MATH 1A SECTION: OCTOBER 25, 2013

Moor, the former dragon ecologist, continues wandering the world. He encounters some calculus problems along the way.

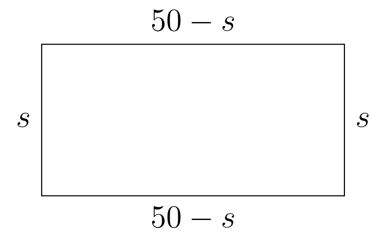
1. Moor discovers a new colony of dragons, and he wants to fence off some land so that the dragons can be protected from the dragon eradicators. He only has 100 km of fencing, however. How does he protect the dragons most optimally? Help him do this by solving a calculus problem:
   Find the dimensions of a rectangle with perimeter 100 whose area is as large as possible.

2. The dragons are angry at Moor for being responsible for the dragon eradication program. A dragon is flying around Moor in an elliptical orbit. How doomed is Moor?
   Find the points on the ellipse $4x^2 + y^2 = 4$ that are closest to the point $(0, 1)$.

3. Moor decides that he needs a new career. He gets a job mooring boats. Unfortunately, mooring boats involves calculus, and Moor isn't so good at calculus. Can you help?
   A boat leaves a dock at 2pm and travels due south at a speed of 20 km/hr. Another boat has been heading due east at 15 km/hr and reaches the same dock at 3pm. At what time were the boats closest together?

4. Mooring boats is too hard. Moor has decided to go back to his dragon colony. While there, he is studying dragon eggs. Unlike normal eggs, dragon eggs are perfectly spherical, with cylindrical yolks. Healthier dragon eggs have bigger egg yolks, and Moor is trying to create an optimized dragon egg. How should he do this?
   A right circular cylinder is inscribed in a sphere with radius r. Find the largest possible volume of such a cylinder.

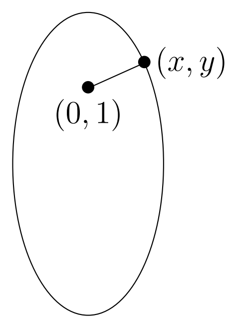
1. Draw a rectangle. Let one of the sides be s. In order to make the perimeter be 100, the other side needs to be $50 - s$; the perimeter is then $s + (50 - s) + s + (50 - s) = 100$.



The area of this rectangle is $s(50-s) = 50s - s^2$, and we need to find the maximum value of this function.

Let $f(s) = 50s - s^2$ be the area function. Then $f'(s) = 50 - 2s$. Solving $f'(s) = 0$ gives $50 - 2s = 0$, so $2s = 50$ and hence $s = 25$. This is the critical number for this function. Note that $f''(s) = -2 < 0$, so this critical number does indeed correspond to a maximum. Therefore, our rectangle has maximum area exactly when it has all sides of length 25, i.e. when it is a square.

2. Draw a picture as follows:



Consider an arbitrary point (x, y) on the ellipse. We are interested in its distance to the point $(0, 1)$. To find this, we use the distance formula. This gives

$$d = \sqrt{(x-0)^2 + (y-1)^2}.$$

We need to minimize this quantity, but it's in terms of both x and y, which is not so good. But since (x, y) is on the ellipse, we also know that $4x^2 + y^2 = 4$, so therefore $4x^2 = 4 - y^2$ and hence $x^2 = 1 - \frac{1}{4}y^2$. So we plug this in:

$$d = \sqrt{x^2 + (y-1)^2} = \sqrt{1 - \frac{1}{4}y^2 + (y-1)^2}.$$

We should now minimize this quantity. To simply the calculation, note that d is minimized exactly when d^2 is also minimized, so it suffices to minimize d^2 and drop the square root.

Now, let

$$f(y) = 1 - \frac{1}{4}y^2 + (y-1)^2.$$

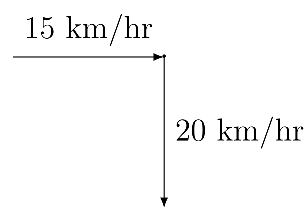
This is the thing that we need to minimize. Differentiate to see that $f'(y) = -\frac{1}{2}y + 2(y - 1) = \frac{3}{2}y - 2$; solving $f'(y) = 0$ yields $\frac{3}{2}y = 2$, so our critical point is $y = \frac{4}{3}$. Note that $f''(y) = \frac{3}{2} > 0$, so we do have a minimum.

Plugging back in, we have that $x^2 = 1 - \frac{1}{4}y^2 = \frac{5}{9}$, so $x = \pm\frac{\sqrt{5}}{3}$. Therefore, the closest points on the ellipse to the point $(0, 1)$ are

$$\left(\frac{\sqrt{5}}{3}, \frac{4}{3}\right), \left(-\frac{\sqrt{5}}{3}, \frac{4}{3}\right).$$

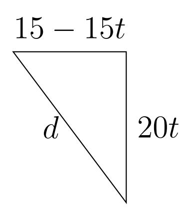
Note that you can also solve this problem by writing everything in terms of x and optimizing for x. By then, you end up needing to optimize a function that has square roots, and the calculation is messier. You'll end up with the right answer, but it might take a bit more work.

3. Again, we draw a picture:



The two boats are at the dock at 2pm and 3pm respectively; let's write everything in terms of a common start time. Let 2pm be the time $t = 0$. At time $t = 0$, the first boat is at the dock and the second boat is at 15 km west of the dock.

At time t, the first boat has moved distance $20t$ (recall that distance is speed $\times$ time), so it is distance $20t$ from the dock. The second boat has moved distance $15t$ toward the dock, but it was originally distance 15 away from the dock, so it is now $15 - 15t$ away from the dock. This gives the following picture:



At time t, the distance between the two boats is then (by the Pythagorean Theorem)

$$d = \sqrt{(15 - 15t)^2 + (20t)^2}.$$

We want to minimize this quantity. As in the previous problem, it suffices to minimize d^2; we therefore need to minimize

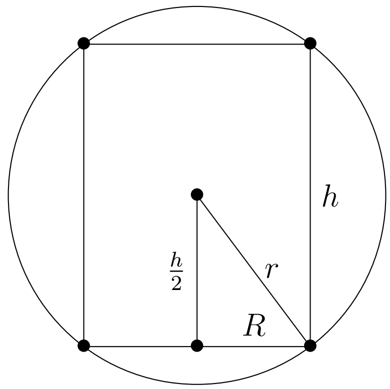
$$d^2 = f(t) = (15 - 15t)^2 + (20t)^2 = 625t^2 - 450t + 225.$$

To minimize, we note that $f'(t) = 1250t - 450$, so $f'(t) = 0$ occurs only at the time $t = \frac{450}{1250} = \frac{9}{25} = 0.36$. This is the critical number, and since $f''(t) = 1250 > 0$, it is indeed a minimum.

So the boats are closest together at time $t = 0.36$, which is 0.36 hours (or 21.6 minutes) after 2pm.

4. Again, we draw a picture.Here, imagine a cylinder inside a sphere, and take a cross section along the axis of the cylinder. The sphere becomes a circle, and the cylinder becomes a rectangle. The rectangle has all four vertices touching the circle (since otherwise we could expand the cylinder and still stay inside the sphere).

   Let the sphere have radius r and let cylinder have height h and radius R. Then the picture looks like this:



   We wish to maximize the volume of the cylinder, which is $V = \pi R^2 h$. We want to turn this into an optimization problem that we can do with the tools that we have so far, so we want to write this in terms of a single variable.

   Happily, we have a connection between R and h. Look at the triangle in the diagram, and apply the Pythagorean theorem. This gives $(\frac{h}{2})^2 + R^2 = r^2$, so $R^2 = r^2 - (\frac{h}{2})^2$. This means that the volume is

$$V = \pi R^2 h = \pi \left(r^2 - \frac{1}{4}h^2 \right) h = \pi r^2 h - \frac{\pi}{4}h^3.$$

   Here, we should think of r as a constant, given to us in the statement of the problem.

   So now, we just have to maximize the volume. To do this, define $f(h) = \pi r^2 h - \frac{\pi}{4}h^3$. Differentiate to see that $f'(h) = \pi r^2 - \frac{3\pi}{4}h^2$. This is zero when $\frac{3\pi}{4}h^2 = \pi r^2$, so $h^2 = \frac{4}{3}r^2$ and $h = \sqrt{\frac{4}{3}}r$. (We can take the positive square root because we know the height needs to be positive.) Since $f''(h) = -\frac{3\pi}{4} < 0$, we know that we indeed have a maximum at our critical point.

   Therefore, we can plug this value of h back into our formula for the volume of the cylinder to see that the maximal volume is

$$V = \pi r^2 h - \frac{\pi}{4}h^3 = \pi r^2 \sqrt{\frac{4}{3}}r - \frac{\pi}{4}\left(\sqrt{\frac{4}{3}}r \right)^3 = \frac{2}{\sqrt{3}}\pi r^3 - \frac{2}{3\sqrt{3}}\pi r^3 = \frac{4\pi}{3\sqrt{3}}r^3.$$