2. Tangents, Areas, Arc Lengths, and Surface Areas

Questions

3.

Whenever the curve is the graph of a function $y = f(x)$, the canonical parameterization of this curve is $x(t) = t$, $y(t) = f(x(t)) = f(t)$. Using this parameterization, we recover the given formula (in the problem) from the more general arc length equation (eqn. 2 on the worksheet).

Problems

1.

(a) This is an ellipse. Notice that

$$\left(\frac{x}{2}\right)^2 + (y)^2 = \cos^2 t + \sin^2 t = 1$$

(b)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-2\sin t}$$

Evaluating at $t = 0, \pi/4, \pi/2$, we have

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{t=0} &= \frac{1}{0} \Rightarrow \text{vertical tangent} \\ \left.\frac{dy}{dx}\right|_{t=\pi/4} &= \frac{1/\sqrt{2}}{-2/\sqrt{2}} \Rightarrow \text{slope } = -\frac{1}{2} \\ \left.\frac{dy}{dx}\right|_{t=\pi/2} &= \frac{0}{-2} \Rightarrow \text{slope } = 0 \end{aligned}$$

(c)

$$\text{Area } = \left|4 \cdot \int_0^{\pi/2} \sin t\,(-2\sin t)\,\mathrm{d}t\right| = |-2\pi| = 2\pi$$

The absolute value is inserted because area must be a non-negative number. (The integral itself is negative because we evaluated from $t = 0$ to $t = \pi/2$, which prescribes a right-to-left directionality.)

2.

$$\text{Arc Length } = \int_0^1 \sqrt{(-e^t \sin(e^t))^2 + (e^t \cos(e^t))^2}\,\mathrm{d}t = \int_0^1 e^t\,\mathrm{d}t = e - 1$$

Alternatively, we could reparameterize the curve (using $s = e^t$) as $x = \cos s$, $y = \sin s$, $1 \le s \le e$. Evaluating the Arc Length of the curve using the new parameterization gives the same answer.

3.

(a) The circle has cartesian equation $(x-2)^2 + y^2 = 1$, centered at $(2, 0)$, with radius 1.

(b) Rotating about the x-axis gives a sphere. Rotating about the y-axis gives a torus.

(c)

$$\text{Surface Area} \;=\; \int_0^{\pi} 2\pi \sin t \sqrt{(-\sin t)^2 + (\cos t)^2}\,\mathrm{d}t = 4\pi$$

We take the bounds of integration only from 0 to π because when we rotate about the x-axis, we only need to revolve the top half of the circle to get the complete sphere. Integrating from 0 to 2π (i.e. rotating the whole circle) gets you the surface twice.

(d)

$$\text{Surface Area} \;=\; \int_0^{2\pi} 2\pi(2+\cos t) \sqrt{(-\sin t)^2 + (\cos t)^2}\,\mathrm{d}t = 8\pi^2$$