**Problem 26, §16.7**

We're trying to integrate $\mathbf{F}(x, y, z) = (xz, x, y)$ over $S$, the hemisphere of radius 5 centered at the origin in the positive $y$ half of $\mathbb{R}^3$, oriented in the positive $y$ direction.

In my 8 AM section, I tried to use $y$ and $\theta$ (in the $xz$ plane) in section. But this doesn't work, since $r$ depends on $y$. $S$ is a sphere of radius 5, so we have the following relationship between $r$ (in the $xz$ plane) and $y$:

$$(x^2 + z^2) + y^2 = 25 \Leftrightarrow r^2 + y^2 = 25 \Leftrightarrow r = \sqrt{25 - y^2}$$

therefore the parameterization I should have used is

$$\mathbf{r}(y, \theta) = \left(\sqrt{25 - y^2}\cos\theta, y, \sqrt{25 - y^2}\sin\theta\right)$$

but this is kind of nasty to calculate derivatives and cross products for. We use $\mathbf{r}_y \times \mathbf{r}_\theta$ because it gives the correct orientation. We'd get

$$\mathbf{r}_y \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{y}{\sqrt{25-y^2}}\cos\theta & 1 & -\frac{y}{\sqrt{25-y^2}}\sin\theta \\ -\sqrt{25-y^2}\sin\theta & 0 & \sqrt{25-y^2}\cos\theta \end{vmatrix} = \left(\sqrt{25 - y^2}\cos\theta, y, \sqrt{25 - y^2}\sin\theta\right)$$

giving us a full integral of

$$\begin{aligned}\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^5 \left((25 - y^2)\cos\theta\sin\theta, \sqrt{25 - y^2}\cos\theta, y\right) \\ &\qquad \cdot \left(\sqrt{25 - y^2}\cos\theta, y, \sqrt{25 - y^2}\sin\theta\right)\, dyd\theta \\ &= \int_0^{2\pi}\int_0^5 (25 - y^2)^{\frac{3}{2}}\cos^2\theta\sin\theta + \sqrt{25 - y^2}y\cos\theta + \sqrt{25 - y^2}y\sin\theta\, dyd\theta \\ &= 0\end{aligned}$$

because every term contains a factor of a trig function, cosine or sine, integrated from zero to $2\pi$, giving zero.

I could have done the problem that way. However, I think when spheres are involved, choosing a parameterization based on spherical coordinates might be a better idea. While it may be difficult to do all the trig derivatives and manipulations, it is easier to be confident that your parameterization is correct. In this case, $\rho = 5$, so we can use

$$\mathbf{r}(\phi, \theta) = (5\sin\phi\cos\theta, 5\sin\phi\sin\theta, 5\cos\phi)$$

We use $\mathbf{r}_\phi \times \mathbf{r}_\theta$ because it gives the correct orientation. The cross product term is

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5\cos\phi\cos\theta & 5\cos\phi\sin\theta & -5\sin\phi \\ -5\sin\phi\cos\theta & 5\sin\phi\cos\theta & 0 \end{pmatrix} = \left(25\sin^2\phi\cos\theta, -25\sin^2\phi\cos\theta, 25\sin\phi\cos\theta\right)$$

giving us a full integral of

$$\begin{aligned}\iint_S \mathbf{F}(x,y,z)\cdot d\mathbf{S} &= \int_0^\pi \int_0^\pi (25\sin\phi\cos\phi\cos\theta, 5\sin\phi\cos\theta, 5\sin\phi\sin\theta) \\ &\qquad\qquad \cdot \left(25\sin^2\phi\cos\theta, -25\sin^2\phi\cos\theta, 25\sin\phi\cos\theta\right)\, d\phi d\theta \\ &= \int_0^\pi \int_0^\pi 25\sin^3\phi\cos\phi\cos^2\theta - 125\sin^3\phi\cos^2\theta + 125\sin^2\phi\sin\theta\cos\theta\, d\phi d\theta \\ &= 0\end{aligned}$$

because every term integrates to something with a factor of sine, which, when evaluated at zero and $\pi$ gives zero.

There is another way to solve this problem, using symmetries. Note that

$$\mathbf{F}(x,y,z) = (xz, x, y) \text{ and } \mathbf{n}(x,y,z) = \frac{(x,y,z)}{5} \text{ so } \mathbf{F}(x,y,z)\cdot\mathbf{n}(x,y,z) = \frac{(x^2z, xy, yz)}{5}$$

where $\mathbf{n}$ denotes the unit normal to $S$. Therefore

$$\mathbf{F}(-x,y,-z) = (xz, -x, y) \text{ and } \mathbf{n}(-x,y,-z) = \frac{(-x,y,-z)}{5}$$

$$\text{so } \mathbf{F}(-x,y,-z)\cdot\mathbf{n}(-x,y,-z) = \frac{(-x^2z, -xy, -yz)}{5} = -\mathbf{F}(x,y,z)\cdot\mathbf{n}(x,y,z)$$

Since sending $(x,y,z)$ to $(-x,y,-z)$ consists of a rotation by $\pi$ in the $xz$ plane, and $S$ is symmetric under rotations, $(x,y,z)$ and $(-x,y,-z)$ occur in canceling pairs, so the integral will be zero.