12. The Chain Rule

Questions

1.

Figure 10(a) encapsulates the information that g is a function of x and y; x and y are both functions of t. In symbols:

$$\begin{aligned} & g(x,y) \\ & x(t) \\ & y(t) \\ \Rightarrow & \frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial t}\frac{dy}{dt} \end{aligned}$$

2.

(a)
Using the chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and similarly

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

(b)
Again using the chain rule,

$$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$$

and similarly

$$\frac{dy}{dt} = \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}$$

Note: we write dv/dt instead of $\partial v/\partial t$ because v is a single variable function with respect to t. (And same for u.)

(c)
By the chain rule

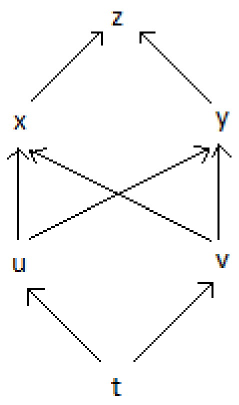
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

At this point, we can plug in the expression we found for dx/dt and dy/dt from part (b). The final answer is

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}\right) \\ &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\frac{dv}{dt} \end{aligned}$$

See Figure 1 for the dependency diagram.

Figure 1: Dependency diagram for 2(c)



Problems

1.

(a)
By the chain rule

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} \\ &= (2xe^y - y^3)(-\sin t) + (x^2e^y - 3xy^2)(\cos t) \\ &= -2\cos t \sin t\, e^{\sin t} + \sin^4 t + \cos^3 t\, e^{\sin t} - 3\cos^2 t \sin^2 t\end{aligned}$$

(where in the last step, we substituted $x = \cos t$ and $y = \sin t$.)

(b)
To find T explicitly as a function of t, we substitute in for x and y:

$$T(t) = (\cos t)^2 e^{\sin t} - (\cos t)(\sin t)^3 = \cos^2 t\, e^{\sin t} - \cos t \sin^3 t$$

Differentiating the above with respect to t, we have

$$\begin{aligned}\frac{dT}{dt} &= -2\cos t \sin t\, e^{\sin t} + \cos^2 t\, e^{\sin t} \cos t - 3\cos^2 t \sin^2 t + \sin^4 t \\ &= -2\cos t \sin t\, e^{\sin t} + \sin^4 t + \cos^3 t\, e^{\sin t} - 3\cos^2 t \sin^2 t\end{aligned}$$

Note that the answers from parts (a) and (b) match.

2.

We have a particle in $\mathbb{R}^3$ that stays on the surface $xy + yz + xz = 0$. The path of the particle is defined by some functions in terms of t, say $(x(t), y(t), z(t))$. The condition that the particle stays on the surface implies that for all t, the particle's position satisfies the equation of the surface, i.e. $x(t)y(t) + y(t)z(t) + x(t)z(t) = 0$. Let's differentiate this equation with respect to t:

$$\frac{dx}{dt}y + x\frac{dy}{dt} + \frac{dy}{dt}z + y\frac{dz}{dt} + \frac{dx}{dt}z + x\frac{dz}{dt} = 0$$

Since the particle is at $(2, -1, 2)$, we can substitute these values into the above equation. The condition that the point is moving along the surface parallel to the xz plane is the condition that $dy/dt = 0$:

$$\frac{dx}{dt}(-1) + (-1)\frac{dz}{dt} + \frac{dx}{dt}(2) + (2)\frac{dz}{dt} = 0$$

Rearranging this equation, we see that

$$\frac{dz}{dx} = \frac{\frac{dz}{dt}}{\frac{dx}{dt}} = -1$$