PATTERN AVOIDANCE IN PERMUTATIONS ON THE BOOLEAN LATTICE

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Abstract. We extend the concept of pattern avoidance in permutations on a totally ordered set to pattern avoidance in permutations on partially ordered sets. The number of permutations on \( P \) that avoid the pattern \( \pi \) is denoted \( \text{Av}_P(\pi) \). We extend a proof of Simion and Schmidt to show that \( \text{Av}_P(132) \leq \text{Av}_P(123) \) for any poset \( P \), and we exactly classify the posets for which equality holds. We also give asymptotically close bounds on the number of permutations on the Boolean lattice that avoid the pattern \( \{1\}\{1,2\}\{2\} \).

1. Motivation

An inversion of a permutation \( \sigma \in S_n \) is a pair of entries \( a < b \in \{1, \ldots, n\} \) such that \( b \) is to the left of \( a \) in \( \sigma \). As Bóna [1] explains, classical pattern containment is “a far-fetching generalization of [inversions of permutations] from pairs of entries to \( k \)-tuples of entries.” A permutation \( \sigma \in S_n \) is said to contain a pattern \( \pi \in S_k \) if some subsequence of \( \sigma \) of length \( k \) is order-isomorphic to \( \pi \). Otherwise, we say \( \sigma \) avoids \( \pi \). Thus an inversion in \( \sigma \) is just a 21 pattern that it contains.

We propose a similar generalization to permutations on posets. A total ordering of the elements of some poset \( P \) that respects its partial order is called a linear extension of \( P \). It is possible also to think of a linear extension of \( P \) as a permutation of the elements of \( P \) that has no inversions. While the only classical permutation in \( S_n \) that has no inversions is \( 12 \ldots n \), in general there may be many linear extensions of \( P \), and counting the number of such extensions is a difficult problem. Since linear extensions are a central object of study in order theory, it is natural to look at avoidance of more complicated patterns than 21 in poset permutations. Pattern avoidance has been defined on structures other than \( S_n \), such as on set partitions in [5], and in [7], Kitaev studies classical permutation avoidance of patterns with incomparable elements. However, we do not believe poset permutations have been considered in the way we define below. We remark that multiset permutations (whose pattern avoidance has been studied for instance in [6] and [11]) are essentially a special case of poset permutations.

Two central themes in classical pattern avoidance are (i) counting the number of permutations of large size which avoid some given permutation of small size; and (ii) demonstrating relationships between the avoidance of different patterns of the same length. In this paper, we demonstrate one
non-trivial relationship between two patterns, and give bounds on the numbers of avoiders of another pattern, thus furthering both of these central themes. We are particularly interested in the Boolean lattice because it is a poset with many nice properties, such as unique complementation and the LYM property \cite{9}, and because counting the number of linear extensions of the Boolean lattice is a well-studied problem, only recently resolved asymptotically by Brightwell and Tetali \cite{2}.

2. Definitions

Let $P$ be a partially ordered set on $n$ elements. A permutation on $P$ is a bijection $\sigma: \{1, \ldots, n\} \rightarrow P$. We define $\sigma_i := \sigma(i)$ and think of $\sigma$ as an ordered list of the elements of $P$. We will use the notation $\sigma = (\sigma_1, \ldots, \sigma_n)$ and we call $\sigma_i$ the entry of $\sigma$ at position $i$. We will use $S_P$ to denote the set of permutations on $P$.

Let $Q$ be a poset on $k$ elements. A permutation $\sigma \in S_P$ is said to contain a permutation $\pi \in S_Q$ (which we call a pattern) if there are $k$ entries $\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}$ so that $i_1 < i_2 < \cdots < i_k$, and for any $1 \leq a < b \leq k$, the order relation ($<$, $>$, or $\sim$ := “incomparable”) between $\sigma_{i_a}$ and $\sigma_{i_b}$ is the same as the order relation between $\pi_a$ and $\pi_b$. Otherwise, we say $\sigma$ avoids $\pi$. When considering permutations as patterns we will suppress the parentheses and write $\pi = \pi_1 \pi_2 \cdots \pi_n$.

We will often consider patterns on chains within partially ordered sets in terms of their representation as patterns from the canonically totally ordered set $[k] := \{1, 2, \ldots, k\}$. As usual, $S_k$ denotes the set of permutations on $[k]$.

**Example 1.** Let $\sigma = (\{2, 3\}, \{2\}, \{1, 3\}, \{1, 2, 3\}, \{1\}, \emptyset, \{1, 2\}, \{3\})$ be an element of $S_3$. Then $\sigma$ avoids $\emptyset\{1\}\{1, 2\}$. However, $\sigma$ contains the pattern $\{1\}\{3\}\{1, 2\}$, as evidenced by the subsequence $(\{2\}, \{1, 3\}, \{1, 2\})$. \hfill $\square$

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**Example 2.** The $\sigma$ from the previous example avoids 123, which is the same pattern as $\emptyset\{1\}\{1, 2\}$. \hfill $\square$

We will use both forms of notation but it will always be clear which notation is being used because of the brackets. Note that $123 \neq \{1\}\{2\}\{3\}$.

For a permutation $\sigma$ on any poset $P$, define the reverse of $\sigma$ to be the permutation $(\sigma_n, \sigma_{n-1}, \ldots, \sigma_1)$. For a permutation $\sigma$ on $P$, define the dual of $\sigma$ to be the same list of entries $(\sigma_1, \ldots, \sigma_n)$ considered as a permutation on the dual poset of $P$. We state the following without proof.

**Fact 1.** Let $P$ be a poset. If $\sigma \in S_P$ avoids $\pi$, then the reverse of $\sigma$ avoids the reverse of $\pi$.\hfill $\square$
Fact 2. Let $P$ be a self-dual poset. Suppose $\Phi$ is an isomorphism between $P$ and its dual. If $\sigma \in S_P$ avoids $\pi$, then $(\Phi(\sigma_1), \ldots, \Phi(\sigma_n))$ avoids the dual of $\pi$.

Remark. The Boolean lattice is self-dual since the map that sends each set to its complement is an isomorphism between $\mathbb{B}_n$ and its dual. □

Denote the number of permutations on $P$ that avoid $\pi$ by $\text{Av}_P(\pi)$. Again, to simplify notation, let $\text{Av}_n(\pi) := \text{Av}_{\mathbb{B}_n}(\pi)$ denote the number of permutations on $\mathbb{B}_n$ that avoid $\pi$. The number $\text{Av}_n(\pi)$ is our main object of study.

In general, we cannot expect to find exact values of $\text{Av}_n(\pi)$: the simplest non-trivial case is where $\pi$ is 12 or 21, and even here we have only asymptotic bounds due to Brightwell and Tetali [2]. However, we may hope to find relations between the $\text{Av}_n(\pi)$ or $\text{Av}_P(\pi)$. If $\pi \neq \pi'$ but $\text{Av}_P(\pi) = \text{Av}_P(\pi')$, we say that $\pi$ and $\pi'$ are Wilf equivalent for $P$. For instance, Facts 1 and 2 establish that any pattern, its reverse, and its dual (and the reverse of its dual) are Wilf equivalent for $\mathbb{B}_n$ for all $n$.

3. Length Two Pattern Avoidance

A linear extension of a poset $P$ on $n$ elements is a total ordering of its elements consistent with its partial ordering. Formally, a linear extension is a bijection $\lambda: \{1, \ldots, n\} \to P$ such that $\lambda_i < \lambda_j \Rightarrow i < j$. (We slightly modify the usual definition of linear extension to match better with permutations, in that what we are terming an extension would in fact be the inverse of an extension under the usual definition. At any rate, the distinction is technical and our extensions are in one-to-one correspondence with the traditional kind.) The 21-avoiding permutations in $S_P$ are exactly the linear extensions of $P$. The 12-avoiding permutations are the reverse of these. The number of linear extensions is traditionally denoted $e(P)$; however, for consistency with our other pattern avoidance notation, we will denote this quantity as $\text{Av}_P(21)$, or equally, as $\text{Av}_P(12)$.

The only other length two poset permutation pattern is $\{1\}{2\}$. However, avoidance of $\{1\}{2\}$ is trivial: there are no $\{1\}{2\}$-avoiding permutations on $P$ unless $P$ is a chain, in which case every permutation in $S_P$ avoids $\{1\}{2\}$.

Brightwell and Tetali [2] established the following by applying the entropy method of Kahn (here and throughout the paper, all logarithms are base 2):

Theorem 3.

$$\frac{\log(\text{Av}_n(12))}{2^n} = \log \left( \frac{n}{\lceil n/2 \rceil} \right) - \frac{3}{2} \log e + O \left( \frac{\ln n}{n} \right).$$

4. Length Three Pattern Avoidance: An Injection

There are 19 non-partial-order-isomorphic length three poset permutation patterns (including the trivial pattern $\{1\}{2\}{3\}$), which we will from now
## Figure 1. Wilf equivalence classes of length three patterns for $B_n$

<table>
<thead>
<tr>
<th>Poset Hasse diagram</th>
<th>Wilf equivalence classes for $B_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>![Hasse diagram 1]</td>
<td>123, 321</td>
</tr>
<tr>
<td>![Hasse diagram 2]</td>
<td>132, 312, 213, 231</td>
</tr>
<tr>
<td>![Hasse diagram 3]</td>
<td>{1}{2}{1,2}, {1}{2}\emptyset, \emptyset{1}{2}, {1,2}{1}{2}</td>
</tr>
<tr>
<td>![Hasse diagram 4]</td>
<td>{1}{1,2}{2}, {1}\emptyset{2}</td>
</tr>
<tr>
<td>![Hasse diagram 5]</td>
<td>{1}{3}{1,2}, {1,2}{3}{1}</td>
</tr>
<tr>
<td>![Hasse diagram 6]</td>
<td>{1}{1,2}{3}, {1,2}{1}{3}, {3}{1,2}, {3}{1,2}{1}</td>
</tr>
<tr>
<td>![Hasse diagram 7]</td>
<td>{1}{2}{3}</td>
</tr>
</tbody>
</table>

By consideration of reverses and duals, we can immediately break these into seven classes which are Wilf equivalent for $B_n$, as Figure 1 demonstrates.

Of course, there may be non-trivial relations between the length three patterns as well; a foundational result of classical permutation pattern avoidance is that there are as many 123-avoiding permutations in $S_n$ as there are 132-avoiding permutations. This has been proved in many ways, for instance in [6, pp. 242-243] and [7, pp. 60-61], [10], [8], etc., but perhaps the simplest proof is due to Simion and Schmidt [12]. One may naturally ask if the same is true of permutations on $B_n$. We answer in the negative for $n \geq 3$ by extending the proof of Simion and Schmidt and exactly classifying those posets for which 123 and 132 are Wilf equivalent patterns.

**Theorem 4.** We have $Av_P(132) \leq Av_P(123)$ for any poset $P$, with strict inequality if and only if $P$ contains one of $Q_1, Q_2,$ or $Q_3$ below as an induced subposet:
Proof. An important tool from classical pattern avoidance is the use of left-to-right minima to demonstrate bijections. An entry of \( \sigma \in S_n \) is called a left-to-right minimum (LRM) if it is less than every entry to its left. Fixing the positions of the LRM of \( \sigma \in S_n \), there is exactly one way to fill in the remaining entries to yield a permutation with those LRM which avoids 132. There is also exactly one way to fill in the entries to yield a permutation with those LRM which avoids 123. Proving this rigorously gives Simion and Schmidt’s bijection between 132- and 123-avoiding permutations.

Example 3. With \( \rho = 67341258 \in S_8 \), the LRM of \( \rho \) are 6, 3, and 1 in positions 1, 3 and 5. The only permutation in \( S_8 \) with those LRM in those positions that avoids 132 is \( \rho \), and the only permutation in \( S_8 \) with those LRM in those positions that avoids 123 is 68371542. \( \square \)

We extend the concept of left-to-right minimum for permutations on a poset \( P \) with what we call a left-to-right minimal element (LRME): an LRME of \( \sigma \in S_P \) is an entry \( \sigma_i \) such that there is no \( j < i \) with \( \sigma_j < \sigma_i \). Informally, an LRME is less than or incomparable to every entry preceding it. Unlike in the classical permutations case, when fixing the positions of the LRME of a poset permutation there may be more than one way to fill in the remaining entries to yield a permutation which avoids either 132 or 123.

Example 4. With \( \sigma = ([2,3],\{1\},\{1,2\},\{2\},\emptyset,\{3\},\{1,3\},\{1,2,3\}) \in B_3 \), the LRME of \( \sigma \) are \([2,3],\{1\},\{2\},\emptyset\) in positions 1, 2, 4, and 5. Note that \( \sigma \) avoids 132 but the following other elements of \( B_3 \) have the same LRME in the same positions as \( \sigma \) and also avoid 132:

- \([\{2,3\},\{1\},\{1,3\},\{2\},\emptyset,\{1,2\},\{3\},\{1,2,3\})\);
- \([\{2,3\},\{1\},\{1,3\},\{2\},\emptyset,\{3\},\{1,2\},\{1,2,3\})\).

Similarly, the following elements of \( B_3 \) all have the same LRME in the same positions and avoid 123:

- \([\{2,3\},\{1\},\{1,2,3\},\{2\},\emptyset,\{1,2\},\{1,3\},\{3\})\);
- \([\{2,3\},\{1\},\{1,2,3\},\{2\},\emptyset,\{1,3\},\{3\},\{1,2\})\). \( \square \)

By an LRME set we mean a list of elements \( x_1, \ldots, x_k \) from \( P \) along with a list of corresponding positions \( 1 \leq \mu_1 < \cdots < \mu_k \leq n \). Call an LRME set \( X \) admissible if there is some permutation \( \sigma \) whose LRME are exactly \( x_1, \ldots, x_k \) in positions \( \mu_1, \ldots, \mu_k \), and in this case we say \( X \) is the LRME set of \( \sigma \). Fix some admissible LRME set \( X \). How many \( \sigma \) have \( X \) as their LRME set and avoid 132?
Denote the positions not among the $\mu_i$ by $1 \leq \nu_1 < \cdots < \nu_l \leq n$. Let $P'$ be the induced subposet of $P$ on the elements that are not among the $x_i$. First note that since $\sigma_{\nu_k}$ is not an LRME of $\sigma$ for any $1 \leq k \leq l$, $\sigma_{\nu_k}$ is greater than some $\sigma_{\mu_j}$, where $\mu_j < \nu_k$. So, for $y \in P'$, define $\omega(y)$ to be the the smallest $i$ such that there exists $\sigma_{\mu_j} < y$ for some $\mu_j < \nu_i$. Call a permutation $\sigma'$ on $P'$ $\omega$-legal when it obeys the condition that $\omega(\sigma'_i) \leq i$ for all $i$, and let $\Lambda^\omega$ be the set of $\omega$-legal permutations in $S_{P'}$. Then, $\sigma$ has $X$ as its LRME set exactly when the subsequence $(\sigma_{\nu_1}, \ldots, \sigma_{\nu_l})$ is in $\Lambda^\omega$.

**Example 5.** With $\sigma = (\{2,3\}, \{1\}, \{1,2\}, \{2\}, \emptyset, \{3\}, \{1,3\}, \{1,2,3\}) \in B_3$ as in the previous example, $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 4$, $\mu_4 = 5$, $\nu_1 = 3$, $\nu_2 = 6$, $\nu_3 = 7$ and $\nu_4 = 8$. We have $\omega(\{1,3\}) = \omega(\{1,2\}) = \omega(\{1,2,3\}) = 1$ since each of these elements is greater than $\sigma_{\mu_2}$, while $\omega(\{3\}) = 2$. Note that $\omega$ values refer to the $\nu$ indices, not to the positions of the elements in $\sigma$. □

If $\sigma$ contains a 132 pattern then it contains a 132 pattern that consists of an LRME followed by two non-LRME, since if the element acting as the 1 is not an LRME then the rightmost LRME to the left of it that it is greater than will also make a 132 pattern with the same elements acting as 3 and 2. Suppose we fill each $\nu_i$ from left to right by choosing a $\omega$-legal, unchosen element of $P'$ to occupy this position. If we ever choose $z$ when $y$ is also an $\omega$-legal choice and $y < z$, we will contain a 132 (with the 32 being $zy$ and the 1 being the LRME they both are greater than). If, on the other hand, we always choose a minimal $\omega$-legal element, we will avoid 132; in any 132 pattern $xzy$, with $x$ an LRME and $z$ and $y$ non-LRME, $y$ was an $\omega$-legal choice for the position $z$ occupies. Let $\Lambda^\omega_{\min} \subset \Lambda^\omega$ be those $\omega$-legal permutations $\sigma'$ for which as we fill in the entries from left to right we always choose a minimal element among the $\omega$-legal choices. Then $\sigma$ has $X$ as its LRME set and avoids 132 exactly when $(\sigma_{\nu_1}, \ldots, \sigma_{\nu_l})$ is in $\Lambda^\omega_{\min}$.

Similarly, $\sigma$ avoids 132 if and only if as we fill in the $\nu_i$ from left to right we always choose a maximal element among $\omega$-legal choices. Let $\Lambda^\omega_{\max} \subset \Lambda^\omega$ be those $\omega$-legal permutations $\sigma'$ for which as we fill in the entries from left to right we always choose a maximal element among the $\omega$-legal choices. Then $\sigma$ has $X$ as its LRME set and avoids 132 exactly when $(\sigma_{\nu_1}, \ldots, \sigma_{\nu_l})$ is in $\Lambda^\omega_{\max}$. In order to complete the proof, we need to show $|\Lambda^\omega_{\min}| \leq |\Lambda^\omega_{\max}|$. The following lemma, embedded within the proof of Theorem 4, will give us just that.

**Lemma 5.** Let $P$ be a poset on $n$ elements and let $\omega: P \to \{1, \ldots, n\}$ be a labeling function such that

1. the number of elements $x$ with $\omega(x) \leq i$ is greater than or equal to $i$ for all $i = 1, \ldots, n$, and,
2. if $x > y$, then $\omega(x) \leq \omega(y)$.

Call a permutation $\sigma \in S_P$ $\omega$-legal if $\omega(\sigma_i) \leq i$ for all $i = 1, \ldots, n$. Let $\Lambda^\omega$ be the set of $\omega$-legal permutations on $P$. 

For permutations $\sigma, \pi \in S_\mathcal{P}$, we say $\sigma > \pi$ if there exists $j$ with $1 \leq j \leq n$ such that $\sigma_j > \pi_j$ and $\sigma_i = \pi_i$ for all $i < j$. Define,

$$\Lambda^\omega_{\max} := \{ \sigma \in \Lambda^\omega: \exists \sigma' \in \Lambda^\omega \text{ such that } \sigma' > \sigma \}$$

$$\Lambda^\omega_{\min} := \{ \sigma \in \Lambda^\omega: \exists \sigma' \in \Lambda^\omega \text{ such that } \sigma > \sigma' \}.$$

Then there is an injection,

$$\phi: \Lambda^\omega_{\min} \rightarrow \Lambda^\omega_{\max}.$$

Further, $\phi$ is a bijection if there does not exist $x, y, z \in \mathcal{P}$ with

- $x < z$;
- $y < z$;
- $x \sim y$;
- $\omega(x) < \omega(y)$.

Proof. Condition (1) on $\omega$ merely guarantees that $\Lambda^\omega$ is nonempty (and consequently $\Lambda^\omega_{\min}$ and $\Lambda^\omega_{\max}$ are also nonempty).

We now define a function $f: \Lambda^\omega \rightarrow \Lambda^\omega_{\max}$, whose restriction to $\Lambda^\omega_{\min}$ will be the $\phi$ we are looking for. The following algorithm defines $f$. Let $\sigma \in \Lambda^\omega$. We will build a series of permutations $\sigma^0, \sigma^1, \ldots, \sigma^n$. Initialize $\sigma^0 := \sigma$. When we are done, $f(\sigma)$ will be defined as $\sigma^n$. We recursively define $\sigma^{i+1}$ from $\sigma^i$:

1. Mark position $i+1$.
2. Consider each position $j$ with $i+2 \leq j \leq n$ from left to right. If the entry $\sigma_j$ at the corresponding position is greater than the entry at the last marked position, mark $j$.
3. Let $\alpha_0, \ldots, \alpha_k$ be the list of marked positions in the order they were marked.
4. Set $\sigma^{i+1}_{\alpha_0} = \sigma^{i}_{\alpha_0}, \sigma^{i+1}_{\alpha_1} = \sigma^{i}_{\alpha_1}, \ldots, \sigma^{i+1}_{\alpha_k} = \sigma^{i}_{\alpha_k}$. For all other positions, set the entry of $\sigma^{i+1}$ to be the same as $\sigma^i$. In other words, let $\sigma^{i+1} = \sigma^i \circ \gamma$, where $\gamma \in S_n$ is the cycle $(\alpha_k, \alpha_{k-1}, \ldots, \alpha_1, \alpha_0)$.

Figure 2 gives an example of one run of the algorithm.

We claim that for any $\sigma \in \Lambda^\omega$, the permutation $f(\sigma)$ is $\omega$-legal and in particular $f(\sigma) \in \Lambda^\omega_{\max}$. Further we claim that there exists some function $g: \Lambda^\omega_{\max} \rightarrow \Lambda^\omega$ such that $g(f(\sigma)) = \sigma$ for all $\sigma \in \Lambda^\omega_{\min}$; i.e., that $\phi$ is injective.

We move an element leftward at any step in the algorithm only if it was greater than the element previously occupying the position to which it moves. Thus if $\sigma$ was an $\omega$-legal permutation then so is $\sigma^{i+1}$ because condition (2) on $\omega$ gives $\omega(\sigma^{i+1}_j) \leq \omega(\sigma^i_j)$ for all $j$. So $\omega$-legality is maintained at every step of the algorithm, and therefore $f(\sigma)$ is an $\omega$-legal permutation.

Clearly $f(\sigma) \in \Lambda^\omega_{\max}$ because at every step in the algorithm we set $\sigma^{i+1}_i$ to be a maximal element among elements to the right of position $i$ in $\sigma^i$ and after this step the entry at position $i$ never changes.
Figure 2. Example run of $f(\sigma)$ algorithm and $g(\pi)$ algorithm. The subscript of each element $x$ in the Hasse diagram of $P$ is $\omega(x)$. For each $\sigma^i$ and $\pi^i$ in the algorithm, the entries at $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_l$, respectively, are bold and underlined.

To show that $\phi$ is injective, we construct its left inverse. The following algorithm defines $g: \Lambda^\omega_{\max} \rightarrow \Lambda^\omega$. Let $\pi \in \Lambda^\omega_{\max}$. We will build a series of permutations $\pi^n, \pi^{n-1}, \ldots, \pi^0$. Initialize $\pi^n := \pi$. When we are done, $g(\pi)$ will be defined as $\pi^0$. We recursively define $\pi^{j-1}$ from $\pi^j$:

1. Mark position $i$.
2. Consider each position $j$ with $n \geq j \geq i + 1$ from right to left. If the entry $\pi^j$ at the corresponding position is less than the entry at the last marked position, and if $\omega(\pi^j) \leq i$, mark $j$.
3. Let $\beta_0, \beta_1, \beta_1, \ldots, \beta_l$ be the list of marked positions in the order they were marked.
4. Set $\pi^{j-1}_{\beta_0} = \pi_{\beta_1}, \pi^{j-1}_{\beta_1} = \pi_{\beta_2}, \ldots, \pi^{j-1}_{\beta_l} = \pi_{\beta_0}$. For all other positions, set the entry of $\pi^{j-1}$ to be the same as $\pi^j$. In other words, let $\pi^{j-1} = \pi^j \circ \gamma$, where $\gamma \in S_n$ is the cycle $(\beta_0, \beta_1, \ldots, \beta_l, \beta_0)$.

We now show that $g(f(\sigma)) = \sigma$ for any $\sigma \in \Lambda^\omega_{\max}$. The proof that follows is technical but necessary. Call a permutation $\sigma$ on $P$ $i$-minimal if for any $k > j > i$ with $\sigma_k < \sigma_j$, we have $\omega(\sigma_k) > j$. This property will be useful for showing that the $(n-i)$-th step of the $g$ algorithm undoes the $i$-th step of the $f$ algorithm because during these steps we consider only the entries in positions $i + 1$ to $n$. First we claim that each $\sigma^i$ in the $f(\sigma)$ algorithm is $i$-minimal. We prove this by induction. The case $i = 0$ holds because $0$-minimality is equivalent to $\sigma$ belonging to $\Lambda^\omega_{\min}$. An element moves leftward at the $i$-th step only if it moves into position $i$. There is no step where element $x$ moves rightward past an element $y$ greater than it, because such a $y$ would be a part of that step’s cycle. Thus if $x$ is to the left of $y$ in $\sigma^i$
with \( x < y \), then \( x \) is to the right of \( y \) in \( \sigma^{i+1} \) only if \( y \) moves into position \( i \) during this step. So the claim follows by induction.

Set \( \pi := f(\sigma) \) and consider the \( \pi^i \) from the \( g(\pi) \) algorithm. We prove by downward induction that \( \pi^i = \sigma^i \) for all \( i \). The case \( i = n \) holds by definition. Assume that \( \pi^i = \sigma^i \). We will show \( \pi^{i-1} = \sigma^{i-1} \). Let \( \alpha_0, \ldots, \alpha_k \) be as defined in the \( f(\sigma) \) algorithm at the step where we go from \( \sigma^{i-1} \) to \( \sigma^i \). Let \( \beta_0, \ldots, \beta_l \) be as defined in the \( g(\pi) \) algorithm at the step where we go from \( \pi^i \) to \( \pi^{i-1} \). Of course, \( \alpha_0 = \beta_0 \). Further, we have that \( \alpha_k = \beta_l \). To see this, suppose that in the \( g(\pi) \) algorithm, as we consider \( j \) with \( n \geq j \geq i + 1 \) from right to left, we mark a position \( \beta_l \) before \( \alpha_k \); that is, suppose \( \beta_l < \alpha_k \). Then, \( \pi^i_{\beta_l} < \pi^i_{\beta_0} \) and \( \omega(\pi^i_{\beta_l}) \leq i < \alpha_k \). But \( \sigma^{i-1}_{\alpha_k} = \sigma^i_{\beta_0} = \pi^i_{\beta_0} \) and also \( \sigma^{i-1}_{\beta_l} = \sigma^i_{\beta_l} \); so we have \( i - 1 < \alpha_k < \beta_l \) such that \( \sigma^{i-1}_{\alpha_k} < \sigma^{i-1}_{\beta_l} \) and \( \omega(\sigma^{i-1}_{\beta_l}) \leq \alpha_k \), a contradiction of the \((i-1)\)-minimality of \( \sigma^{i-1} \). It cannot be that \( \beta_l < \alpha_k \): we definitely mark \( \alpha_k \) when we come to it because \( \pi^i_{\alpha_k} = \sigma^i_{\alpha_k} = \sigma^{i-1}_{\alpha_k} < \sigma^{i-1}_{\alpha_0} = \pi^i_{\beta_0} \) and \( \pi^i_{\alpha_k} = \sigma^{i-1}_{\alpha_{k-1}} > \sigma^{i-1}_{\alpha_0} \), which means \( \omega(\pi^i_{\alpha_k}) \leq \omega(\sigma^{i-1}_{\alpha_0}) \leq i \). So \( \alpha_k = \beta_l \). Applying this argument again gives \( \alpha_{k-1} = \beta_{l-1} \), and so on; it also proves \( k = l \). Thus we have that \( \alpha_i = \beta_i \) for all \( i \). Then,

\[
\pi^{i-1} = \pi^i \circ (\beta_0, \beta_1, \ldots, \beta_{k-1}, \beta_k) = \sigma^i \circ (\alpha_0, \ldots, \alpha_k) = \sigma^{i-1} \circ (\alpha_k, \ldots, \alpha_0) \circ (\alpha_0, \ldots, \alpha_k) = \sigma^{i-1}.
\]

That \( g(f(\sigma)) = \sigma \) follows by induction.

In fact, \( g \) is also injective. That is, we have \( f(g(\pi)) = \pi \) for all \( \pi \in \Lambda^\omega_{\text{min}} \), which can be proved in a manner very similar to the above proof that we have \( g(f(\sigma)) = \sigma \) for any \( \sigma \in \Lambda^\omega_{\text{min}} \). Thus \( \phi \) is a bijection between \( \Lambda^\omega_{\text{min}} \) and \( \Lambda^\omega_{\text{max}} \) if and only if \( g(\pi) \in \Lambda^\omega_{\text{min}} \) for every \( \pi \in \Lambda^\omega_{\text{max}} \). Suppose there exists \( \pi \in \Lambda^\omega_{\text{max}} \) such that \( g(\pi) \) is not in \( \Lambda^\omega_{\text{min}} \). Let \( \pi^0, \ldots, \pi^0 \) be as defined in the \( g(\pi) \) algorithm and let \( i \) be the largest value such that \( \pi^i \) is not \( i \)-minimal. There must be such an \( i \) because \( \pi^0 = g(\pi) \) is not 0-minimal as it is not in \( \Lambda^\omega_{\text{min}} \). Also, \( i \) must be less than \( n \) because any permutation is trivially \( n \)-minimal. Then consider the step of the algorithm that takes us from \( \pi^{i+1} \) to \( \pi^i \). If, as we were marking positions from \( n \) to \( i + 1 \), we marked each position whose entry was less than the entry of the last marked position, we would maintain \( i \)-minimality. So it must be that we skip over some entry \( y \) because \( \omega(y) > i \). Let \( z \) be the entry of the position we had marked before considering \( y \) and let \( x \) be the entry of the next position we mark after \( y \). There must be some such \( x \) so that \( z \) moves to the left of \( y \); in fact \( x \) must be in position \( j \) in \( \pi^{i+1} \) with \( \omega(y) \leq j \) so that \( z \) moving into position \( j \) violates \( i \)-minimality. Then \( y < z, x < z \) and \( \omega(x) < \omega(y) \). Of course \( y \) is not greater than \( x \), but further \( x \) is not greater than \( y \) as \( x \) and \( y \) would then violate the \((i+1)\)-minimality of \( \pi^{i+1} \). So \( x \sim y \) as claimed. □
We now finish the proof of Theorem 4. The $\omega$ defined earlier in the proof of Theorem 4 obeys conditions [1] and [2] from Lemma 5 and the $\Lambda_{\omega_{\min}}$ and $\Lambda_{\omega_{\max}}$ defined earlier are the same as those in Lemma 5. Thus the injection $\phi$ from $\Lambda_{\omega_{\min}}$ to $\Lambda_{\omega_{\max}}$ gives rise to an injection from the set of permutations $\sigma \in S_P$ that have $X$ as their LRME set and avoid 132 and those $\sigma$ that have $X$ as their LRME set and avoid 123. We conclude that $Av_P(132) \leq Av_P(123)$.

If $Av_P(132) < Av_P(123)$, then there has to be an admissible LRME set $X$ such that $|\Lambda_{\omega_{\min}}| < |\Lambda_{\omega_{\max}}|$. In this case, the injection $\phi$ from Lemma 5 must not be a bijection, and so there must be elements $a, b, c \in P$ with

- $a < c$;
- $b < c$;
- $a \sim b$;
- $\omega(a) < \omega(b)$.

But $\omega(a) < \omega(b)$ only if the leftmost LRME that $a$ is greater than, call it $d$, is to the left of the leftmost LRME that $b$ is greater than, call it $c$. Then the induced subposet of $P$ on $\{a, b, c, d, e\}$ matches one of $Q_1$, $Q_2$, or $Q_3$.

On the other hand, if $P$ contains as an induced subposet any of $Q_1$, $Q_2$, or $Q_3$, there is some set of LRME such that there are strictly more 123-avoiding permutations with these LRME than 132-avoiding permutations. If there exists $c' \in P$ with $c' > c$ then the induced subposet on $\{a, b, c', d, e\}$ will be the same as on $\{a, b, c, d, e\}$ and so without loss of generality we may assume $c$ is maximal. Consider the permutation $\sigma = (\theta_1, d, c, \theta_2, e, \theta_3, a, b)$, where $(\theta_1, d, \theta_2, e, \theta_3)$ is a 12-avoiding subsequence of $\sigma$ containing all the elements of $P \setminus \{a, b, c\}$ (here $\theta_1$, $\theta_2$, and $\theta_3$ are themselves permutations). Such a 12-avoiding subsequence exists because $e$ is not greater than $d$. Consider all permutations with the same LRME set as $\sigma$. The non-LRME elements are $a$, $b$, and $c$, with $\omega(c) = \omega(a) = 1$ and $\omega(b) = 2$. It is easily seen that $\Lambda_{\omega_{\min}} = \{abc\}$ while $\Lambda_{\omega_{\max}} = \{cab, cba\}$, so $Av_P(132) < Av_P(123)$. □

**Corollary 6.** $Av_n(132) < Av_n(123)$ for $n \geq 3$.

**Proof.** Consider the induced subposet on the elements $\{1\}, \{3\}, \{1, 2\}, \{2, 3\},$ and $\{1, 2, 3\}$. □

5. The pattern $\{1\}{1, 2}\{2\}$

We now focus our attention on finding bounds for $Av_n(\{1\}{1, 2}\{2\})$ and studying its asymptotic behavior. To that end, we give a characterization of those permutations on $B_n$ that avoid $\{1\}{1, 2}\{2\}$.

**Lemma 7.** Let $\sigma$ be an element of $B_n$. If there exists a subsequence $abc$ of $\sigma$ such that $a < b$, $c < b$ and neither $a$ nor $c$ are the empty set, then $\sigma$ contains a $\{1\}{12}\{2\}$ pattern.

**Proof.** If $a \sim c$, we are done. So suppose first that $c > a$. If $(c \setminus a)$ is after $b$, the subsequence $ab(c \setminus a)$ is a $\{1\}{1, 2}\{2\}$ pattern. Likewise, if $(b \setminus c)$ is
before $b$, then the subsequence $(b \setminus c)b$ is a $\{1\}\{1,2\}\{2\}$ pattern. So assume that $(c \setminus a)$ is before $b$ and $(b \setminus c)$ is after $b$. But in this case $(c \setminus a)b(b \setminus c)$ is a $\{1\}\{1,2\}\{2\}$ pattern. The proof is exactly the same if $a > c$. □

Thus the permutations $\sigma \in B_n$ that avoid $\{1\}\{1,2\}\{2\}$ have the property that, for each $x \in B_n$, every element less than $x$, except possibly the empty set, is to one side of $x$ in $\sigma$. The empty set will never be a part of a $\{1\}\{1,2\}\{2\}$ pattern and so may be in any position in a $\{1\}\{1,2\}\{2\}$-avoiding permutation. Therefore it will be convenient to look at $\sigma - \emptyset$: the subsequence of $\sigma$ on every element other than the empty set. We can think of each element $x$ in $B_n - \emptyset$ as being assigned “left” or “right” according to the relative position of $x$ in $\sigma - \emptyset$ and the elements less than $x$.

Let $P$ be a poset, and $\delta: P \to \{L, R\}$. We say a permutation $\sigma \in S_P$ is $\delta$-legal when for each $x \in P$, if $\delta(x) = L$ then the entry of $x$ in $\sigma$ is to the left of every element less than it, and if $\delta(x) = R$ then the entry of $x$ in $\sigma$ is to the right of every element less than it. Let $\Lambda^\delta$ be the set of $\delta$-legal permutations on $P$.

The permutations $\sigma \in B_n$ that avoid $\{1\}\{1,2\}\{2\}$ are exactly those such that $\sigma - \emptyset \in \Lambda^\delta$ for some $\delta: (B_n - \emptyset) \to \{L, R\}$. The following lemma bounds the number of $\delta$-legal permutations for any choice of $\delta$, and applies to any arbitrary poset $P$.

**Lemma 8.** Let $P$ be a poset. Fix some $\delta: P \to \{L, R\}$. Then $|\Lambda^\delta| \leq |\Lambda^\delta_R|$, where $\delta_R(x) = R$ for all $x \in P$. The inequality is strict if and only if there exist $x, y, z \in P$, with $z < y$, $z < x$, $x \sim y$, and $\delta(x) \neq \delta(y)$.

**Proof.** Suppose $P = (X, <)$. The set $\Lambda^\delta_R$ is the set of linear extensions of $P$. Let $L^\delta := \{x \in P: \delta(x) = L\}$. We define a new poset, $P' = (X, <')$, on the same set of elements as $P$ with $x <' y$ whenever (i) $x < y$ and $y \notin L^\delta$, or (ii) $y < x$ and $x \in L^\delta$. Let $P'$ be the transitive closure of $P'$. It is easy to check that if there does not exist $x, y, z \in P$, with $z < y$, $z < x$, $x \sim y$, and $\delta(x) \neq \delta(y)$, then $P'$ is already transitively closed. On the other hand, if there exists such $x, y, z \in P$, then $P' \neq P'$; in other words, there are more relations in $P'$ than in $P'$.

We claim that $\Lambda^\delta$ is the set of linear extensions of $P'$. Indeed, let $\lambda$ be a linear extension of $P'$. Then $x < y$ in $P$ and $\delta(y) = R$ implies $x <' y$, so $y$ is to the right of $x$ in $\lambda$. On the other hand, $x < y$ in $P$ and $\delta(y) = L$ implies $y <' x$, so $y$ is to the left of $x$ in $\lambda$. Conversely, if $\sigma$ is a $\delta$-legal permutation, then $\sigma_{\alpha_0} <' \sigma_{\alpha_k}$ implies $\alpha_0 < \alpha_k$. To see this, let the entries $\sigma_{\alpha_0} <' \sigma_{\alpha_1} <' \cdots <' \sigma_{\alpha_k}$ be such that $\sigma_{\alpha_i}$ and $\sigma_{\alpha_{i+1}}$ are comparable in $P$ for all $i = 0, \ldots, k - 1$. Then each $\sigma_{\alpha_i}$ must be to the left of $\sigma_{\alpha_{i+1}}$ in any $\delta$-legal permutation, and so $\alpha_0 < \alpha_k$ as claimed.

If $x$ and $y$ are comparable in $P$, they are still comparable in $P'$. That is, the comparability graph of $P$ (as an undirected graph) is a subgraph of the comparability graph of $P'$. By a result of Stachowiak [13], the number of linear extensions of $P'$ is less than or equal to the number of linear extensions of $P$. Stachowiak’s theorem says further that the number of extensions is
equal if and only if the comparability graphs of $P$ and $\overline{P'}$ are equal, that is, they are equal exactly when we need to add no extra relations to $P'$ to obtain its transitive closure. □

Theorem 9.

$$\frac{2^{n-1}}{n+1} \prod_{k=0}^{n} \left( \binom{n}{k} + 1 \right)! \leq \text{Av}_n(\{1\}\{1,2\}\{2\}) \leq 2^{2^n+n-1} \text{Av}_n(12).$$

This means, considering the bound on $\text{Av}_n(12)$ from Theorem 3,

$$o(1) \leq \log(\text{Av}_n(\{1\}\{1,2\}\{2\})) - \log(\text{Av}_n(12)) \leq 1 + o(1).$$

**Proof.** Lower bound: For any poset $P$ on $m$ elements, let us call a permutation $\pi \in S_P$ V-shaped if there exists $1 \leq i \leq m$ such that $(\pi_1, \ldots, \pi_i)$ is 12-avoiding and $(\pi_{i+1}, \ldots, \pi_m)$ is 21-avoiding. Any V-shaped permutations is evidently $\delta$-legal if we let $\delta(x) = L$ for all $x \in \{\pi_1, \ldots, \pi_i\}$ and $\delta(x) = R$ for all other $x$.

We construct a V-shaped permutation $\rho$ on $B_n - \emptyset$ in the following manner. For each $k = 1, \ldots, n$, let $\rho^k$ be a permutation of the elements of the $k$th row of $B_n - \emptyset$. For each $k = 2, \ldots, n$, choose $j_k$ from among $\{0, \ldots, \binom{n}{k}\}$. Let $\rho$ be given by $\rho_1^1 \ldots \rho_{j_1}^1$, followed by $\rho_1^{n-1} \ldots \rho_{j_1}^{n-1}$, and so on, all the way down to $\rho_1^2 \ldots \rho_{j_2}^2$. We then append $\rho_1^1 \ldots \rho_1^{j_1}$, and next $\rho_2^{j_2+1} \ldots \rho_2^{j_1}$, followed by $\rho_3^{j_3+1} \ldots \rho_3^{j_2}$, and so on, up to $\rho_n^{j_{n-1}+1} \ldots \rho_n^{j_n}$. Figure 3 gives a visualization of $\rho$. Each choice of permutation for each row, and each choice of $j_k$, gives a distinct $\rho$. There are $\prod_{k=0}^{n} \binom{n}{k}!$ choices of permutations and $\prod_{k=2}^{n} (\binom{n}{k} + 1)$ choices of $j_k$. Finally, we can place $\emptyset$ in any position in $\rho$ to get a $\sigma \in B_n$ that avoids $\{1\}\{1,2\}\{2\}$, so we multiply by an additional factor of $2^n$. 

Figure 3. The V-shaped permutation $\rho$ is given by reading the entries in this diagram from top-left to bottom-middle to top-right.
Upper bound: There are $2^{2n} - 1$ choices for $\delta: (\mathbb{B}_n - \emptyset) \rightarrow \{L, R\}$, and for any $\delta$, by Lemma 8 we have $|\Lambda^\delta| \leq \text{Av}_n(12)$, so there are at most $2^{2n} - 1 \times \text{Av}_n(12)$ possible $\sigma - \emptyset$. Multiply by an additional factor of $2^n$ to account for the empty set.

Taking the log of both bounds gives our result. Specifically, we refer to Brightwell and Tetali’s [2] demonstration that,

$$\log \left( \prod_{k=0}^{n} \binom{n}{k} \right) + o(1) = \frac{\log(\text{Av}_n(12))}{2^n}. \quad \square$$

6. Open problems

We hope that the simple, combinatorial proofs of our main results will encourage further research into pattern avoidance in permutations on posets. In particular we would like to see a proof of the following:

**Conjecture 10.** We have $\text{Av}_P(\{1\}{1,2}\{2\}) \leq \text{Av}_P(\{1\}{2}\{1,2\})$ for any poset $P$, with strict inequality if and only if $P$ contains either of $R_1$ or $R_2$ below as an induced subposet:

![Diagram of posets](image)

**Remark.** Ideally one would find an injection from the set of $\{1\}{1,2}\{2\}$-avoiding permutations in $S_P$ to the set of $\{1\}{2}\{1,2\}$-avoiding permutations, perhaps using some subsequence similar to LRME. We have not been able to accomplish this. Both Simion and Schmidt’s proof and the proof of our own Theorem 4 rely on the fact that in any permutation of a totally ordered set (or poset), if there exists any 132 pattern then there exists some 132 pattern which begins with a left-to-right minimum (or minimal element). This is not true for $\{1\}{2}\{1,2\}$ and $\{1\}{1,2}\{2\}$. Moreover, it is not always possible to fix the LRME of a $\{1\}{1,2}\{2\}$-avoiding permutation and rearrange the remaining elements in a fashion which avoids $\{1\}{2}\{1,2\}$ as in the case of the permutation $(\{1\}, \{2\}, \{1,2\}, \emptyset)$. We might instead attempt to fix *left-to-right minima* (LRM): entries less than each preceding entry. But it is not always the case that if there exists a $\{1\}{1,2}\{12\}$ pattern then there exists such a pattern beginning with an LRM, as is evidenced by the permutation $(\{1,2,3\}, \{1,2\}, \{1,3\}, \{1\}, \{2\}, \{3\}, \{2,3\}, \emptyset) \in B_3$. Nevertheless, computer tests on all posets with seven or fewer elements do suggest that for any set of LRM, there are at least as many ways to fill in the remaining elements, while preserving that set of LRM, and to avoid $\{1\}{2}\{1,2\}$ as to avoid $\{1\}{1,2}\{2\}$. Therefore we suspect left-to-right minima may be a fruitful line of inquiry.

One can compute the number of $\{1\}{1,2}\{2\}$- and $\{1\}{2}\{1,2\}$-avoiding permutations in $S_{R_1}$ and $S_{R_2}$ to verify that strict inequality holds on these
posets. Computer tests on all posets $P$ with seven or fewer elements indicate that containment of either of $R_1$ and $R_2$ as an induced subposet is equivalent to the inequality between $Av_P(\{1\}, \{2\}, \{3\})$ and $Av_P(\{1\}, \{2\}, \{1, 2\})$ being strict. But even the “if” direction of this claim is more difficult than in the $Av_P(132) \leq Av_P(123)$ case, since it is not obvious, for example, which elements in $R_1$ or $R_2$ must be LRM. □

Example 6. The relationship between avoidance in the last pair of non-trivial length three patterns, $\{1\}, \{1, 2\}, \{3\}$ and $\{1\}, \{3\}, \{1, 2\}$, is more complicated. With posets $T$ and $U$ as in Figure 6, we have:

- $Av_T(\{1\}, \{2\}, \{3\}) < Av_T(\{1\}, \{3\}, \{1, 2\})$;
- $Av_U(\{1\}, \{3\}, \{1, 2\}) < Av_U(\{1\}, \{1, 2\}, \{3\})$. □

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