Limit Laws

Let $f(x)$ be a function and a a number. We say that the number L is the **limit of $f(x)$ as x approaches** a when $f(x)$ can be made arbitrarily close to L for all x sufficiently close (but not equal) to a. In this case, we write

$$\lim_{x\to a} f(x) = L$$

Rephrased, this means that I can make $|f(x) - L|$ as small as I want by choosing an x very close to a. This is because "$f(x)$ is close to L" can be formalized by saying "$|f(x) - L|$ is small."

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, the following are true.

I) If k is a number, $\lim_{x\to a} k \cdot f(x) = k \cdot \lim_{x\to a} f(x)$.

Understanding: If $\lim_{x\to a} f(x) = L$, then I can make $|f(x) - L|$ as small as I want by looking only at x very close to a. I claim that I can also make $|k \cdot f(x) - k \cdot L|$ as small as I want. This is possible because

$$|k \cdot f(x) - k \cdot L| = k \cdot |f(x) - L|$$

Therefore if I want to make $|k \cdot f(x) - k \cdot L|$ smaller than, for example, 2, I can do this by looking at x so close to a that $|f(x) - L|$ smaller than $\frac{2}{k}$, because then I get

$$|k \cdot f(x) - k \cdot L| = k \cdot |f(x) - L| < k \cdot \frac{2}{k} = 2$$

II) If r is a positive number, and $f(x)^r$ is defined for $x \neq a$, $\lim_{x\to a}[f(x)]^r = [\lim_{x\to a} f(x)]^r$.

Understanding: Unfortunately, this requires some more advanced math.

III) $\lim_{x\to a}[f(x) + g(x)] = [\lim_{x\to a} f(x)] + [\lim_{x\to a} g(x)]$.

Understanding: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

Therefore if I want to make $|f(x) + g(x) - (L + M)|$ smaller than, for example, 2, I can do this by looking at x so close to a that both $|f(x) - L|$ and $|g(x) - L|$ smaller than 1, because then I get

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M| = 1 + 1 = 2$$

Note: I did not justify why $|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$. This requires some thinking. If you want to try to understand this yourself, first start with this sub-problem: if a and b are numbers, why is $|a + b| \leq |a| + |b|$? Hint: do it by cases (a and b both positive, both negative, and one positive, the other negative.) You could also research "triangle inequality."

IV) $\lim_{x\to a}[f(x) - g(x)] = [\lim_{x\to a} f(x)] - [\lim_{x\to a} g(x)]$.

Understanding: You can understand this using rules I and III; think about how.

V) $\lim_{x\to a}[f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$

Understanding: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\ &= |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| \end{aligned}$$

For example, say we want to get $|f(x)g(x) - LM| < 2$. It is possible to look only at x close enough to a to make $|g(x) - M|$ very small; say we make it smaller than $\frac{1}{|L|}$. In that case, $|g(x)|$ can be at most $\frac{1}{|L|} + |M|$. I can also look only at x close enough to a to make $|f(x) - L|$ smaller than $\frac{1}{\frac{1}{|L|}+|M|}$. Putting it all together, we get

$$\begin{aligned} |f(x)g(x) - LM| &\leq |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M| \\ &< \left(\frac{1}{|L|} + |M|\right)\left(\frac{1}{\frac{1}{|L|} + |M|}\right) + |L| \cdot \frac{1}{|L|} \\ &= 1 + 1 = 2 \end{aligned}$$

Note that in the step with the $\leq$ I again used the fact that $|a + b| \leq |a| + |b|$, which I used for Limit Law III.

VI) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

Understanding: You can understand this using rules II and V; think about how.

VII) **Limit of a Polynomial** If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

Understanding: You can understand this using rules I, II, and III, and the fact that $\lim_{x\to a} x = a$; think about how.

VIII) **Limit of a Rational Function** If $p(x)$ and $q(x)$ are polynomials, and $q(a) \neq 0$, then $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$.

Understanding: You can understand this using rules VI and VII; think about how.

Differentiation Rules

If a function is differentiable at $x = a$, then it is continuous at $x = a$. If a function is not continuous at $x = a$, then it is not differentiable at $x = a$. We say a function is continuous if it is continuous for all values of x; similarly, we say function is differentiable if it is differentiable for all values of x.

1) Constant-multiple rule: if k is a number, $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx} f(x) = k \cdot f'(x)$.

Understanding: Use the three-step method on $h(x) = k \cdot f(x)$.

2) Sum rule: $\frac{d}{dx}[f(x) + g(x)] = \left[\frac{d}{dx} f(x)\right] + \left[\frac{d}{dx} g(x)\right] = f'(x) + g'(x)$.

Understanding: Use the three-step method on $h(x) = f(x) + g(x)$.

3) General power rule: if r is any number, $\frac{d}{dx} f(x)^r = r \cdot f(x)^{r-1} \cdot \frac{d}{dx} f(x) = r \cdot f(x)^{r-1} \cdot f'(x)$.

Understanding: What is the derivative? It is the limit of $\frac{\text{change in } f(x)}{\text{change in } x}$ as the size of your change in x gets very small. We can think of the generalized power rule as a nest of functions: inside, some $f(x)$, and outside, some function $g(y) = y^r$ for some number r. Then when I want to think about $f(x)^r$, I can instead think about the function $g(f(x))$. The derivative of $g(f(x))$ the limit of $\frac{\text{change in } g(f(x))}{\text{change in } x}$ as the size of your change in x gets very small.

However, I an add another factor:

$$\frac{\text{change in } g(f(x))}{\text{change in } x} = \frac{\text{change in } g(f(x))}{\text{change in } f(x)} \cdot \frac{\text{change in } f(x)}{\text{change in } x}$$

because you can think of them as equal by canceling the "change in $f(x)$" parts.