Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivariable calculus course).

#18. Double Integrals in Polar Coordinates

Questions

1.

For a change in angle of $\Delta\theta$, the corresponding change in the length of the sector traced out by a circle of radius r through the angle $\Delta\theta$ is $r\Delta\theta$.

2.

The square is easier to do in rectangular coordinates (x, y) and it has area $4a^2$. The circle of radius a is easier to do in polar coordinates (r, θ) and it encloses a disc of area πa^2.

Problems

1.

The region traced out by the bounds on the double integral is the region below $x = y$ and above $y = x^2$ in the first quadrant. Putting this in polar coordinates, we have that θ goes from 0 up to $\pi/4$ and r goes from 0 out to the curve $r = f(\theta)$ where $r = f(\theta)$ corresponds to the curve $y = x^2$. So we need to write $y = x^2$ in polar coordinates. Plugging in for r and θ this gives:

$$r\sin\theta = r^2\cos^2\theta \implies r(\theta) = \sin\theta/(\cos^2\theta) = \tan\theta\sec\theta$$

So the integral becomes:

$$\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2+y^2}\; dx\; dy = \int_0^{\pi/4} \int_0^{\tan\theta\sec\theta} r^2\; dr\; d\theta$$

First we do the r integral to get

$$\frac{1}{3}\int_0^{\pi/4} (\tan\theta\sec\theta)^3\; d\theta$$

Now we divide $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ to get $\tan^2\theta + 1 = \sec^2\theta$, hence the above is:

$$\begin{aligned}\frac{1}{3}\int_0^{\pi/4} \tan\theta(\sec^2\theta - 1)(\sec\theta)^3\; d\theta &= \frac{1}{3}\int_0^{\pi/4} (\sec^4\theta - \sec^2\theta)(\sec\theta\tan\theta)\; d\theta \\ &= \frac{1}{3}\left[\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta\right]_0^{\pi/4} \\ &= \frac{2}{45}(\sqrt{2}+1)\end{aligned}$$

where we have used that $\frac{d}{d\theta}(\sec\theta) = \tan\theta\sec\theta$.

2.

Place the cone with its vertex at the origin, and opening up along the positive z-axis. The base of the cone is at height $z = h$. If we had $a = h$, then the cone would be opening up at a 45 degree angle and the equation of the cone would be $z = \sqrt{x^2 + y^2}$. We may not have $a = h$ so we have $z = k\sqrt{x^2 + y^2}$ for some constant k. To find k, look at the line of the cone when $x = 0$, so $z = ky$. This line has slope h/a (easiest to see by drawing a picture) so that $k = h/a$ and the cone has equation

$$z = \frac{h}{a}\sqrt{x^2 + y^2}$$

So to get the volume, we add up the height $h - \frac{h}{a}r$ over the polar region $0 \leq \theta \leq 2\pi$, $0 \leq r \leq a$.

$$\begin{aligned}\int_0^{2\pi} \int_0^a \left(h - \frac{h}{a}r\right) r \, dr \, d\theta &= 2\pi \left[\frac{hr^2}{2} - \frac{hr^3}{3a}\right]_0^a \\ &= 2\pi \left[\frac{ha^2}{2} - \frac{ha^2}{3}\right] \\ &= \pi a^2 \frac{h}{3}\end{aligned}$$

3.

(a) Note that $e^{-(x^2+y^2)}$ can be written as a product of a function only in x and only in y so $\int_a^b \int_a^b e^{-(x^2+y^2)} \, dx \, dy = \int_a^b e^{-y^2} \int_a^b e^{-x^2} \, dx \, dy$. The inner integral $\int_a^b e^{-x^2} \, dx$ doesn't depend on y, i.e. it is a constant, so we can pull it out in front to get $\int_a^b e^{-x^2} \, dx \int_a^b e^{-y^2} \, dy$. The variables x and y are "dummy" variables so we can use the x variable in both integrals to get $\left(\int_a^b e^{-x^2} \, dx\right)^2$.

(b) Converting to polar coordinates, the $xy-$plane can be described by $0 < r < \infty$ and $0 < \theta < 2\pi$.

$$\begin{aligned}\int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta &= \pi[-e^{-r^2}]_0^\infty \\ &= \pi\end{aligned}$$

(c) $\sqrt{\pi}$.