1

34

Details: This is a chain rule problem:

$$\frac{d}{dt}f(\mathbf{r}(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

We now evaluate the above at $t = 2$. When $t = 2$, $\mathbf{r}(2) = \langle 1, 2, (2)^2 \rangle$. Also, $\mathbf{r}'(t) = \langle 0, 1, 2t \rangle$, so that at $t = 2$, $\mathbf{r}'(2) = \langle 0, 1, 2(2) \rangle$. Therefore

$$\left.\frac{d}{dt}f(\mathbf{r}(t))\right|_{t=2} = \left.\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right)\right|_{t=2} = 5 \cdot 0 + 6 \cdot 1 + 7 \cdot 4 = 34$$

2

8π

Details: Combine the equation of the two surfaces, $z = x^2$ and $z = 4 - y^2$, together, to see that $x^2 + y^2 = 4$. What this means is that the intersection of the two surfaces (which is a curve) lies on the cylinder $x^2 + y^2 = 4$, i.e. the surface we are interested in lies entirely on top of the disk $x^2 + y^2 = 4$ in the x-y plane. We will take this as the region of integration.

$$\iint_D (4 - y^2) - (x^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_0^{2\pi} (4 - r^2) r\,\mathrm{d}\theta\,\mathrm{d}r = 8\pi$$

Alternatively, you may integrate the volume over the x-z plane. The answer is the same, but the integral is harder to compute.

$$\int_{-2}^2 \int_{x^2}^4 \sqrt{4 - z} - (-\sqrt{4 - z})\,\mathrm{d}z\,\mathrm{d}x = 8\pi$$

3

The minimum distance is $\sqrt{5}/2$. The set of points closest to the origin is described by $\left\{x^2 + y^2 = 1, z = -\frac{1}{2}\right\}$

Details: Use Lagrange Multipliers. For computational ease, we will try to minimize the squared distance to the origin. $d^2 = f(x, y, z) = x^2 + y^2 + z^2$. The constraint is $g(x, y, z) = x^2 + y^2 - 3/2 - z$. For Lagrange Multipliers, we are trying to solve $\nabla f = \lambda \nabla g$:

$$\begin{cases} 2x = \lambda 2x \\ 2y = \lambda 2y \\ 2z = -\lambda \end{cases}$$

From the first equation

$$\lambda = 1 \text{ or } x = 0$$

If $\lambda = 1$, the third equation tells us $z = -1/2$. Plugging back into the constraint, we have

$$\begin{cases} x^2 + y^2 = 1 \\ z = -\frac{1}{2} \end{cases} \tag{1}$$

as a solution of Lagrange Multiplier.

If on the other hand, $x = 0$, then from the second equation $y = 0$, and plugging back into the constraint, we have

$$\begin{cases} x = y = 0 \\ z = -\frac{3}{2} \end{cases} \tag{2}$$

as another solution of Lagrange Multiplier.

Finally, plug (1) and (2) back into $d = \sqrt{f}$ to find the that (1) is the minimizer, and the minimizing distance is $\sqrt{5}/2$.

4

$\frac{\pi}{16}$

Details: Take the change of coordinates $u = x^2$ and $v = y^2$. The point of taking this change of coordinates is to transform the region of integration into the region S in the u-v plane described by $u^2 + v^2 = 1$, $u \geq 0$, $v \geq 0$. The Jacobian is

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} = \frac{1}{4xy} = \frac{1}{4\sqrt{uv}}$$

The integral is

$$\iint_D xy \, \mathrm{d}x \, \mathrm{d}y = \iint_S \sqrt{uv} \frac{1}{4\sqrt{uv}} \, \mathrm{d}u \, \mathrm{d}v = \frac{1}{4} \iint_S \mathrm{d}u \, \mathrm{d}v = \frac{1}{4} \cdot \text{ area of quarter circle} = \frac{\pi}{16}$$

5

Yes, $\mathbf{F}$ is conservative, because we can find the potential $f(x,y) = \ln(\sqrt{x^2 + y^2}) + C$.

Details: In this problem, you cannot just check $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ to show that $\mathbf{F}$ is conservative, because that theorem requires that the domain D is open and simply connected, which this domain is not, so the Theorem does not apply. To show that $\mathbf{F}$ is conservative, we must produce the function f for which $\nabla f = \mathbf{F}$.

Proceed by the hint: assume $f(x,y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Then

$$\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{dg}{dr} \frac{\partial r}{\partial x}, \frac{dg}{dr} \frac{\partial r}{\partial y} \right\rangle = \frac{dg}{dr} \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$$

Since we want $\nabla f = \mathbf{F}$, this implies that $\frac{dg}{dr} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$. Now, integrate to recover $g(r) = \ln(r) + C$. So that $f(x,y) = g(r) = \ln(r) + C = \ln(\sqrt{x^2 + y^2}) + C$.

6

$-3x + 3y - 2z = -3$

Details: The tangent plane will contain both $\mathbf{r}_u$ and $\mathbf{r}_v$, so we cross them to find the normal vector. At the point $(1, 2, 3)$, $u = 1$ and $v = 2$.

$$\begin{aligned} \mathbf{r}_u\big|_{(1,2)} &= \langle 0, v, 3u^2 \rangle\big|_{(1,2)} = \langle 0, 2, 3 \rangle \\ \mathbf{r}_v\big|_{(1,2)} &= \langle 1, u, 0 \rangle\big|_{(1,2)} = \langle 1, 1, 0 \rangle \\ \Rightarrow \mathbf{n} &= \langle 0, 2, 3 \rangle \times \langle 1, 1, 0 \rangle = \langle -3, 3, -2 \rangle \end{aligned}$$

Therefore, the equation of the plane is $-3x + 3y - 2z = -3$.

7

$\frac{2}{3}\pi \left(2\sqrt{2} - 1\right)$

Details: Parameterize the surface in terms of y, z.

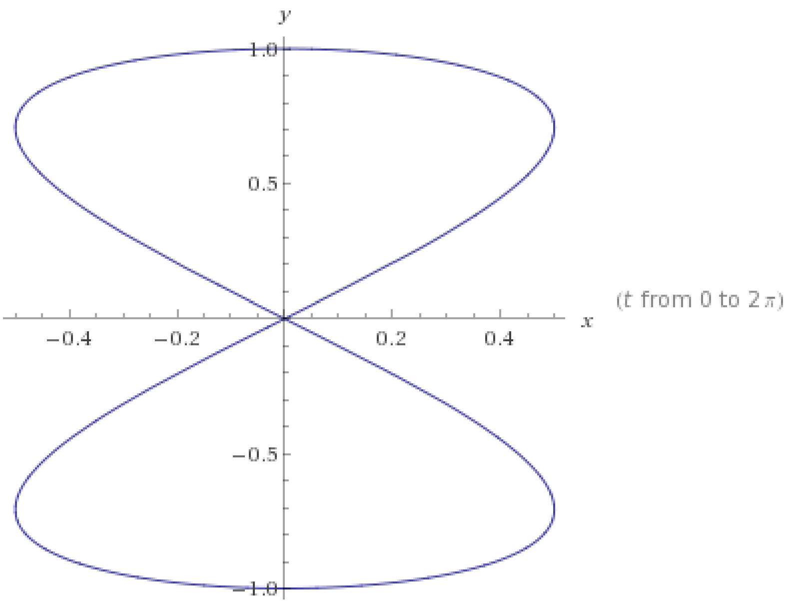
$$\begin{aligned} \mathbf{r}(y, z) &= \langle yz, y, z \rangle \\ \mathbf{r}_y(y, z) &= \langle z, 1, 0 \rangle \\ \mathbf{r}_z(y, z) &= \langle y, 0, 1 \rangle \\ \Rightarrow \mathbf{r}_y \times \mathbf{r}_z &= \langle 1, -z, -y \rangle \end{aligned}$$

The surface integral:

$$\text{Surface Area} = \iint_S 1 \, \mathrm{d}S = \iint_S |\mathbf{r}_y \times \mathbf{r}_z| \, \mathrm{d}y \, \mathrm{d}z = \int_0^1 \int_0^{2\pi} \sqrt{1 + r^2} \, r \, \mathrm{d}\theta \, \mathrm{d}r = \frac{2}{3}\pi \left(2\sqrt{2} - 1\right)$$

8

(a) Courtesy of WolframAlpha



(b) $\frac{1}{3}$

Details:
(a) Sketch the curve by plotting some points for $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, ..., 2\pi$.
(b) Integrate one half of one petal and then multiply by 4 to get the area. For $0 \leq \theta \leq \frac{\pi}{2}$, the part of the curve in the first quadrant is traced out from $(0,0)$ to $(0,1)$. This is a graph of x in terms of y, so we use the formula

$$\int_a^b x(\theta)y'(\theta)\,\mathrm{d}\theta = \int_0^{\pi/2} \sin\theta\cos\theta\cdot\cos\theta\,\mathrm{d}\theta = \frac{1}{3}$$

9

-1

Detail: Use Stokes' Theorem. First calculate Curl$\mathbf{F}\langle x-1, -y, -1\rangle$. Then find the equation of the plane that contain the three points: $x+y+z=1$. Finally, calculate using Stokes' Theorem:

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} &= \iint_S \mathrm{Curl}\mathbf{F}\cdot\mathrm{d}\mathbf{S}, \text{ by Stokes' Theorem} \\
&= \iint_T \mathrm{Curl}\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_T \langle x-1, -y, -1\rangle\cdot\langle 1,1,1\rangle\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1\int_0^{1-y} x-y-2\,\mathrm{d}x\,\mathrm{d}y \\
&= -1
\end{aligned}$$

10

$\frac{7}{6}\pi$

Details: Use Divergence Theorem. Let S be the surface given in the problem, the upper hemisphere oriented upward. Let D be the disk $\{x^2+y^2\leq 1, z=0\}$, oriented downward. Together S and D make a closed surface, to which we apply the Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathrm{d}\mathbf{S} + \iint_D \mathbf{F}\cdot\mathrm{d}\mathbf{S} = \iiint_V \mathrm{Div}\mathbf{F}\,\mathrm{d}V$$

The second integral over D is zero, because on D, $z=0$, and so $\mathbf{F} = \langle xz, yz, z\rangle = 0$.

Therefore

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \iiint_V \mathrm{Div}\mathbf{F} \\
&= \iiint_V 2z + 1 \, \mathrm{d}V \\
&= \iiint_V 2z \, \mathrm{d}V + \iiint_V 1 \, \mathrm{d}V \\
&= \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} 2\rho \cos\phi \cdot \rho^2 \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\rho + \text{ Volume of hemisphere} \\
&= \frac{\pi}{2} + \frac{1}{2} \frac{4\pi}{3} \\
&= \frac{7}{6}\pi
\end{aligned}
$$