TWO DERIVATIONS OF THE DERIVATIVE (SLOPE FUNCTION) OF $y = x^2$

MORGAN WEILER, GSI OF SECTIONS 107, 111, 115

I don't expect the rest of the class to focus on derivations like this one, however it is interesting, and helps your intuition with derivatives. Feel free to ask me questions about this write-up in office hours.

1. My Way

We are trying to look for a general formula, in terms of x, for the slope of the line tangent to the graph of $y = x^2$. For each input x, we will get as an output the slope of the graph at the point (x, x^2).

So let's think about the equation of the line tangent to the graph of $y = x^2$ at the point (a, a^2). (Remember that there is going to be one unique and distinct tangent line for each number a.) In point-slope form, it will look like $y = mx + b$. The requirement that it is tangent to the graph of $y = x^2$ implies that the point (a, a^2) must be on the tangent line, that is,

$$a^2 = ma + b \tag{1}$$

There is another characteristic of the tangent line we must use. As you saw in lecture and section, this tangent line is below the graph of the function at all points, except at (a, a^2). That is, for all nonzero numbers h (they could be positive or negative),

$$(a + h)^2 > m(a + h) + b \tag{2}$$

The left hand side is the value of x^2 at $a + h$ and the right hand side is the value of $mx + b$ at $a + h$. Since the line $mx + b$ is below the curve x^2 for all x except $x = a$, we get (2).

I suggest you draw a picture of the graph of $y = x^2$ and a few sample tangent lines (sections 107 and 111 saw this in class on Tuesday) to understand what is going on.

Solving (2):

$$\begin{aligned}
(a + h)^2 &> m(a + h) + b \\
a^2 + 2h + h^2 &> ma + mh + b \\
a^2 + 2h + h^2 &> (ma + b) + mh \\
a^2 + 2ah + h^2 &> a^2 + mh \qquad\qquad \text{using (1)} \\
2ah + h^2 &> mh \\
h(2a + h) &> mh
\end{aligned}$$

Therefore, if $h > 0$,

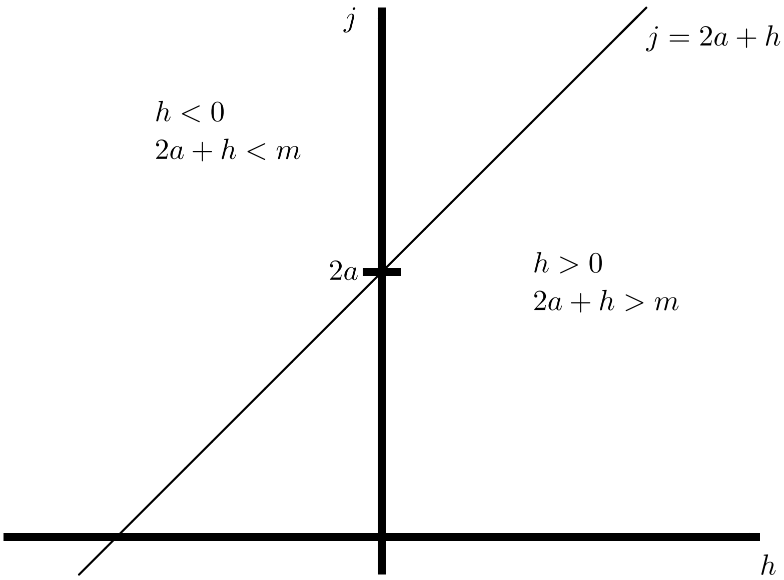
$$2a + h > m$$

and if $h < 0$,

$$2a + h < m$$

because dividing by a negative reverses the inequality.

Since these inequalities hold for all $h \neq 0$, m can only equal 2a. If you'd like a visual (on an h-j coordinate system, instead of x-y, since x and y are being used for different things):



This is intended to show you the areas where m can be located relative to $2a + h$. m depends on a, but not on h; we use h to "squeeze" m close to $2a$ in this proof (remember: m is the slope at a). When $h > 0$ (right of the j axis), m must be below the line $j = 2a + h$. However when $h < 0$ (left of the j axis), m must be above the line $j = 2a + h$.

Since m does not depend on h, there must be some value which satisfies both of those inequalities. The only possible one is $m = 2a$, and on the graph, that is the only j value it is possible to pick which is both lower than any j value on the right-hand (positive) side of the graph of $j = 2a + h$ and higher than any j value on the left-hand (negative h) side of the graph of $j = 2a + h$.

So the slope of the graph of $y = x^2$ at (a, a^2) is $2a$.

2. Professor's Way - My Explanation

This is me re-explaining what the professor did in class. I'm not claiming this is exactly what he said or meant, I'm just trying to give my perspective on his method, which is pretty different from my way above. So if he says anything different, you ought to listen to him.

We are trying to look for a general formula, in terms of x, for the slope of the line tangent to the graph of $y = x^2$. For each input x, we will get as an output the slope of the graph at the point (x, x^2).

Say we want the slope of the graph at the point (a, a^2). Using point-slope form, the tangent line then has the equation

$$y - a^2 = m(x - a) \tag{3}$$

for some slope m which depends on a. This simplifies to

$$y = mx - ma + a^2$$
$$y = mx + (-ma + a^2)$$

Remember the motivation for property (2) in my alternate method, above? We know that the graph of the function $y = x^2$ is always above or equal to the tangent line at (a, a^2), which we have just calculated to have the equation $y = mx + (-ma + a^2)$. So let's solve for the m for which this is *not* true, and eliminate them.

Again, we're trying to find the values of m for which $y = x^2$ is *less than* $y = mx + (-ma + a^2)$. So we set them unequal and solve:

$$
\begin{aligned}
x^2 &< mx + (-ma + a^2) \\
x^2 - mx &< -ma + a^2 \\
x^2 - mx + \left(\frac{m}{2}\right)^2 &< -ma + a^2 + \left(\frac{m}{2}\right)^2 && \text{completing the square} \\
\left(x - \frac{m}{2}\right)^2 &< \left(a - \frac{m}{2}\right)^2 \\
\frac{1}{4}(2x - m)^2 &< \frac{1}{4}(2a - m)^2 \\
(2x - m)^2 &< (2a - m)^2
\end{aligned}
$$

What does it mean for the graph of $y = x^2$ to be below the graph of $y = mx + (-ma + a^2)$? That means that there is some x we can plug in to x^2 and to $mx + (-ma + a^2)$ so that $x^2 < mx + (-ma + a^2)$. We just simplified that to $(2x - m)^2 < (2a - m)^2$.

So when is it possible to plug in some x so that $(2x - m)^2 < (2a - m)^2$? Whenever $2a - m > 0$ or < 0, because then its square will be strictly greater than zero, and we can find some x so that $(2x - m)^2$ is smaller (by picking x really close to m). Therefore if $m \neq 2a$, we contradict the fact that the tangent line is at or below the graph of $y = x^2$ for all x.

So it must be that $m = 2a$. In other words, the tangent line to the graph of $y = x^2$ at (a, a^2) has slope $2a$.