# Periodic orbits of area-preserving annulus diffeomorphisms with bounded mean action

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# Goal

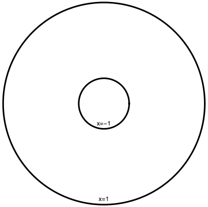
$\psi$ is an area-preserving diffeomorphism of an annulus.

There is a natural "action" function $f$ measuring the area-preserving geometry of $\psi$.

If $\psi^\ell(p) = p$, then $p$ is **periodic**. Its **periodic orbit** is all $\psi^n(p)$.

We show that in most cases, $\psi$ has periodic orbits $\gamma$ for which the average of $f$ over $\gamma \leq$ the average of $f$ over the entire annulus.

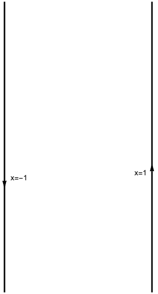
# Specifying $\psi$




$A = [-1, 1] \times (\mathbb{R}/2\pi\mathbb{Z})$ with coordinates $(x, y)$.

$\omega = \frac{1}{2\pi}\, dx \wedge dy$, an area form on $A$.

$\psi$ is an area-preserving diffeomorphism of $(A, \omega)$ which preserves boundary components as sets.




Let $\tilde{\psi}$ be a lift of $\psi$ to $\tilde{A} := [-1, 1] \times \mathbb{R}$. Assume

- near $x = 1$, $\tilde{\psi}(x, y) = (x, y + 2\pi y_+)$
- near $x = -1$, $\tilde{\psi}(x, y) = (x, y + 2\pi y_-)$

$y_+, y_-$, and the choice of lift $\tilde{\psi}$ are all equivalent.

# Periodic points and their geometry

The classical context:

- 1913, Poincaré-Birkhoff: $\psi$ with $y_+ > 0$ and $y_- < 0$ has at least two fixed points.
- 1996, Franks: Any area-preserving homeomorphism of the annulus with at least one periodic point must have infinitely many interior periodic points.

**Note:** the area-preserving assumption is necessary. There are simple counterexamples.

**Question:** Although the geometry of $\psi$ is crucial, these results do not tell us anything about it. What can we say?

# The action function

Let $\beta = \frac{x}{2\pi}\, dy$. Note $d\beta = \omega$.

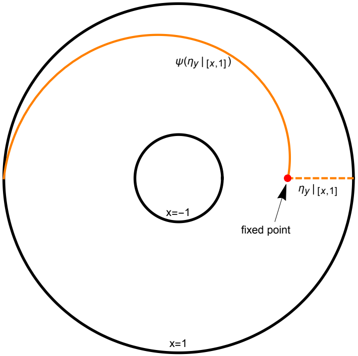
### Definition

*The* **action function** $f : A \to \mathbb{R}$ *is the unique function for which*

- $df = \psi^* \beta - \beta$
- $f(1, y) = y_+$

Throughout, let $\eta_y : [-1, 1] \to A$ be the path $s \mapsto (s, y)$.

Figure: If $\psi(x, y) = (x, y)$, $f(x, y)$ is the area of the wedge determined by $\eta_y|_{[x,1]}$, $\psi(\eta_y|_{[x,1]})$, and the outer boundary of $A$, up to an integer.



# Flux

**Flux** is a number associated to $\psi$.
It's the signed area between the arcs $\tilde{\eta}_0$ and $\tilde{\psi}(\tilde{\eta}_0)$ in $\tilde{A}$. Denote it by $F$.

We lift to $\tilde{A}$ because flux should differentiate between rotation by 0 and by $2\pi$, e.g.

Relationship to action function: $f(-1, y) = -y_- + F$.

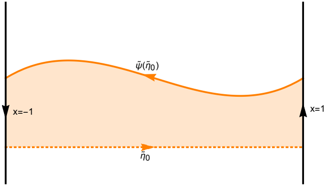



Figure: Part of $\tilde{A}$.

### Definition

*A* **periodic orbit of** $\psi$ *is a tuple* $\gamma = (\gamma_1, \ldots, \gamma_l)$ *of points* $\gamma_i \in A$ *for which* $\gamma_{i+1 \mod l} = \psi(\gamma_i)$.

- *The* **total action** *of a periodic orbit is* $\mathcal{A}(\gamma) = \sum_{i=1}^{l} f(\gamma_i)$.
- *Let* $\ell(\gamma) = l$, *its period. Its* **mean action** *is* $\frac{\mathcal{A}(\gamma)}{\ell(\gamma)} = \frac{\sum_{i=1}^{l} f(\gamma_i)}{l}$.

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## Definition

*The* **Calabi invariant** *of* $\tilde{\psi}$ *is the number*

$$\mathcal{V}(\tilde{\psi}) := \frac{\int_A f\omega}{\int_A \omega} = \frac{1}{2}\int_A f\omega$$

$\mathcal{V}(\tilde{\psi})$ is the average action of $\psi$. When $\psi$ is a rotation, it equals the rotation number.

# Examples

The dotted orange curves are $\eta_0$, and the solid orange curves are $\psi(\eta_0)$. The blue points lie on periodic orbits.

Figure: $\tilde{\psi}(x,y) = \left(x, y + 2\pi\frac{x}{2}\right)$, $\mathcal{V}(\tilde{\psi}) = \frac{1}{3} < y_+, -y_- + F$, each point on blue dotted circle is fixed with action $\frac{1}{4} < \mathcal{V}(\tilde{\psi})$

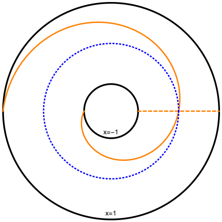
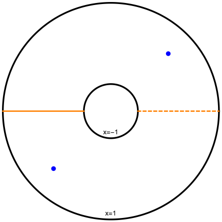



Figure: $\tilde{\psi}(x,y) = \left(x, y + 2\pi\frac{1}{2}\right)$, $\mathcal{V}(\tilde{\psi}) = \frac{1}{2} = y_+, -y_- + F$, each point is part of a two-point orbit with mean action $\frac{1}{2} = \mathcal{V}(\tilde{\psi})$



# Main theorem

Let $\mathcal{P}(\psi)$ denote the set of periodic orbits of $\psi$.

## Theorem (W.)

*Let $\psi$ be an area-preserving diffeomorphism of $(A, \omega)$ which is a rotation by $y_\pm$ near the $x = \pm 1$ boundary. Assume*

$$\mathcal{V}(\tilde{\psi}) < \max\{y_+, -y_- + F\}$$

*or that one of $y_+$ or $y_-$ is rational. Then*

$$\inf\left\{\frac{\mathcal{A}(\gamma)}{\ell(\gamma)}\middle|\gamma \in \mathcal{P}(\psi)\right\} \le \mathcal{V}(\tilde{\psi}) \tag{1}$$

That is, there exists a periodic orbit $\gamma$ of $\psi$ for which the average of $f$ over $\gamma$ is less than or equal to the average of $f$ over the annulus. Hutchings ('16) has proved an analogous theorem for the disk.

# Relationship to contact geometry

**Fundamental example:** $\phi$ a diffeomorphism of a surface $\Sigma$, $M_\phi$ its the mapping torus: $[0,1] \times \Sigma/(1,p) \sim (0,\phi(p))$, coordinates $(\theta, p)$. The flow of $\partial_\theta$ sends $\{0\} \times \Sigma$ to itself.

$$\{\text{closed orbits of flow}\} \overset{1-1}{\leftrightarrow} \{\text{periodic orbits of } \phi\}$$

Contact geometry has excellent tools to study closed orbits of "Reeb" vector fields.

**Strategy:** Realize $A$ as a surface in a 3-manifold which a Reeb vector field sends to itself. Use contact geometry to analyze the closed orbits.

This was Poincaré's motivation for studying annulus diffeomorphisms!

# Review of definitions in contact geometry

**Definition**

*Let $Y$ be a closed oriented three-manifold. A* **contact form** *on $Y$ is a 1-form with $\lambda \wedge d\lambda > 0$.*
*Its* **contact structure** *is $\xi = \ker \lambda \subset TY$.*

Note that contact structures are $\mathbb{R}^2$-bundles because $\lambda \wedge d\lambda > 0$.

**Definition**

*The* **Reeb vector field** *$R$ of $\lambda$ is uniquely determined by $d\lambda(R, \cdot) = 0$, $\lambda(R) = 1$.*
*A* **Reeb orbit** *is a smooth map $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$ for $T > 0$, modulo reparameterization, for which $\dot{\gamma}(s) = R(\gamma(s))$.*

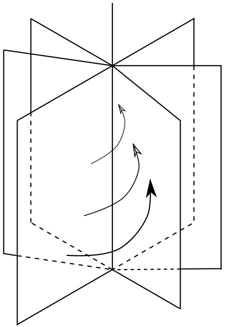


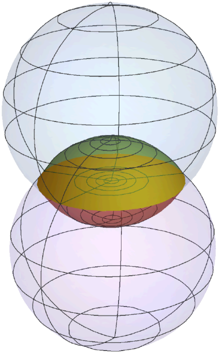
Figure: Near the binding; also, standard OBD on $\mathbb{R}^3$.

## Definition

*An* **open book decomposition** *is a type of fibration by surfaces (***pages***) of a closed oriented 3-manifold minus some oriented 1-manifold (***binding***).*

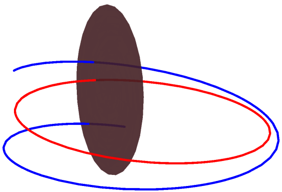
*An open book decomposition and contact form are* **adapted** *to one another if $R$ is transverse to the pages and tangent to $B$.*

Figure: Standard OBD on $S^3$ with disk pages.



$\Pi$, $\lambda$ adapted to one another $\Rightarrow$ Reeb flow determines an area-preserving diffeomorphism on each page.

- Defined at $p \in \Pi^{-1}(\theta)$ by flowing $p$ until it returns to $\Pi^{-1}(\theta)$ for the first time.
- Called the **Poincaré return map** of the pair $(\lambda, \Pi)$.
- The time each point is flowed is its **return time**.



# Realizing the annulus in a contact manifold

## Proposition (W.)

*Under natural technical assumptions*[1] *on* $\psi$*, there is a contact form* $\lambda_{\tilde{\psi}}$ *on a three-manifold* $Y_{\tilde{\psi}}$ *for which*

(a) *An OBD of* $Y_{\tilde{\psi}}$ *with annulus pages is adapted to* $\lambda_{\tilde{\psi}}$*. There is a page* $A_0$ *on which the return time is* $f$ *and the Poincaré return map is* $\psi$*.*

(b) $\{\textit{Reeb orbits of } \lambda_{\tilde{\psi}}\} \overset{1-1}{\leftrightarrow} \mathcal{P}(\psi) \cup \{\textit{binding}\}$

(c) *The bijection in (b) sends* $\mathcal{A}(\gamma), \ell(\gamma)$*, and* $\mathcal{V}(\tilde{\psi})$ *to well-studied contact-geometric quantities.*

[1] ask me about it later!

# Upper bound on total action

We need to find for all $\epsilon$ some periodic orbit $\gamma$ for which

$$\frac{\mathcal{A}(\gamma)}{\ell(\gamma)} \leq \mathcal{V}(\tilde{\psi}) + \epsilon$$

Cristofaro-Gardiner-Hutchings-Ramos ('15) give us a Reeb orbit $\alpha$ of $\lambda_{\tilde{\psi}}$, which is not in the binding, whose action has an upper bound in terms of $\mathcal{V}(\tilde{\psi})$ and some extra terms.

# Lower bound on $\alpha \cdot A_0$:

We use a tool called **knot-filtered embedded contact homology**, which measures $\alpha \cdot A_0$. Computing it gives us the lower bound for $\ell(\gamma_\alpha)$.

Knot-filtered ECH is an invariant of quadruples

$$(Y, \xi, B, \mathrm{rot}(B))$$

where $Y$ is a rational homology sphere, $B$ is an orbit of a contact form $\lambda$ with $\xi = \ker \lambda$, and $\mathrm{rot}(B)$ measures the "twisting" of the Reeb vector field near $B$.

# But it's HARD to compute knot-filtered ECH!

Thankfully, we can simplify:

**Theorem (Giroux ('00))**

*Let $Y$ be a closed oriented three-manifold with contact forms $\lambda$, $\lambda'$. If they are both adapted to the same open book decomposition, then there is a diffeomorphism $(Y, \ker\lambda) \xrightarrow{\cong} (Y, \ker\lambda')$.*

With the right form, the computation is combinatorial. With enough elbow grease[2], these two bounds prove the theorem.

[2] Firstly, the bound isn't quite right, and secondly, we have to remove the technical assumptions!

# Future directions

1. Expand $A$ and $D^2$ to other surfaces. Significantly harder!
2. Use "rational OBDs"; maybe this will make (1) easier?
3. Explore further applications and structural properties of the ECH tools used for the bound.

Thank you!

# References

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