INDEX THEOREMS IN SFT

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Plan of talk:

(1) What are our operators?

(2) Set the stage for the proof of Fredholmness; convince ourselves this is the stage we're on.

(3) Give the sketchiest possible indications towards the proof.

(4) Set the stage for the proof of the index formula.

(5) Give similarly sketchy indications towards the proof.

1. **Outline**

Linearization of the Cauchy-Riemann equations. A quick review.

Let (Σ, j) be a Riemann surface and (W, J) an almost-complex manifold. Consider the space

$$\{u : \Sigma \to W \mid \overline{\partial}_J u = 0 \text{ and } \ldots\}$$

where "..." could refer to regularity/integrability, boundary and/or asymptotic behavior, $[u]$ constraints, etc. (all of which have an effect on what follows, but which we will ignore for the next few paragraphs). Let $\mathcal{B}$ be a Banach manifold containing all such u and $\mathcal{E}$ be a Banach space bundle over $\mathcal{B}$ with fibers

$$\Omega^{0,1}(\Sigma, u^*TW) := \Gamma(\overline{\mathrm{Hom}}_{\mathbb{C}}(T\Sigma, u^*TW))$$

If we've closed $\mathcal{B}$ to k, p-regularity/integrability then we close $\mathcal{E}_u$ to $k-1, p$-regularity/integrability.

Note that $\overline{\partial}_J$ is a section of $\mathcal{E}$. Its zero set is what we're interested in, and this zero set will be a smooth manifold when the **linearization** of $\overline{\partial}_J$ (which is actually the derivative of $\overline{\partial}_J$ *projected* to the fiber) is Fredholm of index zero.

Let's compute this linearization $\mathbf{D}_u$. Let $u \in \overline{\partial}_J^{-1}(0)$, $\eta \in T\mathcal{B}_u = \Gamma(u^*TW)$, $u_\rho \in \mathcal{B}$ with $u_0 = u$ and $\partial_\rho u_\rho|_{\rho=0} = \eta$. Choose local holomorphic coordinates $s + it$ about a point $z \in \Sigma$. Choose a symmetric connection ∇ (since $u \in \overline{\partial}_J^{-1}(0)$, the choice of connection doesn't matter). Then near z we have

$$\begin{aligned}\mathbf{D}_u\eta &= \nabla_\rho(\overline{\partial}_J u_\rho)\big|_{\rho=0} \\ &= \nabla_\rho(du_\rho + J(u_\rho) \circ du_\rho \circ j)\big|_{\rho=0}\end{aligned}$$

Since $\partial_t = j\partial_s$ and $\mathbf{D}_u\eta$ is antilinear, it is determined by its action on ∂_s. Let's compute this:

$$\begin{aligned}
\mathbf{D}_u\eta\partial_s &= \nabla_\rho(\partial_s u_\rho + J(u_\rho)\circ\partial_t u_\rho)\big|_{\rho=0} \\
&= \nabla_s\eta + \nabla_\rho(J(u_\rho)\circ\partial_t u_\rho)\big|_{\rho=0} \text{ by symmetry of } \nabla \\
&= \nabla_s\eta + \nabla_\rho(J(u_\rho))\big|_{\rho=0}\circ\partial_t u + J(u)\,\nabla_\rho(\partial_t u_\rho)\big|_{\rho=0} \text{ product rule} \\
&= \nabla_s\eta + \nabla_\rho(J(u_\rho))\big|_{\rho=0}\circ\partial_t u + J(u)\nabla_t\eta \text{ symmetry again} \\
&= \nabla_s\eta + (\nabla_\eta J)\circ\partial_t u + J(u)\nabla_t\eta \text{ because } \nabla_\rho J(u_\rho)\big|_{\rho=0} = \nabla_{\partial_\rho u_\rho|_{\rho=0}} J
\end{aligned}$$

By antilinearity one can compute the action on ∂_t to get

$$\boxed{\mathbf{D}_u\eta = \overline{\partial}\eta + (\nabla_\eta J)\circ du\circ j}$$

where $\overline{\partial}\eta = \nabla\eta + J(u)\circ\nabla\eta\circ j$. The local form of $\mathbf{D}_u$ motivates the following discussion.

In these lectures we show that operators in a more general class (defined for u with a more general class of domain, which we'll get to posthaste) containing the linearization of the Cauchy-Riemann equations are Fredholm and compute their Fredholm indices.

Let (E, J) be a complex vector bundle over a Riemann surface (Σ, j). A **real linear Cauchy-Riemann type operator** on E is an operator $\mathbf{D} : \Gamma(E) \to \Omega^{0,1}(\Sigma, E)$ which satisfies the Leibniz rule for real-valued smooth $f : \Sigma \to \mathbb{R}$:

$$\mathbf{D}(f\eta) = (\overline{\partial}f)\eta + f\mathbf{D}\eta$$

One can check that $\mathbf{D}_u$ is of this type:

$$\begin{aligned}
\mathbf{D}_u(f\eta) &= \nabla(f\eta) + J(u)\circ\nabla(f\eta)\circ j + (\nabla_{f\eta}J)\circ du\circ j \\
&= df\eta + f\nabla\eta + J(u)\circ(df\eta + f\nabla\eta)\circ j + (f\nabla_\eta J\circ du\circ j \\
&= (df\eta + J(u)\circ df\eta\circ j) + (f\overline{\partial}\eta + f(\nabla_\eta J)\circ du\circ j) \\
&= (\overline{\partial}f)\eta + f\mathbf{D}_u\eta
\end{aligned}$$

From contact action functional to Cauchy-Riemann equations. Let (M, ξ) be a $2n-1$-dimensional contact manifold with contact form α, Reeb vector field R_α, complex structure J on ξ compatible with $d\alpha|_\xi$, and $\pi_\xi : TM \to \xi$ the projection along R_α.

Recall the **contact action functional**

$$\mathcal{A}_\alpha : C^\infty(S^1, M) \to \mathbb{R} \text{ given by } \gamma \mapsto \int_{S^1}\gamma^*\alpha = \int_{S^1}\alpha(\dot\gamma)\,dt$$

Its first variation along $\eta \in \Gamma(\gamma^*TM)$ is given by

$$\begin{aligned}
(d\mathcal{A}_\alpha)_\gamma(\eta) &= \int_{S^1} d\alpha(\eta, \dot\gamma)\,dt \text{ note the } d \text{ hits the } \dot\gamma \text{ too, but } \dot\gamma \sim R_\alpha \\
&= -\int_{S^1} d\alpha(\pi_\xi\dot\gamma, \eta)\,dt
\end{aligned}$$

Restrict to constant-speed T-periodic γ, where the period of γ is $\mathcal{A}_\alpha(\gamma)$. This means $\dot\gamma = TR_\alpha(\gamma)$. These are the generators of the SFT DGA.

Restrict to $\eta \in \Gamma(\gamma^*\xi)$, since those are the directions in which we require γ to be nondegenerate (note that, unlike in Hamiltonian Floer homology, we can't ask for more). Define an L^2-inner product on $\Gamma(\gamma^*\xi)$ by $\langle\cdot,\cdot\rangle := \int_{S^1} d\alpha(\cdot, J\cdot)\,dt$. Then by compatibility of J,

$$(d\mathcal{A}_\alpha)_\gamma(\eta) = \int_{S^1} d\alpha(-J\pi_\xi\dot\gamma, J\eta)\,dt = \langle -J\pi_\xi\dot\gamma, \eta\rangle$$

motivating us to write

$$\nabla\mathcal{A}_\alpha(\gamma) := -J\pi_\xi\dot\gamma \in \Gamma(\gamma^*\xi)$$

Next let's compute this thing's linearization at a Reeb orbit γ. Let $\eta \in \Gamma(\gamma^*\xi)$, $\{\gamma_\rho : S^1 \to M\}_{\rho\in(-\epsilon,\epsilon)}$ with $\gamma_0 = \gamma, \partial_\rho\gamma_\rho|_{\rho=0} = \eta$, and ∇ symmetric on M. Then we have

$$\begin{aligned}
\nabla_\rho(-J\pi_\xi\dot\gamma_\rho)|_{\rho=0} &= -J\,\nabla_\rho(\pi_\xi\dot\gamma_\rho)\big|_{\rho=0} \text{ pick up no } \nabla_\eta J \text{ because that term would act on } \pi_\xi\dot\gamma = 0\\
&= -J\,\nabla_\rho(\dot\gamma_\rho - \alpha(\dot\gamma_\rho)R_\alpha(\gamma_\rho))\big|_{\rho=0} \text{ projecting off the part in the } R_\alpha \text{ direction}\\
&= -J(\nabla_t\eta - T\nabla_\eta R_\alpha - \partial_\rho(\alpha(\dot\gamma_\rho))|_{\rho=0}R_\alpha(\gamma)) \text{ product rule}\\
&= -J(\nabla_t\eta - T\nabla_\eta R_\alpha)
\end{aligned}$$

where the last step is because

$$0 \overset{\partial_t\gamma\sim R_\alpha}{=} d\alpha(\partial_\rho\gamma_\rho, \partial_t\gamma_\rho)|_{\rho=0} = \partial_\rho(\alpha(\partial_t\gamma_\rho))|_{\rho=0} - \partial_t(\alpha(\partial_\rho\gamma_\rho))|_{\rho=0} \overset{\partial_\rho\gamma_\rho|_{\rho=0}\in\xi}{=} \partial_\rho(\alpha(\partial_t\gamma_\rho))|_{\rho=0} - 0$$

All this motivates us to consider

$$\boxed{\mathbf{A}_\gamma : \Gamma(\gamma^*\xi) \to \Gamma(\gamma^*\xi) \text{ given by } \eta \mapsto -J(\nabla_t\eta - T\nabla_\eta R_\alpha)}$$

Recall asymptotic operators:

Definition 1. *An* ***asymptotic operator*** *on a Hermitian vector bundle (E, J, ω) on S^1 is a real-linear differential operator $\mathbf{A} : \Gamma(E) \to \Gamma(E)$ which appears* in any unitary trivialization *as*

$$\eta \mapsto -J_0\partial_t\eta - S(t)\eta$$

where $S : S^1 \to End(\mathbb{R}^{2n})$ is a smooth loop of symmetric matrices.

One can check that $\mathbf{A}_\gamma$ is of this type: symmetry of $S(t)\eta := -JT\nabla_\eta R_\alpha$ follows from symmetry of the connection. One can also check that nondegeneracy of γ as a Reeb orbit is equivalent to nondegeneracy (zero is not in its spectrum) of $\mathbf{A}_\gamma$ as an asymptotic operator.

Cylindrical ends of pseudoholomorphic curves. It is not the case that the cylindrical end of a finite-energy pseudoholomorphic curve $u : [0,\infty) \times S^1 \to \mathbb{R} \times M$ for $J \in \mathcal{J}(\alpha)$ (see §1) is a gradient flow line of $\mathcal{A}_\alpha$. That is, we do NOT have any statement of the form

$$\overline{\partial}_J u = 0 \text{ and } u(s,t) = \exp_{(Ts,\gamma(Tt))} h(s,t) \Leftrightarrow \partial_s u - \nabla\mathcal{A}_\alpha(u(s,\cdot)) = 0$$

where $h(s,t)$ is a vector field along the image of u for which $h(s,\cdot) \to 0$ uniformly as $|s| \to 0$.

What we do have is

$$\pi_\xi\partial_s u + J\pi_\xi\partial_t u = \pi_\xi(-J\partial_t u) + J\pi_\xi\partial_t u = 0$$

which can be thought of as expressing the fact that under projection to ξ, u is a gradient flow line of $\mathcal{A}_\alpha$.

That's not awesome, but it's good enough. We'll return to this idea once we've set up the proper language in which to say precisely how cylindrical ends of pseudoholomorphic curves relate to gradient flow lines of the contact action functional: not on the level of the original PDEs, but on the level of their linearizations.

2. Definitions for Theorem 4.3

Coordinates near punctures. Let (Σ, j) be a closed connected Riemann surface of genus g, with $\Gamma \subset \Sigma$ a finite set partitioned as $\Gamma^+ \cup \Gamma^-$. Let $\dot{\Sigma} = \Sigma \backslash \Gamma$. The points in $\Gamma^{\pm}$ are referred to as **positive/negative punctures** (they correspond to where Σ asymptotes to the Reeb orbits for which (Σ, j) is the domain of a curve exhibiting some term in $\mathbf{H}$ relating all those orbits).

Let $Z_+ := [0, \infty)_s \times S^1_t$ and $Z_- := (-\infty, 0]_s \times S^1_t$, and give each the complex structure $i\partial_s = \partial_t, i\partial_t = -\partial_s$. Fix biholomorphisms $\psi_{\pm}$ from $Z_{\pm}$ to the punctured disk.

Fix a choice of holomorphic cylindrical coordinates near each puncture. By this we mean: choose a neighborhood $\mathcal{U}_z$ of z and a biholomorphism φ_z from $\mathcal{U}_z \backslash \{z\}$ to $\mathbb{Z}_{\pm}$, whose composition with $\psi_{\pm}$ extends holomorphically to a map from $\mathcal{U}_z$ to $\mathbb{D}$. (This is always possible by choosing holomorphic local coordinates near z.) We call the $\mathcal{U}_z \backslash \{z\}$ **cylindrical ends**.

Bundles over punctured surfaces. Let (E, J) be a smooth complex vector bundle of rank m over $(\dot{\Sigma}, j)$. An **asymptotically Hermitian structure** on (E, J) is a choice of Hermitian rank m "limit bundle" (E_z, J_z, ω_z) on S^1 for every $z \in \Gamma$. The sense in which (E_z, J_z, ω_z) is the limit of (E, J) at z is that one also chooses a complex bundle isomorphism $E|_{\mathcal{U}_z \backslash \{z\}} \to \mathrm{pr}_2^* E_z$ covering φ_z.

Let τ be a unitary trivialization of (E_z, J_z, ω_z). An **asymptotic trivializaiton** is complex trivialization $\tau : E|_{\mathcal{U}_z \backslash \{z\}} \to Z_{\pm} \times \mathbb{R}^{2m}$ induced from that of (E_z, J_z, ω_z) via the bundle isomorphism.

Each (E_z, J_z, ω_z) is an **asymptotic bundle** associated to (E, J) near z.

Sobolev spaces of sections. Recall that a function from $\mathbb{R}^n$ to $\mathbb{R}$ is of Sobolev class k, p if it along with its weak derivatives up to order k are all in L^p.

Recall the subscript $_{loc}$ indicates "on all open subsets with compact closure contained in the domain of interest."

Recall that the $W^{k,p}_{loc}$ sections of a vector bundle are those in $W^{k,p}_{loc}$ in a all the open sets in the base of a given bundle atlas of domains of local trivializations. Over compact bases, the resulting space and topology are independent of this choice.

Define
$$W^{k,p}(E) := \left\{ \eta \in W^{k,p}_{loc}(E) \middle| \, \eta_z \in W^{k,p}(\mathrm{int}(Z_{\pm}), \mathbb{R}^{2m}) \text{ for all } z \in \Gamma^{\pm} \right\}$$
where η_z is the expression of η in local cylindrical coordinates about z and we use $ds \wedge dt$ to define the norm on $Z_{\pm}$.

Compactness of S^1 gives that different choices of asymptotic trivializations give equivalent norms. However, over noncompact bases like ours, one doesn't get independence of all choices: different choices of asymptotically Hermitian structures give inequivalent norms. See §A.4 for what else is different in the case of domains with cylindrical ends.

The bundle of forms. The target of a real linear Cauchy-Riemann type operator on E is the complex vector bundle
$$F := \overline{\mathrm{Hom}}_{\mathbb{C}}(T\dot{\Sigma}, E)$$
whose sections are precisely $\Omega^{0,1}(\Sigma, E)$. From asymptotic trivializations of E we get ones for F, as well as the $W^{k,p}(F)$. In these trivializations a real linear Cauchy-Riemann type operator $\Gamma(E) \to \Gamma(F)$ appears over $\mathcal{U}_z \backslash \{z\}$ appears as a linear map
$$\mathbf{D} : C^{\infty}(Z_{\pm}, \mathbb{R}^{2m}) \to C^{\infty}(Z_{\pm}, \mathbb{R}^{2m}) \text{ given by } \mathbf{D}\eta(s,t) = \overline{\partial}\eta(s,t) + S(s,t)\eta(s,t)$$

where J_0 is the standard complex structure (remember this is in a trivialization), and $S \in C^\infty(Z_\pm, \mathrm{End}(\mathbb{R}^{2m}))$.

How operators asymptote. $\mathbf{D}$ is **asymptotic to** $\mathbf{A}_z$ at z if, with respect to the appropriate corresponding trivializations, $\mathbf{D}$ appears in the form above, $\mathbf{A}_z = -J_0\partial_t - S_\infty$, we set $S_\infty(s,t) := S_\infty(t)$, and

$$||S - S_\infty||_{C^k(Z^R_\pm)} \to 0 \text{ as } R \to \infty$$

where $Z^R_+ = [R, \infty) \times S^1$ and Z^R_- is defined analogously to $Z_\pm = Z^0_\pm$. The relation we should think of is "$\mathbf{D}$ limits to $\partial_s - \mathbf{A}$."

SFT operators follow this framework. Let W be a symplectic cobordism and J an almost-complex structure which behaves appropriately at the symplectization ends. Let $\xi_\pm$ be the contact structures with contact forms $\alpha_\pm$ and $J_\pm \in \mathcal{J}(\alpha_\pm)$ at the positive/negative ends. Let $\gamma^i_\pm$ denote nondegenerate Reeb orbits at the positive/negative ends and $z^i_\pm$ enumerate $\Gamma^\pm$.

maps	$u : (\dot{\Sigma}, j) \to (W, J)$
bundles	$(E, J) = (u^*TW, u^*J)$
real-linear Cauchy-Riemann type operators	$\mathbf{D}_u : \Gamma(u^*TW) \to \Omega^{0,1}(\dot{\Sigma}, u^*TW)$
asymptotically Hermitian structures	$(\Gamma({\gamma^i_\pm}^*\xi_\pm, J_\pm, d\alpha_\pm)$
asymptotic operators	$\mathbf{A}_{\gamma^i_\pm}$

In order to apply this organizational scheme, one must show that when u has a cylindrical end positively or negatively asymptotic to a T-periodic Reeb orbit,

$$||(\nabla_{\eta_\pm} J_\pm) \circ du \circ j + JT\nabla_{\eta_\pm} R_{\alpha_\pm}||_{C^k(Z^R_\pm)} \to 0 \text{ as } R \to \infty$$

Remember that $JT\nabla_{\eta_\pm} R_{\alpha_\pm}$ has been extended to the $Z^R_\pm$s by ignoring the $\mathbb{R}$ coordinate s.

Why should we believe this limit? In the simplest case, $u = (Ts, \gamma(t))$ for some T-periodic Reeb orbit (the property we want is $\dot{\gamma} = TR_\alpha(\gamma)$) is a map from (s,t) coordinates on a cylinder to (r,m) coordinates on a symplectization.

We have

$$\begin{aligned} 0 &= \nabla_\eta du \\ &= \nabla_\eta (J \circ du \circ j) \\ &= (\nabla_\eta J) \circ du \circ j + J\nabla_\eta (du \circ j) \end{aligned}$$

so it suffices to show that $\nabla_\eta(du \circ j) = T\nabla_\eta R_\alpha$. This makes sense because no matter what η is, du as given above can only change as R_α changes.

The point, as I said before, is not that pseudoholomorphic *curves* are gradient flow *lines*, but that the linearization $\mathbf{D}_u$ of the Cauchy-Riemann equations near a solution limits to the linearization of the gradient of the contact action functional on cylindrical ends in the domain.

3. **Theorem 4.3**

An asymptotic operator is **nondegenerate** if zero is not in its spectrum, which means it's an isomorphism $H^1(E) \to L^2(E)$ since (last week/§3) asymptotic operators are Fredholm of index zero.

Theorem 1 (4.3)**.** *Suppose (E, J) is an asymptotically Hermitian vector bundle over $(\dot{\Sigma}, j)$, $\mathbf{A}_z$ is a nondegenerate asymptotic operator on the associated asymptotic bundle (E_z, J_z, ω_z) for each $z \in \Gamma$, and $\mathbf{D}$ is a linear Cauchy-Riemann type operator asymptotic to $\mathbf{A}_z$ at each puncture z. Then for every $k \in \mathbb{Z}_{\geq 0}, 1 < p < \infty$,*

- $\mathbf{D} : W^{k,p}(E) \to W^{k-1,p}(F)$ *is Fredholm*
- ind $\mathbf{D}$ *and* $\ker \mathbf{D}$ *(which equals the space of smooth sections whose derivatives of all orders decay to zero at infinity) are each independent of k and p*

In general, the way we prove operators are Fredholm is to do the following:

(1) Use estimates of the form $||\eta||_{W^{k,p}} \leq c||\mathbf{D}\eta||_{W^{k-1,p}} + ||\eta||_{W^{k-1,p}}$, which follow from local elliptic estimates (you can find nice expressions for the Laplace operator when you're near a pseudoholomorphic curve), to show that $\mathbf{D}$ is semi-Fredholm (finite-dimensional kernel and closed range).

(2) Do everything all over again to show that the formal adjoint of $\mathbf{D}$ is also semi-Fredholm (one can define this via L^2 norms and integrate to obtain $\mathbf{D}^* = \partial + S(s,t)^T$).

4. Definitions for Theorem 5.4

Recall μ_{CZ}^{τ}.

Definition 2. *The **Conley-Zehnder index** associates an integer $\mu_{CZ}(\mathbf{A})$ to every trivialized nondegenerate asymptotic operator $\mathbf{A} : H^1(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$, determined uniquely by*

(1) *normalization:* $\mu_{CZ}(-J_0\partial_t - (I \oplus -I)) := 0$

(2) *difference:* $\mu_{CZ}(\mathbf{A}_-) - \mu_{CZ}(\mathbf{A}_+) := \mu^{spec}(\mathbf{A}_-, \mathbf{A}_+)$

We denote the Conley-Zehnder index with respect to a specified trivialization τ by μ_{CZ}^{τ}.

There is a dictionary between asymptotic operators and symplectic arcs: $\mathbf{A} = -J_0\partial_t - S(t)$ corresponds to the arc Ψ defined by

$$(-J_0\partial_t - S(t))\Psi(t) = 0, \Psi(0) = I$$

and, if one is more comfortable with the geometry of symplectic matrices, including how the degenerate ones sit inside, than with the geometry of symmetric matrices, one can think of the Conley-Zehnder index as the Maslov index of the associated loop of symmetric matrices.

Hints towards computing this thing: since our bundles are trivial, they split. Exercise: Conley-Zehnder indices add under Whitney sum/direct sum. So it remains to understand μ_{CZ} for line bundles. Theorem 3.34 introduces the concept of the winding number of an eigenfunction for $\mathbf{A} : H^1(S^1, \mathbb{R}^2) \to L^2(S^1, \mathbb{R}^2)$. Letting $w_{\pm}$ be the minimum winding number of eigenfunctions with positive/negative eigenvalues, and $p = w_+ - w_-$, one can then compute

$$\mu_{CZ}(\mathbf{A}) = 2w_+ - p = 2w_- + p$$

Exercise: $\mu_{CZ}^{\tau_i}(\mathbf{A})$ differ by twice the winding number of the transition function $\tau_1 \circ \tau_2^{-1}$, and so one can classify e.g. Reeb orbits by parity. Moreover, one can show that multiple covers of even Reeb orbits are also even.

Relative Chern numbers. We use this construction to better relate the cases of open and closed domain.

Definition 3. *Let S be a compact oriented surface with boundary. The* ***relative first Chern number*** *associates integers $c_1^\tau(E)$ to complex vector bundles (E, J) over S with trivializations τ of $E|_{\partial S}$, which is uniquely determined by*

(1) *on a line bundle, $c_1^\tau(E)$ is the signed count of zeroes of a generic smooth section which is nonzero constant on ∂S under τ*

(2) *c_1^τ adds under Whitney sum*

These conditions uniquely determine c_1^τ by splitting higher rank bundles and matches $c_1(E)([S])$ when $\partial S = \emptyset$.

One can extend this definition to the case where S is a punctured surface and (E, J) is an asymptotically Hermitian vector bundle with asymptotic trivializations, because asymptotic trivializations are forced to extend to neighborhoods of the punctures too (remove $Z_\pm^R$ in the appropriate cylindrical coordinates about each puncture and take the closure of the remaining surface – since asymptotic trivializations extend by pr_2^*, this construction will give c_1^τ independent of R).

Exercise: Let (E, J) be an asymptotically Hermitian bundle of rank m with two asymptotic trivializations τ_i. Let $\deg(\tau_2 \circ \tau_1^{-1}) \in \mathbb{Z}$ denote the sum over all punctures of winding numbers of determinants of transition maps $S^1 \to U(m)$. Then

$$c_1^{\tau_2}(E) = c_1^{\tau_1}(E) - \deg(\tau_2 \circ \tau_1^{-1})$$

One way to show this on line bundles is to glue the bundles together over a double of S, then use the symplectic definition of c_1 (see little McDuff-Salamon) and the fact that c_1^τ reduces to c_1 in the case when $\partial S = \emptyset$.

Let E be an asymptotically Hermitian vector bundle with a real-linear Cauchy-Riemann type operator $\mathbf{D}$ asymptotic to $\mathbf{A}_z$. Let $\deg(\tau_2 \circ \tau_1^{-1}|_z)$ denote the winding number of the determinant of the transition map at the puncture z. Then

$$\begin{aligned}
2c_1^{\tau_2}(E) + \sum_{z\in\Gamma^+} \mu_{CZ}^{\tau_2}(\mathbf{A}_z) - \sum_{z\in\Gamma^-} \mu_{CZ}^{\tau_2}(\mathbf{A}_z) &= 2c_1^{\tau_1}(E) - 2\deg(\tau_2 \circ \tau_1^{-1}) + \sum_{z\in\Gamma^+} \left(\mu_{CZ}^{\tau_1}(\mathbf{A}_z) + 2\deg(\tau_2 \circ \tau_1^{-1}|_z)\right) \\
&\quad - \sum_{z\in\Gamma^-} \left(\mu_{CZ}^{\tau_1} + 2\deg(\tau_2 \circ \tau_1^{-1}|_z)\right) \\
&= 2c_1^{\tau_1}(E) + \sum_{z\in\Gamma^+} \mu_{CZ}^{\tau_1}(\mathbf{A}_z) - \sum_{z\in\Gamma^-} \mu_{CZ}^{\tau_1}(\mathbf{A}_z)
\end{aligned}$$

Now the following theorem makes sense.

5. **Theorem 5.4**

Theorem 2 (5.4)**.** *The Fredholm index of $\mathbf{D}$ is given by*

$$\text{ind } \mathbf{D} = rk_{\mathbb{C}} E \chi(\dot{\Sigma}) + 2c_1^\tau(E) + \sum_{z\in\Gamma^+} \mu_{CZ}^\tau(\mathbf{A}_z) - \sum_{z\in\Gamma^-} \mu_{CZ}^\tau(\mathbf{A}_z)$$

where τ is an arbitrary choice of asymptotic trivializations.

Relation to the $\mathbb{C}$-linear closed case. Note that were $\mathbf{D}$ to be complex-linear and $\Gamma = \emptyset$, the classical Riemann-Roch formula gives $\operatorname{ind}(\mathbf{D}) = \operatorname{rk}_{\mathbb{C}}(E)\chi(\Sigma) + 2c_1(E)$. We cannot simply extend this to real-linear Cauchy-Riemann type operators, however: though such a $\mathbf{D}$ is a zeroth-order perturbation of a complex-linear one, these perturbations are not compact since $W^{k,p}(\dot{\Sigma}) \hookrightarrow W^{k-1,p}(\dot{\Sigma})$ is not compact when Σ has punctures.

Ideas in the proof.

(1) Restrict to $\Gamma = \emptyset$.

(2) Restrict to line bundles: asymptotically Hermitian bundles always split, and while $\mathbf{D}$ may not respect the splitting, it is always homotopic through Fredholm operators to one that does.

(3) Restrict to $k, p = 1, 2$; regularity results tell us this is okay.

(4) Let $B \in \Gamma(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F))$ be a zeroth-order perturbation whose zeroes are all nondegenerate. For sufficiently large $\sigma > 0$, $\ker(\mathbf{D} + \sigma B)$ is approximately spanned by 1-dimensional subspaces of sections with support near positive zeroes of B.

(5) Do the same thing for $\mathbf{D}^*$. Then by continuity of the Fredholm index as a function to $\mathbb{Z}$ we have

$$\begin{aligned}
\operatorname{ind}(\mathbf{D}) &= \operatorname{ind}(\mathbf{D} + \sigma B) \\
&= \dim\ker(\mathbf{D} + \sigma B) - \dim\ker((\mathbf{D} + \sigma B)^*) \\
&= \#\{\text{pos. zeroes of } B\} - \#\{\text{neg. zeroes of } B\} \\
&= c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, F)) \\
&= c_1(\overline{\operatorname{Hom}}_{\mathbb{C}}(E, \mathbb{C}) \otimes \overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, E)) \\
&= c_1(E \otimes (\overline{\operatorname{Hom}}_{\mathbb{C}}(T\Sigma, \mathbb{C}) \otimes E)) \\
&= c_1(E) + c_1(T\Sigma) + c_1(E) \\
&= \chi(\Sigma) + 2c_1(E)
\end{aligned}$$

which is what we want.

(6) So how do we prove that the dimensions of the kernel/cokernel equals the count of positive/negative zeroes? We can do this using compactness-type analysis:

  (a) §5.5 Let $\eta_\nu \in \ker(\mathbf{D} + \sigma_\nu B)$ for $\sigma_\nu \overset{\nu\to\infty}{\to} \infty$ (assume they satisfy an L^2 bound). Bound the energy of η_ν from above by that of $||\mathbf{D} + \sigma_\nu B||$ away from zeroes of B.

  (b) §5.6 That means that we have to be able to reparameterize near each zero of B to solutions to related PDEs on $\mathbb{C}$. Characterize these: there is only one dimension of solutions near each zero.

  (c) §5.7 We can then upper bound the dimension of the appropriate kernels by the number of zeroes, because we know that sequences of solutions must limit to a direct sum of their behavior near zeroes.

  (d) §5.7 We can lower bound these dimensions by gluing up solutions near zeroes.

(7) We need to let $\Gamma \neq \emptyset$. This is difficult because we've used a lot of compact inclusions which are now off limits because of the cylindrical ends. One has to be more careful about the directions B we choose to deform towards; we can be (ask Chris Gerig).