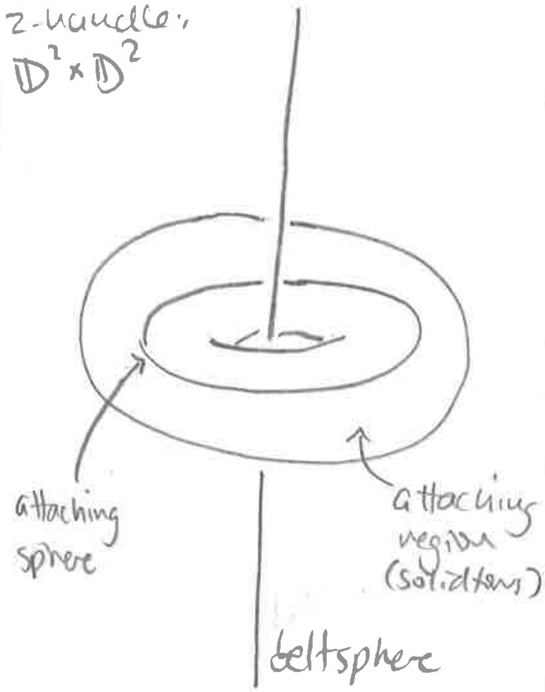
belt sphere is all of S^3

1-handle: $\mathbb{D}^1 \times \mathbb{D}^3$

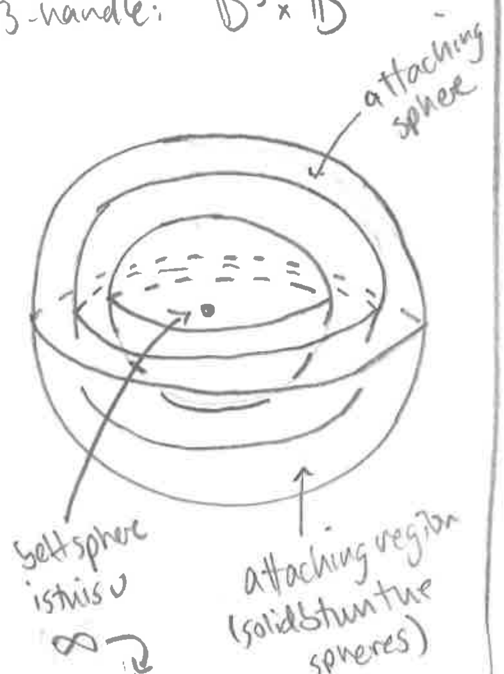


an S^2 splitting these $\mathbb{D}^3$s is the beltsphere

2-handle: $\mathbb{D}^2 \times \mathbb{D}^2$



3-handle: $\mathbb{D}^3 \times \mathbb{D}^1$



4-handle: $\mathbb{D}^4 \times 0$

all of S^3 is the attaching region + sphere

the belt sphere is empty

note:
- the only handle we can attach to $\emptyset$ is a 0-handle
- for the 4-dimensional handles I have only drawn the attaching regions + spheres and beltspheres, as they sit in $S^3 = \partial$(any 4-dim'l handle)

Data of a Handle

isotopy of φ doesn't change diffeo type of $X \cup_\varphi h$:
- use isotopy extension thm to an ambient isotopy $\Phi: I \times \partial X \to \partial X$
- identify $I \times \partial X$ with a collar neighborhood of ∂X in X
- get a diffeo of $X \cup_\varphi h$ to $X \cup_{\varphi'} h$ by the diffeo $id_I \times \Phi$ on this collar

φ can be constructed from
- an embedding $\varphi_0: \partial D^k \times 0 \to \partial X$ with trivial normal bundle
- a trivialization f of the normal bundle, called a <u>framing</u>

because by the tubular nbhd thm, $\varphi_0(\partial D^k \times 0)$ has a neighborhood diffeo to its normal bundle

moreover, φ is determined up to isotopy of framed embeddings

Framings

choose a framing f_0. every other differs at each point in $\partial D^k \times 0 \cong S^{k-1}$ by an element of $GL(n-k)$. compose w/ a self-diffeo of the second factor of $D^k \times D^{n-k}$ to get this elmt $=$ id at a basepoint p

$\Rightarrow$ an elmt of $\pi_{k-1}(O(n-k))$ for each framing

this is a bijection from isotopy classes of framings

it's only a torsor, not in general canonically identified with $\pi_{k-1}(O(n-k))$!

Handle Decomps

X compact n-mfld w/ $\partial X = \partial_+ X \amalg \partial_- X$

compact, may be empty

orient $\partial_\pm X$ so that $\partial X = \partial_+ X \amalg \overline{\partial_- X}$

a handle decomp of X is an identification of X w/ a mfld obtained from $I \times \partial_- X$ by attaching handles, with $\partial_- X$ identified w/ $0 \times \partial_- X$

X is a relative handlebody on $\partial_- X$, or if $\partial_- X = \emptyset$ a handlebody

thm: every compact smooth pair $(X, \partial_- X)$ admits a handle decomp (by identifying descending submanifolds of Morse critical points of index k w/ cores of index k handles), using a proper Morse fxn to $[0,1]$ w/ $f^{-1}(0) = \partial_- X$, $f^{-1}(1) = \partial_+ X$

note: $(X, \partial_- X)$ admits a handle decomp $\Leftrightarrow$ X is smoothable

prop 4.2.7: Any handle decomp of a compact pair $(X, \partial_- X)$ can be modified by isotoping attaching maps so the handles are attached in order of increasing index; handles of the same index can be attached in any order, including simultaneously

the dual handle decomp is obtained on $(X, \overline{\partial_+ X})$ by gluing a collar $I \times \overline{\partial_+ X}$ and removing the collar $I \times \partial_- X$, then reversing all the roles of a k-handle to that of an $(n-k)$-handle one can <u>cancel</u> some handles of adjacent index: the point is that in some configurations their union is a ball attached to X by a ball (the attaching sphere of the $(k-1)$-handle in ∂X is filled by half the attaching sphere of the k-handle in ∂X, which is a $(k-1)$-ball)

prop 4.2.9: A $(k-1)$-handle h_{k-1} and a k-handle h_k $(1 \le k \le n)$ can be cancelled provided the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.

one can <u>slide handles</u> of the same index over one another by isotoping the attaching sphere of h_1 through the belt sphere of h_2; when they intersect at one point p with the sum of their tangent spaces codim 1 in $T_p \partial(X \cup h_2)$ there are precisely 2 directions to go: one is original, other is slid

Thm 4.2.12: Given any two relative handle decomps (ordered by increasing index) of a compact pair $(X, \partial_- X)$ one can get from one to the other by a sequence of handle slides, creating/cancelling pairs, and isotopies.

"pf": Cerf theory

Prop 4.2.13: If X^n is compact, connected then $(X, \partial_- X)$ admits a handle decomp w/ exactly one 0-handle if $\partial_- X = \emptyset$ and none else. Similar statement for n-handles

pf: Only 1-handles can connect 0-handles if there are multiple, b/c S^0 is the only disconnected sphere.

There's a way to get H_* from handle decomps.

Heegaard Splittings (of closed X)

X^3 has a handle decomp w/ a unique 0- and 3-handle

in general, orientable $\Rightarrow$ framings of 1-handles determined b/c $\pi_0(O(n-1)) = \mathbb{Z}_2$ and X orientable $\Leftrightarrow$ around any loop in X, orientation never reverses

$\Rightarrow X_1 = \text{0-handle} \cup_{i=1}^{g} \text{1-handles}_i \cong \natural_g S^1 \times \mathbb{D}^{n-1}$

in general, when $n \neq 2$, $\pi_{n-2}(O(1)) = 0$ and $\pi_{n-1}(O(0)) = 0$, so framings do not matter for $(n-1)$- and n- handles $\Rightarrow$ $\boxed{\text{3-dim'l handlebodies are specified by the attaching circles of the 2-handles in } \partial X_1 = \partial(\text{solid genus } g \text{ sfc})}$

requires: the 3-handle det'd only by gluing map, and $\text{Diff}(S^2)$ is connected

Heegaard splittings $H \cup_\varphi H$ for a diffeo $\varphi: \partial H \to \partial H$ are equivalent up to diffeo

we can represent info of circles in Σ_g by a diagram in $\mathbb{R}^2$ $\leftarrow$ MAIN IDEA

0-handle $\mathbb{D}^3$ has $\partial = S^2 = \mathbb{R}^2 \cup \infty$

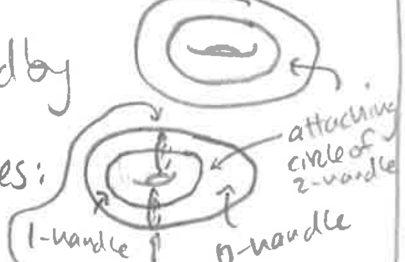
so draw the attaching regions of the 1-handles in $\mathbb{R}^2$ $\leftarrow$ to get X, we add an orientable handle $\Leftrightarrow$ gluing by a reflection

attaching circles of the 2-handles in $\mathbb{R}^2$

examples:

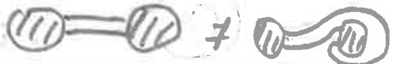
S^3 can be rep'd by

find the handles:



w/ diagram:

rep'ing the ∂ of the 0-handle:

note: 

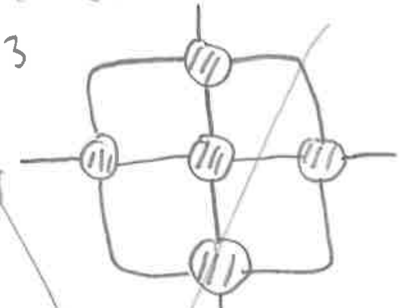
$S^1 \times S^2$

X_1 = solid torus = $S^1 \times$ 0-handle of S^2

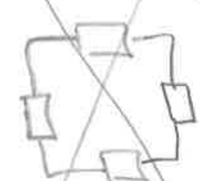
the 2-handle is the 0-handle of $S^1 \times$ the 2-handle of S^2, and the S^1 is the same S^1 as for X_1 so it links like:

T^3

one attaching circle at ∞



think:



reps T^2

when you $\times S^1$ the 0-handle of S^1 replicates the handle decomp of T^2

the 1-handle adds a 1-handle from T^2's 0-handle, and connects it to all others by 2-handles = $h_1(S^1) \times h_1(T^2)$

over

Heegaard Diagram of T^3

$$T^3 = S^1\times S^1\times S^1 = (S^1\times S^1)\times S^1$$

$$= \left(h_0(T^2)\cup h_1^a(T^2)\cup h_1^b(T^2)\cup h_2(T^2)\right)\times\left(h_0(S^1)\cup h_1(S^1)\right)$$

$$= \underbrace{\left(h_0(T^2)\times h_0(S^1)\right)}_{=\,h_0(T^3)} \cup \underbrace{\left(\left(h_1^a(T^2)\times h_0(S^1)\right)\cup\left(h_1^b(T^2)\times h_0(S^1)\right)\cup\left(h_0(T^2)\times h_1(S^1)\right)\right)}_{=\text{ 1-handles of } T^3}$$

$$\cup \underbrace{\left(\left(h_1^a(T^2)\times h_1(S^1)\right)\cup\left(h_1^b(T^2)\times h_1(S^1)\right)\cup\left(h_2(T^2)\times h_0(S^1)\right)\right)}_{=\text{ 2-handles of } T^3}$$

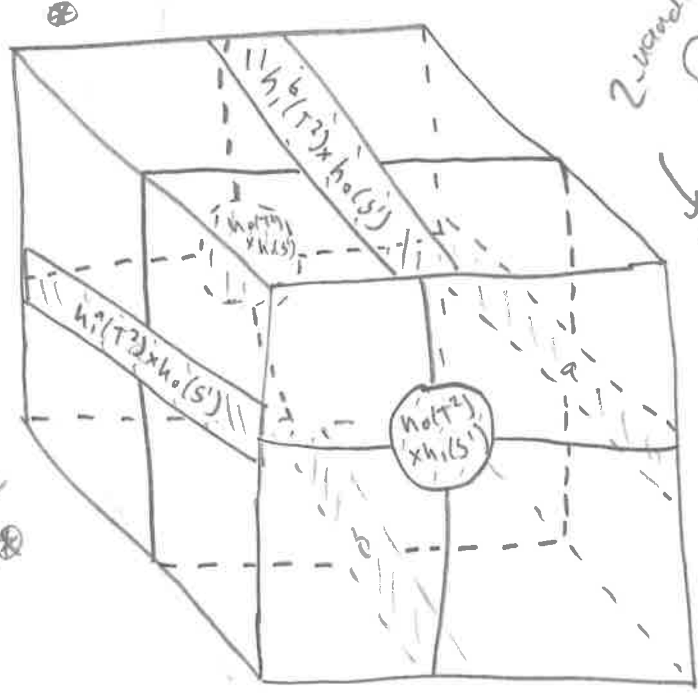
$$\cup \underbrace{\left(h_2(T^2)\times h_1(S^1)\right)}_{=\,h_3(T^3)}$$

think of $h_0(T^3)$ as a cube

$h_1^{a,b}(T^2)\times h_0(S^1)$ attach by the $h_1(T^2)$ maps, $\times h_0(S^1)$

$h_0(T^2)\times h_1(S^1)$ attaches by the $h_1(S^1)$ maps, $\times h_0(T^2)$ } 1-handles

(but we can shrink ⊛ from entire front+back faces to discs)

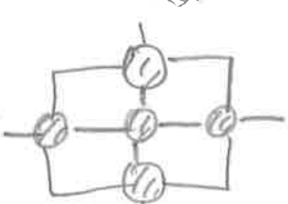


★ cont think of these as the residue of shrinking as in ⊛

2-handles { $h_1^{a,b}(T^2)\times h_1(S^1)$ attach by the product of the maps. the attaching regions are disks and two opp. sides attach as for T^2, other two as for S^1

that's these: $h_1^a(T^2)\times h_1(S^1)$ is  $h_1^b(T^2)\times h_1(S^1)$ is } ★

$h_2(T^2)\times h_0(S^1)$ replicates the 2-handle attachment of T^2 so it's the one 

get  as diagram

Kirby Diagrams

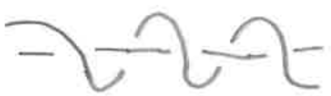
$\partial_- X = \emptyset$
X compact

draw attaching regions of 1-handles
identify boundaries via a reflection in a plane
add 2-handles by circles, but now we have framings as $\pi_1(O(2)) = \mathbb{Z}$!

note: • if $\partial_+ X = \emptyset$ then by dualizing the union of 3- and 4- handles is $\natural_m S^1 \times D^3$
any self-diffeo of $\#_m S^1 \times S^2$ extends over $\natural_m S^1 \times D^3$ (Laudenbach-Poénaru)
so $\exists!$ X w/ $\partial_+ X = \emptyset$ which can be obtained by attaching 3- and 4-handles to X_2
• if $\partial_+ X \neq \emptyset$ but is connected and X is simply connected then X det'd by X_2 and # of 3-handles (Trace)

⇒ for lots of mflds we care about, the complexity is due to 2-handles

framings of 2-handles: a co-orientation on K induced by an orientation of K and S^3 ⇒ to specify a framing we specify a vector field along K
⇒ to specify a framing we specify a parallel copy K'

to specify ± we choose the std orientation on S^3 as ∂D^4 and say "right-handed twists are positive" eg  = +3

can also represent by 

note: sign of a twist depends only on orientation of S^3, not of K (check by going from each direction)
so framings are well-def'd by parallel copies, indep of our orientation of K

the idea is this: if we have K and K' a parallel copy ↔ a vector field along K, an orientation on K + an orientation of S^3 ⇒ the other vector field in the framing up to isotopy. if we change the orientation of K the 2nd vf changes BUT remember we change coords at a basepoint to get all framings to agree there. so we get back to the identity component of $O(2)$.

More about 2-handle framings

def: a **link diagram** is a projection of L to $\mathbb{R}^2$ to an immersion of $\sqcup S^1$ which is 1-1 except at $\uparrow$ double points, together w/ crossing info

if L is oriented then we can give signs to crossings

the signs are indep of the orientation

Linking #s: 3 ways given K_1, K_2 components of an oriented link in S^3 ...

homology: $H_2(S^3, S^3 - K_1; \mathbb{Z}) \cong H_2(\nu K_1, \partial \nu K_1; \mathbb{Z}) \cong \mathbb{Z}$

LES on $H_* \Rightarrow H_1(S^3 - K_1; \mathbb{Z}) \cong \mathbb{Z}\langle \text{meridian of } K_1 \rangle$

$[K_2]$ rep's some multiple of the meridian in $H_1(S^3 - K_1; \mathbb{Z})$

$\underset{\text{linking \# } lk(K_1, K_2)}{\overset{\parallel}{\text{multiple}}}$

prop 4.5.2: $lk(K_1, K_2)$ = signed # of times K_2 crosses under K_1

cor 4.5.3: $lk(K_1, K_2) = lk(K_2, K_1)$

prop 4.5.5: for F a Seifert sfc of K_1, $lk(K_1, K_2) = F \cdot K_2$

2 ways: algorithm (show) or "$H^1(S^3 - K_1; \mathbb{Z}) \cong \mathbb{Z} \leftrightarrow [S^3 - K, \overset{K(\mathbb{Z},1)}{\overset{\parallel}{S^1}}] \Rightarrow \varphi: S^3 - K_1 \to S^1$ induces H^1 iso, $F = \varphi^{-1}(\text{regval})$"

the **0-framing** is the outward normal to any orientable Seifert sfc

the **blackboard framing** is det'd by a link diagram

prop 4.5.8: $lk(K, K_b)$ where K_b is a pushoff by the blackboard framing is the **writhe** of K, that is, the signed # of self-crossings of K

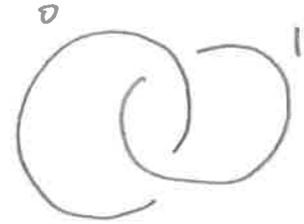
def 4.5.10: the **linking matrix** of a link $L = \bigsqcup_{i=1}^{m} K_i$ is (a_{ij}) where if $i \neq j$,

$a_{ij} = lk(K_i, K_j)$ and $a_{ii} = w(K_i)$

prop 4.5.11: for X a connected handlebody without 1- or 3-handles, Q_X with respect to the canonical ordered basis given by L is given by the linking matrix of L

the canonical ordered basis is given by pushing Seifert sfcs into $\mathbb{D}^4$, then adding the cores of their 2-handles check Q_x!

Examples

$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is 

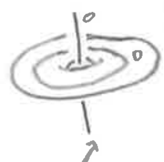
$S^2 \times S^2$ decompose as product of 2 handlebodies $\rightsquigarrow$ two 0-framed unknots

how are they linked?

they're precisely $S^1 \times 0$ and $0 \times S^1$ in S^3

Hopf link

the diagram is 



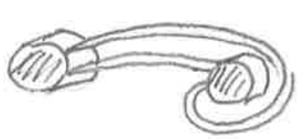
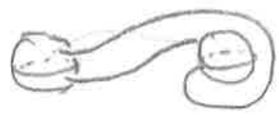
alt, to see the splitting $S^3 = \partial(\mathbb{D}^2 \times \mathbb{D}^2) = \partial\mathbb{D}^2 \times \mathbb{D}^2 \cup \mathbb{D}^2 \times \partial\mathbb{D}^2$

$I \times M^3$

I is just a 0-handle

so we can thicken the Kirby diagram for M, then round the corners of the attaching regions of the 1-handles + collapse the bands of the thickened 2-handles' attaching spheres in M^3 to get the new spheres in the 4-d diagram

eg: $I \times \mathbb{R}P^3 = I \times$ 

$=$  $=$ 

$S^1 \times M^3$

with the 0-handle of S^1 you thicken as for $I \times M^3$

the 1-handle of S^1 bumps all the handles of M^3 up by one

get:
- one more 1-handle from M's 0-handle
- a bunch more 2-handles from M's 1-handles

the 2-handles attach the new 1-handle to all the other 1-handles as in the T^3 Heegaard diagram:

think of the new 2-handles' attaching regions as the result of shrinking the new 1-handle from all of M^3's 0-handle to just 2 balls at ends of the new 0-handle

Dotted Circle Notation

question: how to deal w/ framings on 2-handles which pass over 1-handles?

answer: instead of drawing 1-handles' attaching regions we draw the belt sphere of the cancelling 2-handle and put a dot

it's an unknot b/c it's the cancelling 2-handle (will make more sense once Julian talks about calculus next week)

how to get X? the cocore of the cancelling 2-handle is a disk pushed into int $\mathbb{D}^4$ bounded by the ; remove a tubular nbhd.

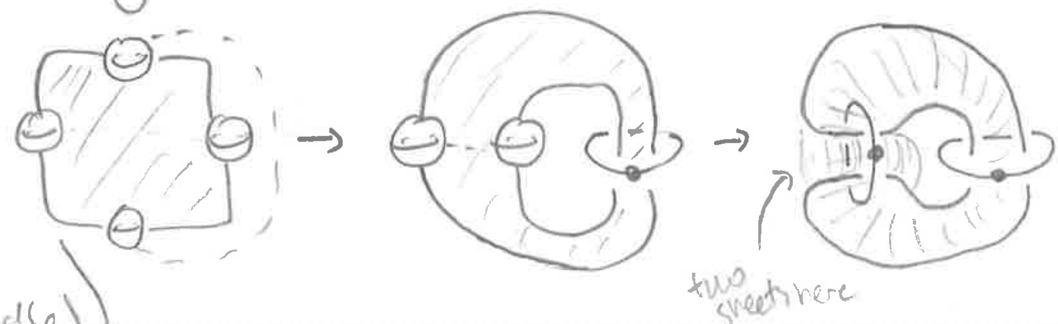
in 3D:

think of sliding all the other data off int ∂(0-cell) then collapsing the 1-handle

w/ diagrams

imagine: squeeze together the 2 balls in the attaching region of a 1-handle you can tell how to attach any 2-handle curves running thru (the framings you get will depend on the path you chose to press the balls together along)

eg $T^2 \times \mathbb{D}^2$ goes from



(shaded region is the core of the 0-handle pushed into ∂(0-handle))