Selected solutions for worksheets from Math 53 (U.C. Berkeley's multivar. calc. course).

1. Curves Defined by Parametric Equations

Problems

1.

(b) Rearrange the parametric equations for x and y as

$$\cos\theta = \frac{x}{a}\,, \sin\theta = \frac{y}{b}$$

and use the identity $\cos^2\theta + \sin^2\theta = 1$ to write the cartesian equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

This describes an ellipse with semi-major axis a and semi-minor axis b.

2.

(a) Consider $x + y$ and $x - y$,

$$\begin{aligned} x + y &= 4\cos t \\ x - y &= -2\sin t \end{aligned}$$

then, rewrite the above as $\cos t = (x + y)/4$ and $\sin t = -(x - y)/2$, and use the identity $\cos^2 t + \sin^2 t = 1$ to write the cartesian equation

$$\left(\frac{x+y}{4}\right)^2 + \left(\frac{x-y}{2}\right)^2 = 1$$

(b) Substitute $x = u + v$ and $y = u - v$ into the cartesian equation from (a):

$$\begin{aligned} &\left(\frac{2u}{4}\right)^2 + \left(\frac{2v}{2}\right)^2 = 1 \\ \Rightarrow &\left(\frac{u}{2}\right)^2 + \left(\frac{v}{1}\right)^2 = 1 \end{aligned}$$

This is the equation of an ellipse.

3.

(a) Consider

$$\begin{aligned} x + y &= t + \frac{1}{t} + t - \frac{1}{t} = 2t \\ x - y &= t + \frac{1}{t} - \left(t - \frac{1}{t}\right) = \frac{2}{t} \end{aligned}$$

Therefore,

$$(x+y)(x-y) = x^2 - y^2 = 4$$

This is a hyperbola, with branches opening along the x-axis.

(b) $t \in (-\infty, 0)$ gives the left branch of the hyperbola; $t \in (0, \infty)$ gives the right branch of the hyperbola. To see this, notice that when $t < 0$, $x < 0$, and when $t > 0$, $x > 0$. The parameterization is undefined for $t = 0$.

(c) D traces out only the right branch of the hyperbola, because for any value of t, $x > 0$.

Additional Problems

1.

The problem specifies that the helix passes through $(0,0,1)$ and $(1,0,1)$. One possible parameterization is

$$\begin{aligned} x(t) &= \frac{t}{2\pi} \\ y(t) &= \sin t \\ z(t) &= \cos t \end{aligned}$$

which passes through $(0,0,1)$ at $t = 0$, and passes through $(1,0,1)$ at $t = 2\pi$.