1

$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$

Details: Use Lagrange Multipliers with $f(x, y, z) = (x-2)^2 + (y-2)^2 + (z-1)^2$ and constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1$. Recall that you are trying to solve $\nabla f = \lambda \nabla g$. This gives you a set of three equations

$$\begin{cases} 2x - 4 = \lambda 2x \\ 2y - 4 = \lambda 2y \\ 2z - 2 = \lambda 2z \end{cases} \Rightarrow \begin{cases} \frac{2x-4}{2x} = \lambda \\ \frac{2y-4}{2y} = \lambda \\ \frac{2z-2}{2z} = \lambda \end{cases} \Rightarrow x = y = 2z$$

Plugging back into the constraint, we have $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right), \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$ as the solutions to the Lagrange Multiplier. Out of this set of points, clearly $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$ is the minimizer of f. (The other point is the maximizer.)

2

$\frac{4}{3}\pi$.

Details: You are not meant to calculate the volume in this problem as an integral; you are suppose to realize that the region described is exactly 1/8 of a sphere of radius 2, and use this fact to find the volume.

3

$\frac{1}{2}(e-1)$

Details: Make the specified change of variables. This transforms the region R in the x-y plane to the region S in the u-v plane, where S is the triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$. The Jacobian of the transformation is $J = 1/2$.

$$\iint_R \frac{e^{x+y}}{x+y}\, dx\, dy = \iint_S \frac{e^u}{u} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv = \int_0^1 \int_0^u \frac{1}{2}\frac{e^u}{u}\, dv\, du = \frac{1}{2}(e-1)$$

4

$y'(0) = -1, z'(0) = 0$

Details: Recall that since $\mathbf{r}(t)$ is the position of the particle, then $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$ is the velocity vector, i.e. the tangent vector. The question is asking for the tangent vector of the curve at $t = 0$. Further recall that when the curve is at the intersection of the two surfaces, then the tangent vector is perpendicular to both the normal vectors of the surfaces. Therefore, we cross the normal vectors to find a vector that is parallel to the tangent vector.

Recall that the gradient is tangent to the surface. For the first surface $\nabla F = \langle 2x, 2y, 4z\rangle$; evaluated at $(1, 1, 1)$, this gives $\langle 2, 2, 4\rangle$ as the normal vector of the first surface. Similarly, for the second surface $\nabla G = \langle y, x, -1\rangle$, which gives $\langle 1, 1, -1\rangle$ as the normal vector of the second surface. Cross them:

$$\langle 2, 2, 4\rangle \times \langle 1, 1, -1\rangle = \langle -6, 6, 0\rangle$$

and this gives a vector that is parallel to $\mathbf{r}'(0)$, i.e. $k\langle -6, 6, 0\rangle = \mathbf{r}'(0)$ for some constant scalar multiple k. Since $x'(0) = 1$, this means $k = -1/6$, and so $y'(0) = -1$ and $z'(0) = 0$.

5

(a) $f_x(x, y) = x + y$, $f_y(x, y) = x - y$.
(b) $f(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + xy$

Details:
(a) Recall for a unit vector v, $D_v f = v \cdot \nabla f$. Therefore, the information in the problem gives you a set of two equations

$$\begin{cases} \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \cdot \langle f_x, f_y\rangle = \sqrt{2}x \\ \langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle \cdot \langle f_x, f_y\rangle = \sqrt{2}y \end{cases} \Rightarrow \begin{cases} f_x + f_y = 2x \\ f_x - f_y = 2y \end{cases}$$

the solution of which is $f_x = x + y$, $f_y = x - y$.
(b) From the previous part, integrate $f_x = x + y$ with respect to x to see that

$$f(x, y) = \frac{x^2}{2} + xy + g(y)$$

Then differentiate the above with respect to y to see that

$$f_y(x, y) = x + g'(y)$$

This implies that $g'(y) = -y$. Now integrate against y to find

$$g(y) = -\frac{y^2}{2} + K$$

which means that

$$f(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + xy + K$$

Finally, the information that $f(0, 0) = 0$ tells you that $K = 0$.

6

-1

Details: You will need to parameterize the surface S. The easiest way to do this is to find the equation of the plane spanned by the three vertices. To do this, make two vectors, and cross them:

$$\langle 1, 0, 1\rangle \times \langle 1, 1, 2\rangle = \langle -1, -1, 1\rangle$$

so the equation of the plane is $-x-y+z=0$. Then, compute the surface integral

$$\begin{aligned}\iint_S \langle 3,4,5\rangle \cdot \mathrm{d}\mathbf{S} &= \iint_D \langle 3,4,5\rangle \cdot \langle -1,-1,1\rangle \,\mathrm{d}A \\ &= \int_0^1 \int_0^x -2 \,\mathrm{d}y \,\mathrm{d}x \\ &= -1\end{aligned}$$

NB: In the above calculation, D is the region in the x-y plane with vertices $(0,0)$, $(1,0)$, $(1,1)$. This is the region that the surface S projects to in the x-y plane.

7

(a) $c=4$
(b) $\pi(2-\sqrt{2})$

Details:
(a) Compute Div$\mathbf{F}$ with the given expression for $\mathbf{F}$: Div$\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, where

$$\begin{aligned}\frac{\partial F_1}{\partial x} &= \sqrt{x^2+y^2+z^2} + \frac{x^2}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial F_2}{\partial y} &= \sqrt{x^2+y^2+z^2} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} \\ \frac{\partial F_3}{\partial z} &= \sqrt{x^2+y^2+z^2} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}\end{aligned}$$

so that Div$\mathbf{F} = 4\sqrt{x^2+y^2+z^2}$, i.e. $c=4$.
(b) Use the Divergence Theorem.

$$\begin{aligned}\text{Flux} &= \iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} \\ &= \iiint_V \mathrm{Div}\mathbf{F}\,\mathrm{d}V, \text{ by Divergence Theorem} \\ &= \iint_V 4\sqrt{x^2+y^2+z^2}\,\mathrm{d}V, \text{ by part (a)} \\ &= \int_0^{2\pi}\int_0^1\int_0^{\pi/4} 4\rho\rho^2 \sin\phi \,\mathrm{d}\phi\,\mathrm{d}\rho\,\mathrm{d}\theta, \text{ change to spherical coord.} \\ &= \pi(2-\sqrt{2})\end{aligned}$$

8

$-\frac{3}{2}\pi$

Details: As suggested, use Stokes' Theorem, where for the surface we choose D, the unit disk in the x-z plane, whose boundary is the curve C:

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \iint_S \mathrm{Curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S}, \text{ by Stokes' Theorem} \\
&= \iint_D \mathrm{Curl}\mathbf{F} \cdot \mathbf{n} \mathrm{d}S \\
&= \iint_D \langle 0, 3(x^2+z^2), 0 \rangle \cdot \langle 0, -1, 0 \rangle \, \mathrm{d}A \\
&= \int_0^{2\pi} \int_0^1 -3r^2 r \, \mathrm{d}r \, \mathrm{d}\theta, \text{ change to Polar coord.} \\
&= -\frac{3}{2}\pi
\end{aligned}$$

NB: $\mathrm{Curl}\mathbf{F} = \langle 0, 3(x^2+z^2), 0 \rangle$.
NB2: $\mathbf{r}(t)$ is a the unit circle in the x-z plane, going in the direction from the positive x axis to the positive z axis. This induces an orientation on D, the unit disk, such that the normal is pointing in the direction of the negative y-axis, hence $\mathbf{n} = \langle 0, -1, 0 \rangle$.

9

(a) S_2 = unit disk in the x-y plane, oriented with normal pointing up. (One of many possible choices for S_2, but the most logical and computational straightforward choice.)
(b) $\frac{\pi}{2}$

Details:
(a) Let S_1 be the given surface, the upper hemisphere of the unit sphere oriented upward, and let S_2 be the surface we have chosen, the unit disk in the x-y with orientation upward. Let $-S_2$ denote the same surface as S_2, except with the opposite orientation, i.e. orientation downward.

Divergence Theorem says:

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \iint_{-S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_V \mathrm{Div}\mathbf{F} \, \mathrm{d}V$$

Note that the Divergence Theorem requires a completely closed surface, where the normal vector is pointing outward everywhere on the surface. In this problem, for the $\mathbf{F}$ give, $\mathrm{Div}\mathbf{F} = 0$. Therefore, the above simplifies to

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = -\iint_{-S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

And finally since $-\iint_{-S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ (switching orientation adds a negative sign), we have

$$\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

(b)

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iint_{S_2} \mathbf{F} \cdot \mathrm{d}\mathbf{S}, \text{ from part (a)} \\
&= \iint_D \langle x + y^2, x^2 - y, x^2 + y^2 \rangle \cdot \langle 0, 0, 1 \rangle \, \mathrm{d}A, \text{ since } z = 0 \text{ on } S_2 \\
&= \int_0^{2\pi} \int_0^1 r^2 r \, \mathrm{d}r \, \mathrm{d}\theta, \text{ change to Polar coord.} \\
&= \frac{\pi}{2}
\end{aligned}$$

10

$\pi - 2\sin(1)$

Details: Using Green's Theorem, let C be the curve given in the problem. Let L be the line segment from $(-1, 0)$ to $(1, 0)$. Together, C and L form a closed, counterclockwise oriented curve, to which we apply Green's Theorem:

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, \mathrm{d}A$$

Next, separately calculate the line integral over L and the double integral:

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, \mathrm{d}A = \iint_D 2 \, \mathrm{d}A = \pi, \text{ i.e. 2 times area of half unit circle}$$

$$\int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{-1}^{1} \langle \cos t, t \rangle \cdot \langle 1, 0 \rangle \, \mathrm{d}t = \sin(1) - \sin(-1) = 2\sin(1), \text{ since sin is odd}$$

Therefore, from Green's Theorem, we have

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \pi - 2\sin(1)$$