SOME TRUE STATEMENTS ABOUT K3 SURFACES

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1. Motivation and Outline

What 4-dimensional orientable closed smooth manifolds do you know? I know $S^4, S^2 \times S^2, \mathbb{CP}^2, \overline{\mathbb{CP}}^2$, and connected sums thereof. But what comes "next"?

One way we can answer this question is as follows. The above manifolds have pretty basic intersection forms: $(0), H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, (1), (-1)$. The next most basic bilinear form appearing as a summand in intersection forms for 4-manifolds is the matrix of the Dynkin diagram for E_8 (there is a bit of a story as to why, which we will not get into). You can build a manifold with E_8 as its intersection form but it can never be smoothable (Rohlin). In the right basis, $K3$ has intersection form $2(-E_8) \oplus 3H$.

One can also ask for what comes "next" symplectically or complex-geometrically. $K3$ surfaces are definitely not easy to understand but they do for example have $c_1 = 0$, and we know which cohomology classes are realizable as the class of a symplectic form. They're also the lowest-dimensional nontrivial hyperkähler manifolds. We also know some fibrations on $K3$ surfaces which play nicely with symplectic geometry (including Lagrangian fibrations).

Since there is at least half a century's worth of work on $K3$ from many directions I will be happy if I can simply point out some interesting phenomena with any clarity at all. Apologies in advance for omitting your favorite fact about $K3$ (though if you want to talk about why it can't split as a connected sum I'm your woman). For example, it's a great starting place to understand mirror symmetry [4].

Plan of talk:

(1) constructions

(2) homeomorphism, diffeomorphism, and deformation-equivalence type

(3) brief remarks on geometry

2. Three Constructions

Complex Projective Surfaces in $\mathbb{CP}^3$.

This perspective makes the role of complex and symplectic geometry obvious.

Let $d \in \mathbb{Z}_{>0}$. Consider

$$S_d := \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \mid z_0^d + z_1^d + z_2^d + z_3^d = 0\}$$

Facts about the S_d:

- The S_d are smooth: use the implicit function theorem. One better: they're complex, because they're cut out by holomorphic functions.
- The S_d are simply connected: according to the Lefschetz hyperplane theorem, if H is a hyperplane in $\mathbb{CP}^3$, $\pi_1(S_d) = \pi_1(S_d \cap H)$ because $1 < 4 - 1$. Let H be the hyperplane with $z_3 = 0$. Now we can reduce to $\mathbb{CP}^2$: let s_d consist of those $[z_0 : z_1 : z_2] \in \mathbb{CP}^2$ for which $[z_0 : z_1 : z_2 : 0] \in S_d$. Using the Lefschetz hyperplane theorem again with h the hyperplane with $z_2 = 0$ gives a surjection from $\pi_1(s_d \cap h)$ to $\pi_1(s_d)$ because $1 = 2 - 1$. Now we can compute

$$\pi_1(s_d \cap h) = \pi_1(\{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_0^2 + z_1^2 = 0\}) = \pi_1(\{[1 : i]\}) = 0$$

- $c_1(S_d) = (4-d)x$ and $c_2(S_d) = (d^2 - 4d + 6)x^2$ where x is the pullback of the generator of $H^*(\mathbb{CP}^3; \mathbb{Z})$ (proof: adjunction)
- If p_1, p_2 are homogeneous polynomials of degree d with smooth zero sets then their zero sets F_1, F_2 are diffeomorphic. (We can even do better: monomials correspond to points in $\mathbb{C}^{N(n,d)}$ via specifying the coefficients or the roots. Their zero sets are invariant under multiplication from $\mathbb{C}^\times$ so their zero sets correspond to points in $\mathbb{CP}^{N-1}$. "A zero set is singular" is captured in an equation on derivatives, so the singular zero sets form a complex codimension-1 subspace of $\mathbb{CP}^{N-1}$, and so in fact by connecting the zero sets via a path in $\mathbb{CP}^{N-1}$ avoiding the singular codimension-1 subspace one can isotope F_1 to F_2.)

The first four S_d are given as follows:

(1) $S_1 = \mathbb{CP}^2$ by using the polynomial $p(z) = z_3$.

(2) $S_2 = \mathbb{CP}^1 \times \mathbb{CP}^1$ by using the polynomial $p(z) = z_0 z_3 - z_1 z_2$: use the map

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \to S_2 \text{ given by } ([s_0 : s_1], [t_0 : t_1]) \mapsto [s_0 t_0 : s_0 t_1 : s_1 t_0 : s_1 t_1]$$

(3) $S_3 = \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$ (this would take us on a tangent: exercise)

(4) $S_4 = K3$, where

Definition 1. *A $K3$ **surface** is a simply-connected complex surface with $c_1 = 0$.*

First fibration.

We can put a holomorphic $\mathbb{CP}^1$-valued fibration on S_4 with elliptic fibers.

Use instead $z_0^4 - z_1^4 + z_2^4 - z_3^4 = 0$ as the defining polynomial. Let $L_1 = \{z_0 = z_1, z_2 = z_3\}$ and $L_2 = \{z_0 = -z_1, z_2 = -z_3\}$ be skew projective lines in $\mathbb{CP}^3$ (skew: no hyperplane contains them both). Note $L_i \subset S_4$. We'll construct a map $S_4 \to L_2$.

Let $p \in S_4$. Define $\pi(p)$ as follows:

- for $p \notin L_1$, let H_p be the unique hyperplane spanned by p and L_1, and set $\pi(p) = H_p \cap L_2$
- for $p \in L_1$, let H_p be the tangent to S_4 at p in $\mathbb{CP}^3$, and set $\pi(p) = H_p \cap L_2$

One can show [3] that this is an elliptic fibration (compute the fibers directly to show they're cut out by cubics and use the degree-genus formula, which is just adjunction).

Kummer Construction.

This perspective makes the hands-on topology clear. Consider $T^4 = S^1 \times S^1 \times S^1 \times S^1$, where we think of S^1 as $U(1) \subset \mathbb{C}$. $\mathbb{Z}_2$ acts on T^4 by conjugating each factor. This action has the sixteen fixed points $\{(\pm 1, \pm 1, \pm 1, \pm 1)\}$, so if we quotient we won't get a smooth manifold. We can obtain $K3$ either by

- blowing up at the sixteen points, extending the action by the identity along the exceptional spheres, then quotienting
- or quotienting, excising small neighborhoods $\{[(e^{i\theta_1}, \ldots, e^{i\theta_4})] \in T^4/\mathbb{Z}_2 \mid \sum_i \theta_i^2 \leq \epsilon\} \cong C\mathbb{RP}^3$ of the image of each fixed point (because each layer $\sum_i \theta_i = const$ is a copy of S^3 mod $\mathbb{Z}^2$, so $\mathbb{RP}^3$), and gluing in D^*S^2 along the common $\mathbb{RP}^3$ boundary; D^*S^2 has this boundary because
  - we can compute $e(D^*S^2)([S^2]) = -2$ since it's the unit disk bundle of T^*S^2
  - on the other hand, the Euler number of the bundle H obtained by gluing in disks along the fibers of the Hopf map is the tautological bundle on $\mathbb{CP}^1$ and so $e(H)([S^2]) = -1$
  - when we construct a new bundle E by quotienting by $\mathbb{Z}_2$ before gluing in the disks we get twice the zeroes of a generic section, so $e(E)([S^2])$
  - by the fact that $H^2(S^2; \mathbb{Z})$ is one-dimensional we can identify disk bundles by their Euler numbers, so we've identified D^*S^2 with E, which has boundary $\mathbb{RP}^3$ because of how it was constructed)

Second fibration.

From this construction we can put another fibration on $K3$ with elliptic fibers.

We take advantage of the hyperelliptic involution $T^2 \to S^2$. These exist in all genuses – arrange the the surface so that a skewer threaded through the "center" pierces all the "holes" – but we can easily describe it explicitly as a holomorphic map here because we can conjugate on $T^2 = S^1 \times S^1$. Then $T^2/\mathbb{Z}_2 \cong S^2$ (the induced metric is singular but one can extend the smooth and complex structures over the images of the fixed points).

Let $[]_4$ denote $/\mathbb{Z}_2$ on T^4 and $[]_2$ on T^2. The map $f : K3 \to S^2$ is then defined by

$$f : [(z_1, z_2, z_3, z_4)]_4 \mapsto [(z_1, z_2)]_2$$

Its regular fibers are toruses: if $(z_1, z_2) \notin \{(\pm 1, \pm 1)\}$ then its preimage under f consists of the image under $[]_4$ of $\{(z_1, z_2) \times T^2\} \cup \{(\bar{z}_1, \bar{z}_2) \times T^2\}$. These two toruses get identified under the $\mathbb{Z}_2$-action on T^4, so they are the same torus in $K3$.

Its singular fibers are the images under $[]_4$ of toruses containing one of the fixed points. In fact, the fixed points split in T^4 into four groups of four based on their images under f, so one can see that the image in $K3$ of these toruses are spheres (the image of the second T^2 factor under the hyperelliptic involution to $K3$ from T^4) blown up at 4 points. All these spheres have self-intersection -2: the exceptional divisors from the fact that they're the zero sections of D^*S^2s, and the spheres persisting from T^4 by the fact that if F denotes the class of a regular (near a singular fiber) then $F = 2S + \sum_i S_i$ and $F^2 = 0$.

It's a $K3$.

The involution is holomorphic and if you quotient first, the complex structure can be resolved as the singularities get resolved. Or you can blow up first.

Note that we can from this perspective also compute $\pi_1(K3) = 0$: move any loop into a singular fiber (possible since the base is simply-connected) and shrink there, because the singular fibers are unions of spheres.

We can also compute $c_1(K3) = 0$: the existence of a nonvanishing holomorphic 2-form is enough; the form $dz_1 \wedge dz_2$ (now we're thinking of T^4 as $\mathbb{C}^2/\mathbb{Z}^4$ and the involution as $(z_1, z_2) \mapsto (-z_1, -z_2)$ – don't worry too much about it) can be resolved at the singular points of the quotient.

Elliptic Fibrations.

This perspective gives us insight into the myriad ways elliptic surfaces can be organized in $K3$.

Let C_1 be a smooth cubic in $\mathbb{CP}^2$ cut out by the polynomial p_1 and C_2 any other cubic, cut out by p_2. Their intersection number is nine; it's possible that this intersection is realized in fewer than nine distinct points (some may be multiple intersections). But for now assume they're all distinct (and complex, so transverse and positive). Call the union of the intersection points B. They generate what's called a **pencil**: we can define a map $f : \mathbb{CP}^2 - B \to \mathbb{CP}^1$ by sending all points on the cubic $t_1p_1 + t_2p_2$ to $[t_1 : t_2] \in \mathbb{CP}^1$ – every point in $\mathbb{CP}^2 - B$ lies on one of these cubics – and though we can't extend the map to $\mathbb{CP}^2$, we can blow up at each of the base points to get a map $f : \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2 \to \mathbb{CP}^1$.

The fibers are the cubics $t_1p_1 + t_2p_2 = 0$; by the degree-genus formula (this is just adjunction) the fibers have to be elliptic.

Varying C_2 gives us many possibilities for the fiber over $[0 : 1]$, which is going to be the proper transform of C_2 (remove the basepoints from $C_2 \subset \mathbb{CP}^2$, consider the image of what remains in the blown-up manifold, then close it – this will consist of adding in the point in the exceptional divisor corresponding to the direction C_2 approached the basepoint). Therefore the fiber over $[0 : 1]$ could be...

- if C_2 is smooth then the fiber is smooth
- if C_2 is nodal, e.g. cut out by $zy^2 = x^3 + zx^2$, then the fiber is called a **fishtail** and is obtained from nearby fibers by collapsing a meridian circle to a point
- if C_2 is cuspidal, e.g. cut out by $zy^2 = x^3$, then the fiber is called a **cusp** and is obtained from nearby fibers by collapsing both meridian circles to a point
- if C_2 is a conic plus a line and
  - they are not tangent, then they intersect in two points transversely
  - they are tangent, then they intersect once to degree two
  - you're welcome for those tautologies – the point is that we're realizing most of Kodaira's possible singular fibers for elliptic fibrations
- if C_2 is three lines and
  - they are not the same, then we get a triangle of lines (each pair intersects transversely once) (in the moment polytope, they can be arranged to be the boundary triangle)

– they are the same, then we get a triple of lines all intersecting in a single point (in the moment polytope, pick any three lines emanating from a single vertex)

Let's stop being coy: what we're talking about is the first **elliptic surface** $E(1)$. An **elliptic fibration** is a proper connected holomorphic map to $\mathbb{CP}^1$ with regular fiber an elliptic curve.

There are a couple of ways to figure out how "many" singular fibers a map $E(1) \to \mathbb{CP}^1$ must admit. One way is in Miranda [6]. Another way is to show that one can always demand (that is, for generic choices of the p_i it is the case) that the singular fibers are ambiently isotopic to fishtail fibers. See [3]. Now fishtails have Euler characteristic one – they admit a cell decomposition with one cell in each dimension. We can build $E(1)$ by gluing together a bunch of $T^2 \times \mathbb{D}^2$s together along their boundaries and along these fishtail fibers, and $\chi(T^2 \times \mathbb{D}^2) = \chi(T^2) \cdot \chi(\mathbb{D}^2) = 0 \cdot 1$, so $12 = \chi(\mathbb{CP}^2) + 9\chi(\overline{\mathbb{CP}}^2) - 18 = \chi(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2) = \#\{\text{fishtail fibers}\} \cdot 1$.

Definition 2. *The **fiber sum** of two (C^∞ only, not necessarily complex) elliptic fibrations E_i is given as follows: identify neighborhoods νF_i of regular fibers with $T^2 \times \mathbb{D}^2$, select a fiber-preserving, orientation-reversing diffeomorphism φ along their boundaries, and glue the $E_i - \nu F_i$ using φ to obtain $E_1 \#_f E_2$.*

One can show that the diffeomorphism type of $E_1 \#_f E_2$ is independent of the choice of φ so long as one E_i contains a cusp fiber (part of the idea is that the monodromy around a cusp fiber can be factored into monodromies around two fishtail fibers which generate $SL(2, \mathbb{Z}) = \mathrm{Mod}(T^2)$).

Let $E(n)$ denote the n-fold fiber sum of $E(1)$. **Claim:** $E(2)$ is a $K3$.

$E(2)$ is simply-connected: use Van Kampen's theorem and the fact that $E(1) - \nu F$ is simply-connected. The latter goes as follows. $E(1)$ is simply-connected, so any fundamental group must come from circles which need to collapse in the $T^2 \times \mathbb{D}^2$. The two circles from the fiber factor never got to collapse in νF even in $E(1)$, and the circle from the $\partial \mathbb{D}^2$ factor can collapse along any of the nine exceptional sphere sections of $E(1)$.

$c_1(E(2)) = 0$: one can define fiber sum holomorphically as the pullback of the map $z \mapsto z^n$ on the base $\mathbb{CP}^1$ and show that $c_1(E(n)) = PD([(2-n)F])$.

A Lagrangian Fibration.

The fibration of $E(1)$ with 12 fishtail fibers provides a fibration of $E(2)$ with 24 fishtail fibers. One can construct a Kähler form on $E(2)$ for which this fibration is Lagrangian [1], or construct this fibration given a hyperkähler structure on $K3$ [4], and this has allowed us to understand a lot about the mirror symmetry of $K3$.

(Quick aside: a **hyperkähler structure** on a Riemannian manifold is a triple of complex structures which multiply like the quaternions and for which the associated 2-forms are symplectic. Sweet fact: the only compact complex hyperkähler surfaces are $K3$s and T^4. Another sweet fact: these are basically the only ones which admit nice Lagrangian fibrations, see [8].)

3. They are all diffeomorphic

One can compute whether or not a pair of 4-manifolds is homeomorphic by computing its intersection form and using Freedman's theorem.

All $K3$s are diffeomorphic by Kodaira's classification of complex surfaces up to deformation equivalence (exhibiting them as regular fibers of proper holomorphic fibrations) and the fact that deformation equivalent implies diffeomorphic (essentially because critical values are codimension two in complex-land, so one can always parallel transport around any singularities to connect regular fibers of a holomorphic map through a path of regular fibers).

There *are* exotic $K3$s, that is, homeomorphic but not diffeomorphic: see Gompf and Mrowka, [2].

Elliptic surfaces are also determined up to diffeomorphism by the fibration structures they admit, so there's another way to show that $K3$s are diffeomorphic as well.

4. They are all Kähler

If you start with a $K3$ (now that we know what to say, I mean something diffeomorphic to any of the above constructions with a complex structure), you can always find a symplectic structure to make it Kähler: [7].

How can $K3$s be "the same" or not symplectically? This takes us into a discussion of the **symplectic cone**, those positive H^2 classes which admit symplectic forms. It turns out that all of them do: see [5].

I'm also going to hazard a guess that for any symplectic structure you could want you can always put on an integrable complex structure, though I'm not familiar enough with hyperkähler/Calabi-Yau geometry to be definitive. I think just pick a Ricci-flat metric and you should be good.

5. There are More Fibrations

See e.g. [8], in which Smith uses an explicit fibration on $K3$ to construct genus three Lefschetz pencils on torus bundles on toruses.

References

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