Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivariable calculus course).

#32. The Divergence Theorem

Questions

1.

Zero, since incompressible means zero divergence. So apply the divergence theorem to get zero.

2.

Use the divergence theorem to get the triple integral of 1 over E, which is the volume of E.

Problems

1.

Divergence theorem: $\int\int\int_E \nabla^2(f) = 6\int\int\int_E dV = 6\cdot\text{vol}(E)$.

2.

a) Both equal $\int_C \overrightarrow{F}\cdot d\overrightarrow{r}$ by applying Stokes' theorem.

b) S_1 and S_2 have the same orientation on their boundary curve C, so if we just glue S_1 to S_2 along this boundary curve, the surface orientations won't be compatible. (Think of C as a ring, and S_1 as the surface obtained by blowing a bubble through the ring. We have two rings and blow out two different surfaces S_1, S_2, then put them together by putting the rings together. This now encloses a volume, so we can apply the Divergence theorem.)

Thus to form a closed surface with outward orientation everywhere, we need to take $S_1 \cup -S_2$. This will then enclose a volume E bounded by the surface S defined to be $S_1 \cup -S_2$. (Note that we couldn't apply the divergence theorem to S_1 or S_2 individually since they didn't enclose a volume.) Now we apply the divergence theorem to E:

$$\int\int_{S_1\cup -S_2} \text{curl}\,\overrightarrow{F}\cdot d\overrightarrow{S} = \int\int\int_E \text{div}\,\text{curl}\,\overrightarrow{F}\,dV = 0$$
$$\implies \int\int_{S_1} \text{curl}\,\overrightarrow{F}\cdot d\overrightarrow{S} = \int\int_{S_2} \text{curl}\,\overrightarrow{F}\cdot d\overrightarrow{S}$$

Additional problem

Apply Green's theorem with the vector field $\overrightarrow{G} = \langle -Q, P\rangle$. The statement of the two-dimensional divergence theorem is:

$$\int\int_D \text{div}\,\overrightarrow{F}\,dA = \int_C \overrightarrow{F}\cdot d\overrightarrow{n}$$

where $d\overrightarrow{n}$ is the normal $\langle dy, -dx\rangle$ to the curve. (One can check this is perpendicular to $d\overrightarrow{r} = \langle dx, dy\rangle$ by taking the dot product. The convention is that $d\overrightarrow{n}$ points out of the area enclosed by the curve when oriented positively.)

$$\begin{aligned}\int\int_D \left(\frac{\partial P}{\partial x} - \frac{\partial(-Q)}{\partial y}\right) dA &= \int_C \overrightarrow{G} \cdot d\overrightarrow{r} \\ \implies \int\int_D \text{div}\, \overrightarrow{F}\; dA &= \int_C \langle -Q, P\rangle \cdot \langle dx, dy\rangle \\ &= \int_C \overrightarrow{F} \cdot d\overrightarrow{n}\end{aligned}$$