(1) Outline

Let \((V,\omega)\) denote a closed symplectic 4-manifold. Let \(C\) be a symplectically embedded compact 2-manifold. If \(C\) is \(S^2\) then we call \(C\) rational. \((V,C,\omega)\) is minimal if \(V-C\) contains no exceptional curves, that is, symplectically embedded \(S^2\)s with self-intersection \(-1\).

Theorem 1. Let \((V,C,\omega)\) be a minimal symplectic pair where \(C\) is a rational curve with self-intersection \(p \geq 0\).

(i) The diffeomorphism type of the pair \((V,C)\) is determined by \(p\) when \(p \neq 0,4\).

(ii) When \(p = 4\), either \((V,C) \cong (\mathbb{CP}^2, Q)\) where \(Q\) is a quadric, or \(\cong (S^2 \times S^2, \Gamma_2)\), where \(\Gamma_2\) is the graph of a holomorphic self-map of \(S^2\) of degree 2. When \(p = 0\), \(C\) is a fiber of a symplectic \(S^2\)-bundle.

(iii) \((V,C,\omega)\) is determined up to symplectomorphism by the cohomology class of \(\omega\) (and in many cases up to isotopy).

Unfortunately I won’t be able to prove this theorem at all satisfactorily in 40 minutes. However, I hope to provide a thrilling trailer. The key steps are as follows.

Lemma 1. Let \(C\) be a rational curve with \(C \cdot C \geq 0\). There is a \(J\) for which \([C]\) may be represented by union of \(J\)-holomorphic rational curves \(S = S_1 \cup \cdots \cup S_m\), where for each \(i\) the class \([S_i]\) is \(J\)-simple and \(J\) is regular for curves in that class.

Lemma 2. Let \(J\) be a regular value for \(D_A\) and suppose that \(A\) is a \(J\)-simple class which may be represented by an embedded \(J\)-holomorphic \(S^2\). Then \(p = A \cdot A \leq 1\).

Lemma 3. Let \((V,C,\omega)\) be minimal and suppose \(C\) is a rational curve with \(C \cdot C \geq 0\).

(i) We may assume in Lemma 5.1 that the \(S_i\) are distinct embedded curves with self-intersections \(-1,0,\text{ or } 1\).

(ii) If \(S_i \cdot S_i = 1\) for some \(i\) then \(V = \mathbb{CP}^2\) and \(m = 1\) or 2.

(iii) If \(S_i \cdot S_i = -1\) for some \(i\) then there is only one such \(i\), and all the other \(S_j\) are homologous with \(S_j \cdot S_j = 0\). Furthermore, \(V = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\).
(iv) If $S_i \cdot S_i = 0$ for all $i$ then all the $S_i$ except perhaps one are homologous, and $V$ is an $S^2$-bundle.

(2) Tools

We will use $\mathcal{J}(\omega)$ to denote the space of all $\omega$-tame $J$ of Sobolev regularity $s$, $2$ for $s$ as large as we need; it is nonempty and contractible.

The universal moduli space of $J$-holomorphic spheres in the class $A$ is denoted $\mathcal{M}_A = \{(f, J) \in W^{k,p}(S^2, V) \times \mathcal{J}(\omega) | \bar{\partial}_J(f) = 0\}$ and $D_A : \mathcal{M}_A \to \mathcal{J}(\omega)$ denotes projection onto the second factor. It is Fredholm, and so preimages of its regular values are manifolds. Denote the space of parameterized $J$-holomorphic curves in the class $A$ by $\tilde{\mathcal{M}}(J, A) := D_A^{-1}(J)$. Their dimensions are $\text{ind}(D_A) = 2c_1(A)+4$.

Simplicity.

Definition 1. A class $A \in H_2(V; \mathbb{Z})$ is $J$-simple if it cannot be written as a sum $A_1 + \cdots + A_k$ of classes which can all be represented by $J$-holomorphic curves and if $A_i \neq A_j$ then there is a sequence $A_i = A_{i_1}, \ldots, A_{i_n} = A_j$ with $A_{i_p} \cdot A_{i_{p+1}} \geq 1$ for $p = 1, \ldots, n-1$.

Definition 2. $A$ is simple if it is $J$-simple for all $\omega$-tame $J$.

Why do we like simplicity? By the mean value inequality, nonconstant solutions to the Cauchy-Riemann equation must have positive energy. Since for $\omega$-tame $J$, energy is controlled by homology, it is in particular impossible to bubble off nonconstant $J$-holomorphic spheres from a $J$-holomorphic curve in the class $A$ if $A$ is $J$-simple. Therefore if $A$ is simple then we always know that $\mathcal{M}(J, A) := \tilde{\mathcal{M}}(J, A)/\text{Aut}(\mathbb{C}P^1)$ is compact.

Note that the dimension of $\text{Aut}(\mathbb{C}P^1)$ is six, so $\dim \mathcal{M}(J, A) = \text{ind}(D_A) - 6$.

Adjunction. If $J$ is smooth then the virtual genus of a $J$-holomorphic sphere $C$ is defined as

$$g(C) = 1 + \frac{1}{2}([C] \cdot [C] - c_1(C))$$

and in [4] it is shown that $g(C) \geq 0$ with equality if and only if $C$ is embedded.

Consequences of rational curves with self-intersection 0 and 1. These two results are the main tools in understanding the consequence of having a rational curve with specified self-intersection. The proofs follow the same game: mark enough points on your moduli space of curves to get a degree one evaluation map; this has geometric consequences for how rational curves can sit in $V$.

Proposition 1. Let $F$ be a symplectically embedded $S^2$ in the class $B$ in a symplectic 4-manifold $(V, \omega)$ where $B$ is a simple homology class of self-intersection zero. Then there is a fibration $\pi : V \to M$ which is compatible with $\omega$ and has one fiber equal to $F$.  

2
Proof. Let $J'$ be any $\omega$-tame almost complex structure which splits near $F$ (one can prescribe such local behavior by defining $J'$ locally and then using cutoff functions to extend the compatible Riemannian metric). If $f$ is a $J'$-holomorphic parameterization of $F$ then since $J'$ is split, $c_1(F) = 0$, so by [2] $(f, J')$ is a regular point for the projection operator $D_B$ from the universal moduli space to $\mathcal{J}$. Pick a nearby regular value $J$ (so that its preimage under $D_B$ is a manifold; we don’t know that $\tilde{M}(J, B)$ is a manifold). If $J$ is near enough to $J'$ it admits a $J$-holomorphic curve in the class $B$ which is close to $F$ and so symplectically isotopic to $F$ (above a path $J_t$ of regular values, the regular points in the fibers $\tilde{M}(J_t, B)$ do not change materially). Change $J$ by this symplectic isotopy (doesn’t affect regularity) to get $F$ itself a $J$-holomorphic curve.

By the virtual genus formula,

$$c_1(B) = 2 + B \cdot B + g(F) = 2 + 0 + 0$$

Since $B$ is simple, the space $\tilde{M}(J, B)$ of parameterized $J$-holomorphic curves in the class $B$ is compact. Its dimension is

$$\text{ind}(P_B) = 2c_1(B) + 4 = 8$$

Consider the evaluation map

$$e_B(J) : \tilde{M}(J, B) \times_{\text{Aut}(\mathbb{CP}^1)} S^2 \to V$$

given by $(f, z) \mapsto f(z)$

It is a map between 4-dimensional spaces. Since $B \cdot B = 0$, there can only be one $J$-holomorphic curve in the class $B$ through any point lest the curves violate intersection positivity. Therefore the degree is at most one. Computing the degree locally near the fiber $F$ gives degree at least one. Since the map is continuous, the total degree is one, so there is precisely one $J$-holomorphic curve in the class $B$ through each point. And these curves are embedded by the virtual genus formula when $J$ is smooth. So they form the fibers of a continuous surjection $\pi : V \to M$, which is smooth by [1]. I punt on the proof of symplectomorphism.

Proposition 2. Let $F$ be a symplectically embedded $S^2$ in the class $B$ in a symplectic 4-manifold $(V, \omega)$ where $B$ is a simple homology class of self-intersection one. Then $(V, \omega) \cong (\mathbb{CP}^2, \omega_{FS})$ with $F$ going to a complex line.

Proof. I will only indicate some ingredients in the proof. The idea is essentially the same as for the self-intersection zero case. We consider the evaluation map

$$e_B^2(J) : \tilde{M}(J, A) \times_{\text{Aut}(\mathbb{CP}^1)} (S^2 \times S^2) \to V \times V$$

where $\text{Aut}(\mathbb{CP}^1)$ acts diagonally on $S^2 \times S^2$. By similar reasoning to the above, it is degree one, meaning there is precisely one holomorphic sphere in the class $B$ through any pair of points in $V$. Fix one point in $V$ off $F$ and let the other point run through $F$; with a little work this provides a diffeomorphism to $\mathbb{CP}^2$, and with more work a symplectomorphism. □
Let’s analyze the consequences of Lemma 3.

In the case (ii), a single self-intersection one sphere gives us the diffeomorphism to \(\mathbb{C}P^2\) as in the recognition of \(\mathbb{C}P^2\), above. Moreover, in the case \(m = 1\) we get the \(p = 1\) case and in the case \(m = 2\) we get one of the \(p = 4\) cases, the \((\mathbb{C}P^2, Q)\).

In the case (iv), \(C\) itself is a fiber of an \(S^2\)-bundle under the diffeomorphism as in the recognition of \(S^2\)-bundles, above. Note \(p\) can be anything here: if there are two homology classes, one obtains nonzero \(p\) from their intersection number.

One can obtain all even \(p\) by gluing \(\mathcal{O}(p)\) to \(\mathcal{O}(-p)\); the topological type of this manifold is \(S^2 \times S^2\), and \(C\) can be in the class of the zero section of \(\mathcal{O}(p)\). This degenerates as a sum of simple classes into a self-intersection zero section plus \(\frac{p}{2}\) times the class of the fiber. Here’s where the \((S^2 \times S^2, \Gamma_2)\) shows up when \(p = 4\).

The case (iii) is more complicated. The \(S_j\)s exhibit \(V\) as an \(S^2\)-bundle and the fact that there is any simple homology class with negative self-intersection shows that \(V\) is a nontrivial bundle. It remains to show that \(V\) has a sphere as a section. Let \([C] = E + kA\), where \(E \cdot E = -1\) and \(A \cdot A = 0\). Then

\[
C \cdot C = E \cdot E + 2kE \cdot A + A \cdot A = -1 + 2kE \cdot A
\]

Meanwhile

\[
C \cdot C + 2 = c_1(C) = c_1(E) + kc_1(A) = E \cdot E + 2 + k(A \cdot A + 2) = 1 + 2k
\]

so \(E \cdot A = 1\). Therefore \(E\) is the section we are looking for.

One can obtain all odd \(p\) by gluing \(\mathcal{O}(p)\) to \(\mathcal{O}(-p)\); the topological type of this manifold is \(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\), and \(C\) can be in the class of the zero section of \(\mathcal{O}(p)\). This degenerates as a sum of simple classes into a self-intersection \(-1\) section plus plus \(\frac{p+1}{2}\) times the class of the fiber.

References


