ON MCDUFF'S "THE STRUCTURE OF RATIONAL AND RULED SYMPLECTIC 4-MANIFOLDS"

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(1) Outline

Let (V, ω) denote a closed symplectic 4-manifold. Let C be a symplectically embedded compact 2-manifold. If C is S^2 then we call C **rational**. (V, C, ω) is **minimal** if $V - C$ contains no **exceptional curves**, that is, symplectically embedded S^2s with self-intersection -1.

Theorem 1. *Let (V, C, ω) be a minimal symplectic pair where C is a rational curve with self-intersection $p \geq 0$.*

(i) *The diffeomorphism type of the pair (V, C) is determined by p when $p \neq 0, 4$.*

(ii) *When $p = 4$, either $(V, C) \cong (\mathbb{CP}^2, Q)$ where Q is a quadric, or $\cong (S^2 \times S^2, \Gamma_2)$, where Γ_2 is the graph of a holomorphic self-map of S^2 of degree 2. When $p = 0$, C is a fiber of a symplectic S^2-bundle.*

(iii) *(V, C, ω) is determined up to symplectomorphism by the cohomology class of ω (and in many cases up to isotopy).*

Unfortunately I won't be able to prove this theorem at all satisfactorily in 40 minutes. However, I hope to provide a thrilling trailer. The key steps are as follows.

Lemma 1. *Let C be a rational curve with $C \cdot C \geq 0$. There is a J for which $[C]$ may be represented by union of J-holomorphic rational curves $S = S_1 \cup \cdots \cup S_m$, where for each i the class $[S_i]$ is J-simple and J is regular for curves in that class.*

Lemma 2. *Let J be a regular value for D_A and suppose that A is a J-simple class which may be represented by an embedded J-holomorphic S^2. Then $p = A \cdot A \leq 1$.*

Lemma 3. *Let (V, C, ω) be minimal and suppose C is a rational curve with $C \cdot C \geq 0$.*

(i) *We may assume in Lemma 5.1 that the S_i are distinct embedded curves with self-intersections $-1, 0,$ or 1.*

(ii) *If $S_i \cdot S_i = 1$ for some i then $V = \mathbb{CP}^2$ and $m = 1$ or 2.*

(iii) *If $S_i \cdot S_i = -1$ for some i then there is only one such i, and all the other S_j are homologous with $S_j \cdot S_j = 0$. Furthermore, $V = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.*

(iv) *If $S_i \cdot S_i = 0$ for all i then all the S_i except perhaps one are homologous, and V is an S^2-bundle.*

(2) Tools

We will use $\mathscr{J}(\omega)$ to denote the space of all ω-tame J of Sobolev regularity $s, 2$ for s as large as we need; it is nonempty and contractible.

The **universal moduli space** of J-holomorphic spheres in the class A is denoted $\mathcal{M}_A = \{(f, J) \in W^{k,p}(S^2, V) \times \mathscr{J}(\omega) \mid \bar{\partial}_J(f) = 0\}$ and $D_A : \mathcal{M}_A \to \mathscr{J}(\omega)$ denotes projection onto the second factor. It is Fredholm, and so preimages of its regular values are manifolds. Denote the space of parameterized J-holomorphic curves in the class A by $\tilde{\mathcal{M}}(J, A) := D_A^{-1}(J)$. Their dimensions are $\operatorname{ind}(D_A) = 2c_1(A)+4$.

Simplicity.

Definition 1. *A class $A \in H_2(V; \mathbb{Z})$ is J-**simple** if it cannot be written as a sum $A_1 + \cdots + A_k$ of classes which can all be represented by J-holomorphic curves and if $A_i \neq A_j$ then there is a sequence $A_i = A_{i_1}, \ldots, A_{i_n} = A_j$ with $A_{i_p} \cdot A_{i_{p+1}} \geq 1$ for $p = 1, \ldots, n-1$.*

Definition 2. *A is **simple** if it is J-simple for all ω-tame J.*

Why do we like simplicity? By the mean value inequality, nonconstant solutions to the Cauchy-Riemmann equation must have positive energy. Since for ω-tame J, energy is controlled by homology, it is in particular impossible to bubble off nonconstant J-holomorphic spheres from a J-holomorphic curve in the class A if A is J-simple. Therefore if A is simple then we always know that $\mathcal{M}(J, A) := \tilde{\mathcal{M}}(J, A)/\operatorname{Aut}(\mathbb{CP}^1)$ is compact.

Note that the dimension of $\operatorname{Aut}(\mathbb{CP}^1)$ is six, so $\dim \mathcal{M}(J, A) = \operatorname{ind}(D_A) - 6$.

Adjunction. If J is smooth then the virtual genus of a J-holomorphic sphere C is defined as

$$g(C) = 1 + \frac{1}{2}([C] \cdot [C] - c_1(C))$$

and in [4] it is shown that $g(C) \geq 0$ with equality if and only if C is embedded.

Consequences of rational curves with self-intersection 0 and 1. These two results are the main tools in understanding the consequence of having a rational curve with specified self-intersection. The proofs follow the same game: mark enough points on your moduli space of curves to get a degree one evaluation map; this has geometric consequences for how rational curves can sit in V.

Proposition 1. *Let F be a symplectically embedded S^2 in the class B in a symplectic 4-manifold (V, ω) where B is a simple homology class of self-intersection zero. Then there is a fibration $\pi : V \to M$ which is compatible with ω and has one fiber equal to F.*

Proof. Let J' be any ω-tame almost complex structure which splits near F (one can proscribe such local behavior by defining J' locally and then using cutoff functions to extend the compatible Riemannian metric). If f is a J'-holomorphic parameterization of F then since J' is split, $c_1(F) = 0$, so by [2] (f, J') is a regular point for the projection operator D_B from the universal moduli space to $\mathscr{J}(\omega)$. Pick a nearby regular value J (so that its preimage under D_B is a manifold; we don't know that $\tilde{\mathcal{M}}(J', B)$ is a manifold). If J is near enough to J' it admits a J-holomorphic curve in the class B which is close to F and so symplectically isotopic to F (above a path J_t of regular values, the regular points in the fibers $\tilde{\mathcal{M}}(J_t, B)$ do not change materially). Change J by this symplectic isotopy (doesn't affect regularity) to get F itself a J-holomorphic curve.

By the virtual genus formula,

$$c_1(B) = 2 + B \cdot B + g(F) = 2 + 0 + 0$$

Since B is simple, the space $\tilde{\mathcal{M}}(J, B)$ of parameterized J-holomorphic curves in the class B is compact. Its dimension is

$$\mathrm{ind}(P_B) = 2c_1(B) + 4 = 8$$

Consider the evaluation map

$$e_B(J) : \tilde{\mathcal{M}}(J, B) \times_{\mathrm{Aut}(\mathbb{CP}^1)} S^2 \to V \text{ given by } (f, z) \mapsto f(z)$$

It is a map between 4-dimensional spaces. Since $B \cdot B = 0$, there can only be one J-holomorphic curve in the class B through any point lest the curves violate intersection positivity. Therefore the degree is at most one. Computing the degree locally near the fiber F gives degree at least one. Since the map is continuous, the total degree is one, so there is *precisely* one J-holomorphic curve in the class B through each point. And these curves are embedded by the virtual genus formula when J is smooth. So they form the fibers of a continuous surjection $\pi : V \to M$, which is smooth by [1]. I punt on the proof of symplectomorphism. □

Proposition 2. *Let F be a symplectically embedded S^2 in the class B in a symplectic 4-manifold (V, ω) where B is a simple homology class of self-intersection one. Then $(V, \omega) \cong (\mathbb{CP}^2, \omega_{FS})$ with F going to a complex line.*

Proof. I will only indicate some ingredients in the proof. The idea is essentially the same as for the self-intersection zero case. We consider the evaluation map

$$e_B^2(J) : \tilde{\mathcal{M}}(J, A) \times_{\mathrm{Aut}(\mathbb{CP}^1)} (S^2 \times S^2) \to V \times V$$

where $\mathrm{Aut}(\mathbb{CP}^1)$ acts diagonally on $S^2 \times S^2$. By similar reasoning to the above, it is degree one, meaning there is precisely one holomorphic sphere in the class B through any pair of points in V. Fix one point in V off F and let the other point run through F; with a little work this provides a diffeomorphism to $\mathbb{CP}^2$, and with more work a symplectomorphism. □

(3) Diffeomorphism Types

Let's analyze the consequences of Lemma 3.

In the case (ii), a single self-intersection one sphere gives us the diffeomorphism to $\mathbb{CP}^2$ as in the recognition of $\mathbb{CP}^2$, above. Moreover, in the case $m = 1$ we get the $p = 1$ case and in the case $m = 2$ we get one of the $p = 4$ cases, the $(\mathbb{CP}^2, Q)$.

In the case (iv), C itself is a fiber of an S^2-bundle under the diffeomorphism as in the recognition of S^2-bundles, above. Note p can be anything here: if there are two homology classes, one obtains nonzero p from their intersection number.

One can obtain all even p by gluing $\mathcal{O}(p)$ to $\mathcal{O}(-p)$; the topological type of this manifold is $S^2 \times S^2$, and C can be in the class of the zero section of $\mathcal{O}(p)$. This degenerates as a sum of simple classes into a self-intersection zero section plus $\frac{p}{2}$ times the class of the fiber. Here's where the $(S^2 \times S^2, \Gamma_2)$ shows up when $p = 4$.

The case (iii) is more complicated. The S_js exhibit V as an S^2-bundle and the fact that there is any simple homology class with negative self-intersection shows that V is a nontrivial bundle. It remains to show that V has a sphere as a section. Let $[C] = E + kA$, where $E \cdot E = -1$ and $A \cdot A = 0$. Then

$$C \cdot C = E \cdot E + 2kE \cdot A + A \cdot A = -1 + 2kE \cdot A$$

Meanwhile

$$C \cdot C + 2 \overset{\text{adjunction}}{=} c_1(C) = c_1(E) + kc_1(A) \overset{\text{adjunction}}{=} E \cdot E + 2 + k(A \cdot A + 2) = 1 + 2k$$

so $E \cdot A = 1$. Therefore E is the section we are looking for.

One can obtain all odd p by gluing $\mathcal{O}(p)$ to $\mathcal{O}(-p)$; the topological type of this manifold is $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, and C can be in the class of the zero section of $\mathcal{O}(p)$. This degenerates as a sum of simple classes into a self-intersection -1 section plus plus $\frac{p+1}{2}$ times the class of the fiber.

References

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