Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivariable calculus course).

27. Green's Theorem

Questions

2.

(a) 0

(b) 0 (this is because $f_{xy} = f_{yx}$, since $\mathbf{F} = \nabla f$).

3.

Integrating over a curve dx is not the same as integrating over $\mathbb{R}$ dx.

Problems

1.

$$\int_C y^2\,dx + x\,dy = \int\int_D -2y + 1\,dA$$

where D is the interior of the ellipse. Making the change of coordinates $u = \frac{x}{a}$ and $v = \frac{y}{b}$ we find that the integral equals $ab\pi$.

2.

(b) You should get zero because C_2 and C_4 are vertical, while $\mathbf{F}$ has no $\hat{\mathbf{j}}$ component.

(c) C_1 can be parameterized as $\mathbf{r}(t) = (t, c)$ for $a \leq t \leq b$ so

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_a^b (P(t,c), 0) \cdot (1, 0)\,dt = \int_a^b P(t,c)\,dt$$

The rest of the answer follows by parameterizing C_3 as $\mathbf{r}(t) = (b + a - t, d)$ for $a \leq t \leq b$.

(d) $P(x,c) - P(x,d)$

(e)

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_a^b P(x,c) - P(x,d)\,dx \text{ by parts } \textit{(a)-(c)} \\ &= \int_a^b \int_c^d -\frac{\partial P}{\partial y}(x,y)\,dy \text{ by part } \textit{(d)} \\ &= \int\int_D -\frac{\partial P}{\partial y}\,dx\,dy\end{aligned}$$

3.

In this case $Q_x = 0$, so we get the expected result from Green's theorem.