13. Directional Derivatives and the Gradient Vector

Questions

1.

False. The gradient of the function $f(x, y) = x^2 + y^2$ at the point $(1, 1)$ is $\langle 2, 2\rangle$.

2.

$\nabla f = 0$ at this point.

3.

First graph: Curves of constant concentration are vertical lines. Gradient vectors point horizontally to the left. No points where the gradient is zero.
Second graph: Curves of constant concentration are vertical lines. Gradient vectors point horizontally to the center. All points on the vertical line in the center have gradient zero.
Third graph: Curves of constant concentration are concentric circles. Gradient vectors point radially inward. The center of the circles is the only point with gradient zero.

Problems

1.

(a) $\nabla f = \langle 2x - y, 2y - x\rangle$. Evaluated at $(1, 3)$, $\nabla f(1, 3) = \langle -1, 5\rangle$.
(b) $\mathbf{D}_u f = \nabla f \cdot u = \langle -1, 5\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle = -3\sqrt{2}$.

2.

Let $F(x, y, z) = x^2+y^2+z^2$ and $G(x, y, z) = x^2+3y^2+2z^2$. Then, the sphere $x^2+y^2+z^2 = 6$ is a level set of F and the ellipsoid $x^2 + 3y^2 + 2z^2 = 9$ is a level set of G. We know that the gradient of a function is perpendicular to its level sets. This means that $\nabla F(2, 1, 1) = \langle 4, 2, 2\rangle$ is perpendicular to the sphere at $(2, 1, 1)$ and hence is a normal of the tangent plane. By the same reasoning, $\nabla G(2, 1, 1) = \langle 4, 6, 4\rangle$ is a normal of the tangent plane to the ellipse. To find the angle between the tangent planes, we use the dot product to find the angle between the two normal vectors:

$$\theta = \cos^{-1}\left(\frac{\langle 4, 2, 2\rangle \cdot \langle 4, 6, 4\rangle}{|\langle 4, 2, 2\rangle||\langle 4, 6, 4\rangle|}\right) \approx 27^\circ$$

3.

$\nabla f(0) = 0$. (To show this, you must use the fact that f is differentiable at the origin. This implies that the left hand limit must coincide with the right hand limit, both of which must be zero. For intuition, think about the case when $f : \mathbb{R} \to \mathbb{R}$ is an even function. The derivative at $x = 0$ must be 0 because the graph is reflected across the y axis.)