Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivar. calc. course).

3. Vectors, Dot/Cross Products, Lines & Planes

Questions

1.

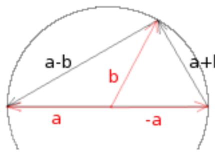
No. We know that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$. If $\vec{u} \times \vec{v}$ is also a scalar multiple (i.e. parallel) to $\vec{u}$, then $\vec{u}$ must be zero.

2.

$$\begin{aligned} |\vec{u}+\vec{v}|^2 &= (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) \\ &= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} \\ &= |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2 \end{aligned}$$

From the above, we see that $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2$ if and only if $2\vec{u}\cdot\vec{v} = 0$, if and only if u is perpendicular to $\vec{v}$. This is the Pythagorean theorem. (View $\vec{u}$, $\vec{v}$, and $\vec{u}+\vec{v}$ as the three sides of a right triangle.)

3.



Reference the diagram above, and let the radius of the circle be denoted by r:

$$(\vec{a}+\vec{b}) \cdot (\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2 = r^2 - r^2 = 0$$

This shows that the two vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

Problems

1.

Given $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (2, 1, -1)$. The goal of this problem is to find the parametric equations for the line L in $\mathbb{R}^3$ passing through $\mathbf{a}$ and $\mathbf{b}$.

(a) We want a vector that is parallel to the line L. Any vector starting from one point on the line and ending at another point on the line will be parallel to the line. In particular, we could take $\vec{u} = \mathbf{b} - \mathbf{a} = \langle 1, 1, -2\rangle$.

(b) $\mathbf{c}$ can be any point on the line. Let's take $\mathbf{c} = \mathbf{a} = (1, 0, 1)$. $L = \vec{c} + t\vec{u}$, where t is the parameter. The parametric equations come from writing out each component for the above vector equation for L:

$$\begin{cases} x(t) &= 1 + t \\ y(t) &= 0 + t \\ z(t) &= 1 - 2t \end{cases}$$

(c) Yes. A new parameterization results from taking any scalar multiple of $\vec{u}$, or any other point $\mathbf{c}$ on the line. In particular, this means that each line in $\mathbb{R}^3$ has an infinite number of valid parameterizations.

2.

Given the plane $x + y - z = 4$. The goal of this problem is to show that the equation of any plane in $\mathbb{R}^3$ can be written in the form $ax + by + cz = d$, where $\langle a, b, c\rangle$ is a normal vector to the plane.

(a) Suppose we have a point $\mathbf{a} = (a_1, a_2, a_3)$ that is a point on the plane. (This means that $\mathbf{a}$ satisfies the equation for the plane, so, in particular, we know that $a_1 + a_2 - a_3 = 4$.) Let $\mathbf{x} = (x, y, z)$.

$$\begin{aligned} &(\mathbf{x} - \mathbf{a}) \cdot (1, 1, -1) = 0 \\ \Rightarrow &(x - a_1, y - a_2, z - a_3) \cdot (1, 1, -1) = 0 \\ \Rightarrow &x + y - z = a_1 + a_2 - a_3 \\ \Rightarrow &x + y - z = 4 \end{aligned}$$

This shows that $(\mathbf{x} - \mathbf{a}) \cdot (1, 1, -1) = 0$ is exactly the equation of the plane.

(b) Any vector in the plane can be written in the form $(\mathbf{x} - \mathbf{a})$. The equation $(\mathbf{x} - \mathbf{a}) \cdot (1, 1, -1) = 0$ says exactly that any vector in the plane is normal to $\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(c) Suppose $ax + by + cz = d$ is the equation of a plane. We want to show that $\langle a, b, c\rangle$ is a normal vector to the plane. This is equivalent to showing that for any vector $\vec{v}$ in the plane, $\vec{v} \cdot \langle a, b, c\rangle = 0$. Let $\mathbf{p} = (x_1, y_1, z_1)$ and $\mathbf{q} = (x_2, y_2, z_2)$ be the starting and ending point respectively for $\vec{v}$. In particular, this means that $\mathbf{p}, \mathbf{q}$ both line in the plane and so satisfies the equation for the plane. We have

$$\begin{aligned} \vec{v} \cdot \langle a, b, c\rangle &= (\mathbf{q} - \mathbf{p}) \cdot \langle a, b, c\rangle \\ &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle \cdot \langle a, b, c\rangle \\ &= (ax_2 + by_2 + cz_2) - (ax_1 + by_1 + cz_1) \\ &= d - d \\ &= 0 \end{aligned}$$

This shows that $\langle a, b, c\rangle$ is the normal vector to the plane.

3.

Given $\mathbf{a} = (0, 0, 1)$, $\mathbf{b} = (0, 1, 2)$, and $\mathbf{c} = (1, 2, 3)$. The goal of this problem is to find the equation of the plane containing those points. Recall that the equation of a plane

$ax + by + cz = d$ is determined fully by the normal vector $\vec{n} = \langle a, b, c \rangle$, and any point $\mathbf{p}$ on the plane.

(a) $\vec{u} = \mathbf{b} - \mathbf{a} = \langle 0, 1, 1 \rangle$. $\vec{v} = \mathbf{c} - \mathbf{a} = \langle 1, 2, 2 \rangle$.
(b) We can find a vector perpendicular/normal to the plane by finding a vector perpendicular to both $\vec{u}$ and $\vec{v}$. The corss product $\vec{u} \times \vec{v}$ gives a vector that is perpendicular to both $\vec{u}$ and $\vec{v}$.

$$\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \langle 0, 1, -1 \rangle$$

(c) We know the plane is of the form $0x + 1y - 1z = d$. To determine d, plug in any point, say $\mathbf{a} = (0, 0, 1)$, into the equation to get $d = 0 \cdot 0 + 1 \cdot 0 - 1 \cdot 1 = -1$. Therefore, the equation of the plane is $y - z = -1$.