Review: Secant Definition of the Derivative

The **derivative** of a function $f(x)$ at a is called $f'(a)$ and is the slope of the graph of $f(x)$ at $x = a$. This is also the slope of the tangent line to the graph of $f(x)$ at $(a, f(a))$.

The **three step method** for calculating $f'(x)$ is as follows. Example: $f(x) = x^2 + x$, $x = 2$.
1) Write $\frac{f(x+h)-f(x)}{h}$ using your function f. Example: $\frac{[(x+h)^2+(x+h)]-[x^2+x]}{h}$.
2) Simplify. Example: $\frac{f(x+h)-f(x)}{h} = 2x + h + 1$.
3) Take the limit as h approaches zero. This will give you $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Example: $2x + 1$.

Problems

(1) Use the three-step method for calculating $f'(x)$, where $f(x) = 3x^2 + x + 5$. Draw the associated secant line picture.
(2) Use the three-step method for calculating $f'(x)$, where $f(x) = 2x + \frac{1}{x^2}$. Draw the associated secant line picture.

1.4: Limit Rules

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, the following are true.

I) If k is a number, $\lim_{x\to a} k \cdot f(x) = k \cdot \lim_{x\to a} f(x)$.

II) If r is a positive number, and $f(x)^r$ is defined for $x \neq a$ (counterexample: $f(x) = -x^2$ and $r = \frac{1}{2}$ does NOT work), $\lim_{x\to a}[f(x)]^r = [\lim_{x\to a} f(x)]^r$.

III) $\lim_{x\to a}[f(x) + g(x)] = [\lim_{x\to a} f(x)] + [\lim_{x\to a} g(x)]$.

IV) $\lim_{x\to a}[f(x) - g(x)] = [\lim_{x\to a} f(x)] - [\lim_{x\to a} g(x)]$. This follows from I and III; how?

V) $\lim_{x\to a}[f(x) \cdot g(x)] = [\lim_{x\to a} f(x)] \cdot [\lim_{x\to a} g(x)]$

VI) If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$. (Counterexample: $f(x) = x + 5$, $g(x) = x + 3$, and $a = -3$ does NOT work.)

VII) **Limit of a Polynomial** If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

VIII) **Limit of a Rational Function** If $p(x)$ and $q(x)$ are polynomials, and $q(a) \neq 0$, then $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$.

Problems

(1) Come up with examples for the above limit rules: I, III, VI, VII. An example consists of an appropriate $f(x), g(x)$, and x, and possibly a k or r or $p(x)$ or $q(x)$ if necessary.

(2) Come up with examples for the above limit rules: II, IV, V, VIII. An example consists of an appropriate $f(x), g(x)$, and x, and possibly a k or r or $p(x)$ or $q(x)$ if necessary.

(3) Compute the following limits:
   (a) $\lim_{x\to 3} \frac{x^2+6x+9}{x+3}$
   (b) $\lim_{x\to 3} 1-(x+1)^{\frac{1}{2}}$

(4) Compute the following limits:
   (a) $\lim_{x\to 0} \frac{(x^2-4)(2x+1)}{x+2}$
   (b) $\lim_{x\to 0} 2x+\frac{1}{x}$

1.5: Continuity and Differentiability

A function $f(x)$ is **continuous at *a*** if $f(a) = \lim_{x\to a} f(x)$.

A function $f(x)$ is **differentiable at *a*** if the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

If $f(x)$ is differentiable at a then $f(x)$ is continuous at a. If $f(x)$ is not continuous at a then $f(x)$ is not differentiable at a.

Problems

(1) $f(x) = \begin{cases} \sqrt{x-3} \text{ when } x \geq 3 \\ x-3 \text{ when } x < 3 \end{cases}$ Is f continuous at $x = 3$? Is f differentiable at $x = 3$?

(2) $f(x) = \begin{cases} (x-1)^3 \text{ when } x \leq 0 \\ (x-1)^2 \text{ when } x > 0 \end{cases}$. Is f continuous at $x = 1$? Is f differentiable at $x = 1$?

(3) $f(x) = \frac{x^2(x+2)}{x^2-4}$. Is f continuous at $x = -2$? If not, how could you define f at $x = -2$ in a way that makes f continuous?

1.6: Differentiation Rules

Constant-multiple rule: if k is a number, $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx} f(x)$.

Sum rule: $\frac{d}{dx}[f(x) + g(x)] = \left[\frac{d}{dx} f(x)\right] + \left[\frac{d}{dx} g(x)\right]$.

General power rule: if r is any number, $\frac{d}{dx} f(x)^r = r \cdot f(x)^{r-1} \cdot \frac{d}{dx} f(x)$.

Problems

(1) Come up with examples for all of the above differentiation rules.

(2) Differentiate $f(x) = 4x^{\frac{1}{2}} + (x^3-2)^2$ *without* using difference quotients. Show which rules you use at each step.

(3) Differentiate $f(x) = \sqrt[3]{(2x+3)^2 + x}$ *without* using difference quotients. Show which rules you use at each step.