Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivariable calculus course).

#31. Stokes' Theorem

Problems

1.

curl $\overrightarrow{F} = \langle 2, -2, -2, \rangle$. Treating the surface as a graph of $z = 4 - x^2 - y^2 = f(x, y)$ above the disc $x^2 + y^2 \leq 4$ in the xy-plane, we use that $d\overrightarrow{S} = \pm \langle -f_x, -f_y, 1\rangle \ dx \ dy$. The normal should point away from the origin, i.e. have positive $\hat{\mathbf{k}}$ component, so we take +. The integral is then

$$\int\int_{x^2+y^2\leq 4} \langle 2, -2, -2, \rangle \cdot \langle 2x, 2y, 1\rangle \ dx \ dy$$

which we can convert into polars and evaluate to get -8π.

2.

curl $\overrightarrow{F} = \langle -y, -1, -1\rangle$, $d\overrightarrow{S} = \langle 1, 1/2, 1\rangle \, dy \, dx$, where we think of the triangle is lying on a plane, which is a graph of a function. This plane is the graph of $z = 1 - x - \frac{1}{2}y$ once we get the normal by taking the cross product of two vectors on the plane.

$$\int_0^1 \int_0^{2-2x} \langle -y, -1, -1\rangle \cdot \langle 1, 1/2, 1\rangle \ dy \ dx = -13/6$$

3.

$\overrightarrow{r}(u, v) = \langle uv, u + v, u^2 + v^2\rangle$ and $\overrightarrow{r}_u \times \overrightarrow{r}_v = \langle 2v - 2u, 2u^2 - 2v^2, v - u\rangle$. Lastly, curl $\overrightarrow{F} = \langle 0, -1, 0, \rangle$. Putting everything together we get:

$$\int_0^1 \int_0^u \langle 0, -1, 0, \rangle \cdot \left\langle 2v - 2u, 2u^2 - 2v^2, v - u\right\rangle \ dv \ du = -1/3$$