Solutions to Professor Hutchings' 12/18/03 final

1) Use the Chain Rule.

$$\begin{aligned}\frac{d}{dt} f(\overrightarrow{r}(t))\ |_{t=0} &= \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)\ |_{t=0} \\ &= (\nabla f \cdot \overrightarrow{r}'(t))|_{t=0} \\ &= \nabla f\ |_{\langle 1,2\rangle} \cdot \langle 3,4\rangle \\ &= \langle ye^{xy}, xe^{xy}\rangle_{\langle 1,2\rangle} \cdot \langle 3,4\rangle \\ &= \left\langle 2e^2, e^2\right\rangle \cdot \langle 3,4\rangle \\ &= 10e^2\end{aligned}$$

2a) The gradient is perpendicular to level surfaces. The normal to $G(x,y,z) = x^2+y^2+z^2 = 14$ is $\nabla G = \langle 2x, 2y, 2z\rangle$ which is $\langle 6,4,2\rangle$ at $(3,2,1)$.

b)

$$\begin{aligned} z - z_0 &= f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= 2x_0(x-x_0) - 2y_0(y-y_0) \\ \implies z &= 3 + 4(x-2) - 2(y-1)\end{aligned}$$

3) Use the second derivative test inside the region. $\nabla f = \langle 2x-4, 2y\rangle = \langle 0,0\rangle$ implies $(x,y) = (2,0)$. This is not inside the region $x^2+2y^2 \leq 1$.

Use Lagrange multipliers on the boundary $g(x,y) = x^2+2y^2 = 1$. We get

$$\begin{aligned} 2x-4 &= 2\lambda x \\ 2y &= 4\lambda y \\ \implies (2x-4)y &= xy \\ \implies 4y &= xy\end{aligned}$$

If $y = 0$ then the constraint implies $x = \pm 1$. If $y \neq 0$ then $x = 4$ but then there is no solution to $4^2+2y^2 = 1$ in y. So the maximum is $f(-1,0) = 1+4 = 5$ and the minimum is $f(1,0) = 1-4 = -3$.

4) Do directly, treating S as the graph of the function $f(x,y) = 9-x^2-y^2$ over the xy-plane.

$$\overrightarrow{n}\,dS = \langle -f_x, -f_y, 1\rangle\ dx\ dy = \langle 2x, 2y, 1\rangle\ dx\ dy$$

Note that the normal is upward pointing so we take positive the z-component in $d\overrightarrow{S}$. The surface is a graph over the region in the xy-plane given by the disc $x^2+y^2 \leq 9$.

$$\begin{aligned}\int\int_{x^2+y^2\leq 9} \left\langle x, y, 9-x^2-y^2\right\rangle \cdot \langle 2x, 2y, 1\rangle\ dx\ dy &= \int_0^{2\pi}\int_0^3 (2r^2+9-r^2)r\ dr\ d\theta \\ &= 2\pi\int_0^3 9r + r^3\ dr \\ &= 243\pi/2\end{aligned}$$

5) Swap the order of integration.

$$\int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} y \sin(x^2)\ dy\ dx = \frac{1}{2}\int_0^4 [y^2 \sin(x^2)]_{-\sqrt{x}}^{\sqrt{x}}\ dx = 0$$

6) There are several ways to parametrize this. We use the parametrization $\overrightarrow{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ and $|dS| = |\overrightarrow{r}_r \times \overrightarrow{r}_\theta|\ dr\ d\theta = \sqrt{2}r\ dr\ d\theta$. So the surface area is

$$\int_0^{2\pi}\int_1^2 \sqrt{2}r\ dr\ d\theta = 3\sqrt{2}\pi$$

7)

$$\int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^2 \rho^2 \sin\phi\ d\rho d\phi d\theta = 16\pi\sqrt{2}/3$$

8) S is a closed surface so we can use the Divergence theorem. The divergence of $\overrightarrow{F}$ is 3 so we get

$$\int\int_S \overrightarrow{F}\cdot d\overrightarrow{S} = \int\int\int_E 3\ dV = 3\cdot vol(E) = 3(4\pi/3) = 4\pi$$

9a) Check that curl $\langle y, x + z\cos y, \sin y\rangle = \overrightarrow{0}$ on a simply connected region (on all of $\mathbb{R}^3$ here).

$$\int P dx = yx + g(y,z) = f(x,y,z)$$
$$\implies f_y = x + g_y = x + z\cos y$$
$$\implies g(y,z) = z\sin y + h(z)$$
$$\implies f_z = \sin y + h'(z) = \sin y$$
$$\implies h(z) = C \text{ constant}$$
$$\implies f(x,y,z) = xy + z\sin y + C$$

b) Does not exist since $\text{div}(\langle z, y, x\rangle) = 1 \neq 0$.

10) Let $u = 3x$ and $v = 2y$. The Jacobian is the reciprocal of $\det\begin{pmatrix}3 & 0\\ 0 & 2\end{pmatrix} = 6$.

$$\int\int_{u^2+v^2\leq 1} (u^2+v^2)^{5/2}\frac{1}{6}\ du\ dv = \frac{1}{6}\int_0^{2\pi}\int_0^1 r^5\cdot r\ dr\ d\theta$$
$$= \frac{2\pi}{6}\cdot\frac{1}{7} = \frac{\pi}{21}$$

11) We would like to use Stokes' theorem but the curve C is not closed. Note that curl$(\overrightarrow{F}) = 0$. The curve C starts at $(0,0,0)$ and ends at $(\pi,0,0)$ so let L be the straight line along the x-axis from $(\pi,0,0)$ to $(0,0,0)$. Then $C\cup L$ is an oriented closed curve, so we can apply Stokes' theorem to get:

$$\int_{C\cup L}\overrightarrow{F}\cdot d\overrightarrow{r} = \int\int_S \text{curl}(\overrightarrow{F})\cdot d\overrightarrow{S} = 0$$
$$\implies \int_C \overrightarrow{F}\cdot d\overrightarrow{r} = \int_{-L}\overrightarrow{F}\cdot d\overrightarrow{r}$$

We do the integral over L directly by parametrization with $\overrightarrow{r}(t) = \langle t, 0, 0 \rangle$ where, since we are considering $-L$, t goes from 0 to 1.

$$\begin{aligned}
\int_{-L} \overrightarrow{F} \cdot d\overrightarrow{r} &= \int_0^1 \overrightarrow{F}(\overrightarrow{r}(t)) \cdot \overrightarrow{r}'(t) dt \\
&= \int_0^1 \langle t, 0, \cos 1 \rangle \cdot \langle 1, 0, 0 \rangle \, dt \\
&= \int_0^1 t \, dt = \frac{1}{2}
\end{aligned}$$

12) Let D be the disc $\overrightarrow{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, 4 \rangle$ for $0 \le r \le 3$ and $0 \le \theta \le 2\pi$. Let D have upward orientation Then $S \cup -D$ is a closed surface with outward pointing normal everywhere so we can apply the Divergence theorem. Note that div $\overrightarrow{F} = 0$.

$$\begin{aligned}
&\int\int_{S \cup -D} (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \int\int\int_E \operatorname{div} \operatorname{curl} \overrightarrow{F} dV = 0 \\
\implies &\int\int_S (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S} = \int\int_D (\nabla \times \overrightarrow{F}) \cdot d\overrightarrow{S}
\end{aligned}$$

Then we can do the integral over D directly. $\overrightarrow{n} dS = \langle 0, 0, r \rangle \ dr \ d\theta$ for the disc D, since we have upward normal. And curl $\overrightarrow{F} = 3 \langle y^2 + z^2, z^2 + x^2, x^2 + y^2 \rangle$. So curl $\overrightarrow{F} \cdot d\overrightarrow{S} = 3r^3 \ dr \ d\theta$.

$$\int\int_D \operatorname{curl} \overrightarrow{F} \cdot d\overrightarrow{S} = \int_0^{2\pi} \int_0^3 3r^3 \ dr \ d\theta = \frac{3^5}{2}\pi$$