MATH 215A NOTES

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ABSTRACT. These notes were taken during Math 215A (Complex Analysis) taught by Kannan Soundararajan in Fall 2011 at Stanford University. They were live-TEXed during lectures in `vim` and compiled using `latexmk`. Each lecture gets its own section. The notes are not edited afterward, so there may be typos; please email corrections to `moorxu@stanford.edu`.

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We will think about functions defined on either the complex numbers $\mathbb{C}$ or some region $\Omega \subseteq \mathbb{C}$. Throughout, a region will be an open and connected subset of $\mathbb{C}$. It may have some holes.

We are interested in considering functions $f : \Omega \to \mathbb{C}$. We'd like to understand functions that are nice.

Definition 1.1. f is said to be *holomorphic* at a point $z_0 \in \Omega$ if it is differentiable there, i.e.

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists. If it exists, we denote it as $f'(z_0)$.

f is *holomorphic* on all of Ω if it is holomorphic at every point in Ω. We can also say that f is *complex differentiable*.

Example 1.2. Polynomials in z are holomorphic on $\mathbb{C}$.

Example 1.3. Rational functions $f = \frac{P(z)}{Q(z)}$ are holomorphic except at the zeros of Q.

You can check that if f and g are holomorphic at some point z_0, then $f+g$ and $f\cdot g$ are holomorphic, and f/g is too so long as $g(z_0) \neq 0$.

Example 1.4. $f(z) = \frac{1}{z}$ is holomorphic on the punctured unit disc $= \{0 < |z| < 1\}$.

Another basic example is the idea of a *power series*. To start, let's center the power series at 0. We are interested in $f(z) = \sum_{n=0}^\infty a_n z^n$ for some sequence $a_n \in \mathbb{C}$. A priori, this might not converge. We can ask when this converges. It should be familiar that there is a number R, called the *radius of convergence*, defined by $R = \liminf_{n\to\infty} |a_n|^{-1/n}$. This means that f converges absolutely in $|z| < R$, and the series will diverge for $|z| > R$. It could do anything on $|z| = R$.

Example 1.5. Consider $\sum_{n=0}^\infty z^n$, which converges in $|z| < 1$ and diverges as $|z| \to 1$. Or $\sum_{n=0}^\infty \frac{z^n}{n^2}$, which converges on $|z| = 1$. Or $\sum_{n=0}^\infty z^{n!}$, which also has $R = 1$, but it $\to \infty$ as $z \to e^{2\pi i a/q}$.

Example 1.6. For the exponential function $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, we have $R = \infty$.

Proposition 1.7. *In the region $\{|z| < R\}$, the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic, and its derivative is given by term-by-term differentiation $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Since $n^{1/n} \to 1$ as $n \to \infty$, this also has radius of convergence R. Therefore, f is infinitely differentiable. Then*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k}.$$

This shows that $a_k = \frac{f^{(k)}(0)}{k!}$, and this is just the Taylor series.

Proof. We'll sketch the proof. It suffices to consider the first derivative.

Consider z such that $|z| < R$, and consider h very small, so $|z+h| < R$. We compute

$$\frac{f(z+h) - f(z)}{h} = \lim_{N\to\infty} \sum_{n=0}^{N} a_n \frac{(z+h)^n - z^n}{h}.$$

We can now use the binomial theorem to work out what this is. The numerator is

$$\begin{aligned}(z+h)^n - z^n &= h(z^{n-1} + (z+h)z^{n-2} + \cdots + (z+h)^{n-1}) \\ &= h(nz^{n-1} + \text{stuff that will go to zero as } h \to 0),\end{aligned}$$

and the rest should be easy. □

This is certainly true for real-valued power series too. This is what we do with Taylor series expansions; the point here is that f is represented by its Taylor series.

In complex analysis, there is a remarkable converse to this result. In a certain sense, holomorphic functions are exactly given by power series.

Theorem 1.8. *Suppose that f is holomorphic in a region Ω. If $z_0 \in \Omega$ and the disc with center z_0 and radius r is contained in Ω, then in that disc $\{|z - z_0| < R\}$ we have $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. In particular, f is infinitely differentiable on Ω and locally has a Taylor expansion.*

Example 1.9. From real analysis, we remember bad examples like

$$\begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

which is infinitely differentiable but not analytic at zero. This cannot happen in the complex case.

The point is that being holomorphic is more restrictive than simply being differentiable in the real variable case.

Think about what a complex differentiable function looks like. Let $z = x + iy$. We have $f(x+iy) = u(x,y) + iv(x,y)$, as a function in two real variables, but the partial derivatives interact as in the *Cauchy-Riemann equations.*

Consider h being purely real. Then

$$\frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} = u_x(x,y) + iv_x(x,y).$$

But if we want f to be complex differentiable, this must be the same as in the case where h is complex imaginary. Then

$$\frac{u(x,y+ih)+iv(x,y+h)-u(x,y)-iv(x,y)}{ih}=\frac{1}{i}(u_y(x,y)+iv_y(x,y)).$$

Comparing yields the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y \\ v_x &= -u_y. \end{aligned}$$

We know that this is necessary to be complex differentiable; it is also sufficient.

Theorem 1.10 (Looman and Menchoff)**.** *The Cauchy-Riemann equations imply that f is holomorphic.*

This is kind of technical and we won't prove it.

Here's what we will prove;

Proposition 1.11. *If $u(x,y)$ and $v(x,y)$ have* continuous *partial derivatives and satisfy the Cauchy-Riemann equations, then $f(z)=u(x,y)+iv(x,y)$ is holomorphic.*

Proof. We want to compute

$$\frac{f(z+h)-f(z)}{h}$$

where $h=k+il$. This requires that we understand quantities like $u(x+k,y+l)-u(x,y)$ and $v(x+k,y+l)-v(x,y)$.

Note that

$$u(x+k,y+l)-u(x,y)=ku_x(x,y)+lu_y(x,y)+o(|k|+|l|).$$

Here, $o(|k|+|l|)$ means that a quantity that is $<\varepsilon(|k|+|l|)$ if $|k|+|l|$ is sufficiently small. This is where we use the continuity of partial derivatives. Similarly,

$$v(x+k,y+l)-v(x,y)=kv_x(x,y)+lv_y(x,y)+o(|k|+|l|).$$

Adding these two equations and dividing by $k+il$ gives

$$\frac{ku_x(x,y)+ilv_y(x,y)+lu_y(x,y)+ikv_x(x,y)+o(|k|+|l|)}{k+il},$$

and applying the Cauchy-Riemann equations finishes the proof. □

Instead of thinking of $f(z)$ as a function of the real and imaginary parts x and y, we can think of it as a function of z and $\overline{z}$. Here, $x=\frac{z+\overline{z}}{2}$ and $y=\frac{z-\overline{z}}{2i}$. We want to consider differential operators $\frac{d}{dz}f$ and $\frac{d}{d\overline{z}}f$. Then we have

$$\begin{aligned} \frac{d}{dz}f &= \frac{df}{dx}\frac{dx}{dz}+\frac{df}{dy}\frac{dy}{dz}=\frac{1}{2}\left(\frac{df}{dx}+\frac{1}{i}\frac{df}{dy}\right) \\ \frac{d}{d\overline{z}}f &= \frac{df}{dx}\frac{dx}{dz}-\frac{df}{dy}\frac{dy}{dz}=\frac{1}{2}\left(\frac{df}{dx}-\frac{1}{i}\frac{df}{dy}\right). \end{aligned}$$

Then the Cauchy-Riemann equations simply say that $\frac{d}{dz}f=f'(z)$ and $\frac{d}{d\overline{z}}f=0$, which means that f is only a function of z and not of $\overline{z}$.

If f is holomorphic and $f(x,y) = u(x,y) + iv(x,y)$ then the real and imaginary parts are *harmonic*, which means that $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. We'll come back to this later. This operator is called the *Laplacian* and is denoted $\triangle$.

We will now define *path integrals*, or integrals over curves. We will only deal with curves that are piecewise smooth. Think of a *parametrized curve* as $z : [a,b] \to \mathbb{C}$, connecting $z(a)$ to $z(b)$. We also get an orientation, pointing in the direction of increasing time.

We are going to assume that z is continuously differentiable except at the endpoints, where we want one-sided limits, i.e.

$$\lim_{h\to 0^+} \frac{z(a+h) - z(a)}{h}.$$

We also insist that $z'(t) \neq 0$ for all t, so the curve doesn't get stuck at any point. This is not strictly necessary.

We can imagine some seemingly nice curves that are piecewise smooth, and we will allow these curves. This means that we divide $[a,b]$ into intervals $[a,a_1], [a_1,a_2], \ldots, [a_n,b]$, and the curve is smooth on each interval.

Given a parametrized curve γ (which is the curve in $\mathbb{C}$ together with an orientation), we can think of $\int_\gamma f(z)\,dz = \int_a^b f(z(t))z'(t)\,dt$.

We also have path integrals $\int_\gamma (p\,dx + q\,dy) = \int_a^b (p(x,y)x' + q(x,y)y')\,dt$.

This is independent of parametrization. This means that if we have $z_1 : [a,b] \to \mathbb{C}$ and $z_2 : [c,d] \to \mathbb{C}$, they are *equivalent* if there exists a continuously differentiable function $t : [c,d] \to [a,b]$ with $z_2(s) = z_1(t(s))$. The point is that the curve looks exactly the same in $\mathbb{C}$, but it is parametrized differently. We require that $t'(s) > 0$ in order to preserve the orientation.

Example 1.12. $t + it^3$ with $0 \leq t \leq 1$ and $t^2 + it^6$ with $0 \leq t \leq 1$ are equivalent.

If we have a curve γ parametrized by $z : [a,b] \to \mathbb{C}$ then $\tilde{z}(t) = z(a+b-t) : [a,b] \to \mathbb{C}$ gives a curve $-\gamma$, which is the same curve as γ but with the opposite orientation.

We can check that

$$\int_c^d f(z_2(s))z_2'(s)\,ds = \int_c^d f(z_1(t(s)))z_2'(t(s))t'(s)\,ds = \int_a^b f(z_1(x))z_1'(x)\,dx,$$

so this definition of path integral makes sense

We can also check the basic things, so for example $\int_\gamma (f+g)\,dz = \int_\gamma f\,dz + \int_\gamma g\,dz$.

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We have bounds of the form

$$\left|\int_\gamma f\,dz\right| \leq \left(\sup_{z \text{ on } \gamma} |f(z)|\right) \int_a^b |z'(t)|\,dt.$$

There is a basic question that we can ask, and we will come back to this repeatedly.

Question. If we have a curve γ from z_0 to z_1, when does $\int_\gamma f\,dz$ depend only on the endpoints z_0 and z_1 (and not on the actual path)?

A related question is as follows. If γ is a closed curve, then is $\int_\gamma f\,dz = 0$?

We'll begin with the first result on this question.

Remark. Imagine that we have a region Ω and a continuous function f. We say that F is a *primitive* for f if F is holomorphic on Ω and $F' = f$.

Proposition 2.1. *If there is such a primitive F, then $\int_g f\,dz$ depends only on the endpoints of γ, and the integral over a closed curve is zero.*

Proof. We can parametrize γ, so the path integral is

$$\int_a^b f(z(t))z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt = \int_a^b \left(\frac{d}{dt}F(z(t))\right)\,dt = F(z(b)) - F(z(a)). \qquad \square$$

Example 2.2. For any polynomial f, the path integral $\int_\gamma f(z)\,dz$ is path-independent.

Example 2.3. Consider $f(z) = z^{-n}$ for $n \geq 2$; then $F(z) = \frac{z^{1-n}}{1-n}$. Therefore, if γ is a closed curve not containing the origin, then $\int_\gamma z^{-n}\,dz = 0$. (We exclude the origin because F is not holomorphic there.)

Example 2.4. Let $f(z) = \frac{1}{z}$, and let γ be the unit circle, which we will parametrize as $e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Then

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta}\,d\theta = 2\pi i.$$

This means that there is no region containing the unit circle with a primitive for $f(z) = \frac{1}{z}$.

The converse is also true.

Proposition 2.5. *If f is continuous, and for every curve γ in Ω, the integral $\int_\gamma f\,dz$ depends only on the endpoints, then there exists a primitive F for f in the region Ω.*

Proof. Take any two points z_0 and z_1 in Ω. Since Ω is path-connected, we have find a path between them using only segments parallel to the axes. Fix the point z_0, and define

$$F(z) = \int_{\zeta_0 \text{ to } z} f(w)\,dw.$$

Note that our hypotheses imply that this is well-defined. If we choose a path where z_1 is on a segment parallel to the x-axis, we should have $F_x(z) = f(z)$. On the other hand, we might choose a path where z_1 is on a segment parallel to the y-axis, so that $\frac{1}{i}F_y(z) = f(z)$. Therefore, F satisfies the Cauchy-Riemann equations. Also, we assumed that f is continuous, so from the converse 1.11 that we proved last time, the continuity of partial derivatives implies that F is holomorphic. $\square$

Corollary 2.6. *If f is a holomorphic function on Ω with $f' = 0$, then f is constant.*

Proof. Since Ω is connected, we can fix a point z_0, and for every point z we have

$$\int_{z_0 \text{ to } z} f' = 0 = f(z) - f(z_0).$$

$\square$

Let's recall what our main goal is.

Theorem 2.7. *If f is holomorphic on Ω, then f is infinitely differentiable in Ω, and it is represented by its Taylor series locally.*

The theme of the proof will be that the disc D has the property that for any holomorphic function on D, the integral $\int_{\gamma \text{ closed}} f\,dz = 0$. This is called Cauchy's theorem.

We start with a particularly nice form of the theorem.

Theorem 2.8 (Goursat's Theorem)**.** *Suppose f is holomorphic on Ω. Fix some triangle T; this clearly has an inside and an outside, and we assume that the interior of T and T both lie in Ω. Then $\int_T f\,dz = 0$.*

Proof. Take the triangle. Connect its midpoints to produce four similar triangles, and repeat for each of the smaller triangles. At the nth stage, we have 4^n triangles. The integral over T can be done as an integral over each small triangle; all of the internal lines will cancel. Therefore, $\int_T f\,dz$ is a sum over $\int_{T_n}$, where the T_n range over 4^n little triangles.

Suppose to the contrary that $\int_T f\,dz \neq 0$. Then we can find a triangle T_1 with

$$\left|\int_{T_1} f\,dz\right| \geq \frac{1}{4}\left|\int_T f\,dz\right|.$$

We can divide T_1 into smaller triangles to get a triangle T_2 inside T_1 such that

$$\left|\int_{T_2} f\,dz\right| \geq \frac{1}{4^2}\left|\int_T f\,dz\right|.$$

Continuing this process, we get a nested sequence of smaller and smaller triangles; in general, we have

$$\left|\int_{T_n} f\,dz\right| \geq \frac{1}{4^n}\left|\int_T f\,dz\right|.$$

The triangles are getting smaller in two ways that we can quantify. In particular, $\operatorname{diameter}(T_n) = \frac{1}{2^n}\operatorname{diameter}(T)$ and $\operatorname{perimeter}(T_n) = \frac{1}{2^n}\operatorname{perimeter}(T)$. This means that there is a *unique* z_0 contained in every T_n. (This is because we can pick a point from each T_n, and this forms a Cauchy sequence.)

As n becomes large, we see that (by holomorphicity)

$$\begin{aligned}\int_{T_n} f(z)\,dz &= \int_{T_n} \left(f(z_0) + (z-z_0)f'(z_0) + o(|z-z_0|)\right)\,dz = \int_{T_n} o(|z-z_0|)\,dz\\ &= o\left(\frac{1}{2^n}\operatorname{diameter}(T)\cdot\frac{1}{2^n}\operatorname{perimeter}(T)\right) = o\left(\frac{1}{4^n}\right),\end{aligned}$$

which implies that $\int_T f\,dz = 0$. $\square$

All that we used was some way of dissecting a triangle into smaller and smaller triangles; the same argument works for a rectangle. We could also dissect a rectangle into two triangles.

Now let's talk about a particular special case, where Ω is a disc. Here, we can strengthen Goursat's theorem to any closed curve.

Theorem 2.9. *If f is holomorphic on the disc Ω, then f has a primitive. In particular, this means that $\int_\gamma f\,dz = 0$ for any closed curve γ inside the disc.*

Proof. Suppose that a fixed point in the disc is z_0. For any point z in the disc, we can find a path from z_0 to z by first going parallel to the x-axis and then parallel to the y-axis. We then have a well-defined

$$F(z) = \int_{\text{this path from } z_0 \text{ to } z} f(w)\,dw.$$

This tells us that $\frac{1}{i}F_y = f$. We can define another path from z_0 to z by first going parallel to the y-axis and then parallel to the x-axis. This gives us

$$G(z) = \int_{\text{second path}} f(w)\,dw = F(z)$$

by Goursat's theorem 2.8 for the rectangle (since the two paths form a rectangle). This implies by the Cauchy-Riemann equations that F is holomorphic. $\square$

This was a special case of Cauchy's Theorem: If γ is a closed curve in the disc, then $\int_\gamma f(z)\,dz = 0$. Note that the preceding argument applies to any convex region, or any star-shaped region. Our goal is to understand for what regions it is true, and for what versions it is not true.

Example 2.10. Consider an annulus. We can define paths of this type (with only segments parallel to the axes), but there is clearly no way to use Goursat's theorem to make this work (for a path going around the hole).

There are some regions for which we can clearly carry out this proof, where it is clear what the inside is, and it is easy to write down a path from one point to any other point. One example of a region of this type is the keyhole.

Next, we want to prove the main theorem 2.7. The next ingredient that we need is Cauchy's integral formula.

Theorem 2.11 (Cauchy's integral formula)**.** *Consider some region Ω and a circle C contained in Ω. (This means that the interior of C is also contained in Ω.) Suppose that f is holomorphic on Ω; given a point z in the circle, we then have*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw.$$

Proof. The proof is to use a contour that looks like a keyhole.

Consider a circle C, and use a keyhole with z in the interior of the hole. Instead of integrating around the circle, we integrate around the keyhole. There are a couple of parameters: ε is the radius of the keyhole, and δ is the width of the keyhole corridor; both will be small. As in the proof of Goursat's theorem, we see that

$$\int_{\text{keyhole contour}} \frac{f(w)}{w-z}\,dw = 0.$$

Here, the point is that z lies outside the keyhole region, so everything is holomorphic there. Now, let $\delta \to 0$. The outside of the keyhole is approximately the same as the circle, and the two parallel lines of the keyhole corridor will cancel out. What remains is the small circle C_ε of radius ε. Therefore,

$$\int_{\text{keyhole}} \frac{f(w)}{w-z}\,dw = 0 = \int_C \frac{f(w)}{w-z}\,dw + \int_{C_\varepsilon} \frac{f(w)}{w-z}\,dw.$$

Observe that C_ε now has opposite orientation as C. Parametrize it as $z+\varepsilon e^{i\theta}$ for $0 \le \theta \le 2\pi$. We now let $\varepsilon \to 0$. Then $f(w) = f(z) + f'(z)(w-z) + o(w-z)$. Therefore,

$$\int_{C_\varepsilon} \frac{f(w)}{w-z}\,dw = \int_{C_\varepsilon} \left(\frac{f(z)}{z-w} + f'(z) + o(1)\right)\,dw \approx f(z)\int_0^{2\pi} \frac{d(z+\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = 2\pi i f(z).$$

This is the same calculation as where we computed $\int_C \frac{dz}{z} = 2\pi i$. This completes the proof. $\square$

We can now prove the main theorem 2.7.

Proof of Theorem 2.7. We can just differentiate Cauchy's integral formula repeatedly. That is, for every $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw,$$

so f is infinitely differentiable. $\square$

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We can now prove the main theorem 2.7:

Theorem 3.1. *If f is holomorphic on Ω then f is infinitely differentiable and has a power series representation.*

Proof. Consider the set Ω, and take any point $z \in \Omega$. Since Ω is open, there is a circle in Ω centered at z. There, we have

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw.$$

Suppose we have another point $z+h$ inside the circle. We can compute the derivative

$$\frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i} \int_C f(w) \left(\frac{\frac{1}{w-z-h} - \frac{1}{w-z}}{h} \right) dw = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)(w-z-h)}\, dw.$$

As we let $h \to 0$, we find that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw,$$

and now we can do the same process over and over again. Effectively, we are repeatedly differentiating $\frac{1}{(w-z)^k}$. From this argument, we obtain that f is infinitely differentiable, and we have that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw.$$

Proposition 3.2 (Cauchy's Inequalities)**.** *If the circle with radius R around z is contained in Ω, then we have*

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{|f(w)|}{R^{n+1}}\, |dz| \leq \frac{n!}{R^n} \max_{w \in C} |f(w)|.$$

This gives a bound on the size of Taylor series coefficients. Also,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} (w-z)^n$$

converges absolutely on $|w-z|<R$. A natural question is if this is equal to $f(w)$. This is not too hard to check. Take

$$\sum_{k=0}^{N} \frac{f^{(n)}(z)}{n!}(w-z)^n = \frac{1}{2\pi i}\int_C f(\zeta)\sum_{n=0}^{N}\frac{(w-z)^n}{(\zeta-z)^{n+1}}\,d\zeta.$$

Observe that

$$\left|\frac{w-z}{\zeta-z}\right| = \frac{|w-z|}{R} < 1,$$

so therefore the sum in the integral above is a geometric progression that converges. Hence,

$$\begin{aligned}\sum_{k=0}^{N} \frac{f^{(n)}(z)}{n!}(w-z)^n &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\frac{1-(\frac{w-z}{\zeta-z})^{N+1}}{1-\frac{w-z}{\zeta-z}}\,d\zeta \\ &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-w}\left(1-\left(\frac{w-z}{\zeta-z}\right)^{N+1}\right)d\zeta. \qquad \square\end{aligned}$$

Here is a nice classical application.

Theorem 3.3 (Liouville's Theorem). *If f is holomorphic on $\mathbb{C}$ and f is bounded, then f is a constant.*

Proof. If f is bounded, we can use the Cauchy's inequalities 3.2 for the first derivative. That is,

$$|f'(z)| \le \frac{1}{2\pi}\int_C \frac{|f(w)|}{|w-z|^2}\,|dw| \le \frac{\text{constant}}{2\pi R^2}(2\pi R) \le \frac{\text{constant}}{R}.$$

Letting $R\to\infty$, we see that $f'(z)=0$ for all z, which shows that f is constant. $\square$

This also gives a quick proof of the fundamental theorem of algebra.

Theorem 3.4 (Fundamental Theorem of Algebra). *Every polynomial (of degree ≥ 1) has a root.*

Proof. Suppose to the contrary that $P(z)\neq 0$. Then $\frac{1}{P(z)}$ is holomorphic on $\mathbb{C}$. Note that $|P(z)|\to\infty$ as $|z|\to\infty$, so therefore $\frac{1}{P(z)}$ is bounded. This contradicts Liouville's theorem! By induction, we can show that a polynomial of degree n has exactly n roots. $\square$

Another application of what we have so far is Morera's Theorem. This is like a converse to Cauchy's Theorem. This says that if an integral is path-independent then the function is analytic.

Theorem 3.5 (Morera's Theorem). *Suppose that f is continuous on the disk Ω, and suppose that $\int_T f(z)\,dz = 0$ for any triangle T contained in Ω. Then f is holomorphic on Ω.*

Proof. As in the proof of Cauchy, there exists a primitive F with $F'=f$, where F is holomorphic on Ω. But F is infinitely differentiable, which implies that f is holomorphic. $\square$

For the next several lectures, we will study local properties of holomorphic functions.

What can we say about the zeros of holomorphic functions? For example, for polynomials, we can factor a polynomial and get a small collection of zeros. Here, we can prove a nice result.

Proposition 3.6. *If we have a region Ω and f is holomorphic on Ω and not constant, then the points where $f(z) = 0$ are isolated. (This means that for any zero, there is a neighborhood $|w - z| < \delta$ of z such that $f(w) \neq 0$ unless $w = z$.)*

Again, this is different from real analysis; for example, functions of compact support cannot exist here.

Proof. Take a point $z_0 \in \Omega$. We can expand $f(z)$ near z_0 as a power series, so that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

A couple of things can happen here:

(1) $a_n = 0$ for all $n \geq 0$.
(2) there exists a least $m \geq 0$ with $a_m \neq 0$, and we say that f has a zero of order m at z_0.

First, consider case (2). Now,

$$f(z) = a_m (z - z_0)^m \left(1 + \sum_{k=1}^{\infty} b_k (z - z_0)^k \right).$$

Now, we know from our Cauchy estimates that the quantity in parentheses is not zero for all $|z - z_0|$ small enough, which shows that z_0 is an isolated zero.

Now, we consider case (1). In this case, we claim that $f(z) \equiv 0$ on Ω. Here, we use the fact that Ω is connected and open. Let $\mathcal{O} = \{z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \dots\}$, and let $\mathcal{N} = \{z \in \Omega : \text{case (2) happens}\}$. Observe that $\mathcal{O}$ is open, and similarly, $\mathcal{N}$ is also open. Then $\Omega = \mathcal{O} \cup \mathcal{N}$ is the union of two disjoint open sets, which is impossible if Ω is connected. $\square$

Now, we know that the set of zeros of a holomorphic function is quite nice. The set of zeros of $f \neq 0$ in Ω cannot have any limit points in Ω.

Corollary 3.7. *Suppose that we have two regions $\Omega_1 \subseteq \Omega_2$, and suppose that we have two functions f and g that are holomorphic on Ω_2, and $f = g$ on Ω_1. This implies that $f = g$ on Ω_2.*

Proof. Just consider $f - g$. $\square$

This is a very useful principle, known as *analytic continuation*.

Example 3.8. Consider the power series $1 + z + z^2 + \cdots$. A priori, we know that this is a holomorphic function on $|z| < 1$. We can also recognize that this is a geometric progression, so therefore this is equal to $\frac{1}{1-z}$, which is holomorphic for $z \neq 1$. These two functions agree on the disc, so this is actually the unique holomorphic function that matches with our power series.

We'll come back to analytic continuation later. Consider two regions Ω_1 and Ω_2 that have overlap. If f_1 is holomorphic on Ω_1 and f_2 is holomorphic on Ω_2, and if $f_1 = f_2$ on the intersection $\Omega_1 \cap \Omega_2$, then we can think of a function f on $\Omega_1 \cup \Omega_2$. These is a warning here: Suppose that there is another region Ω_3 intersecting both Ω_1 and Ω_2; beware that f_3 may not agree with f_1 on $\Omega_1 \cap \Omega_3$.

We say one more thing about Cauchy's formula. It actually gives us a lot of information. Given a holomorphic function, we can find its value inside a circle just by knowing its values on the circle. Again, this is something that we cannot hope to have in real variables. Now, what about the case where f is holomorphic on all of the inside of the circle except at one or two points: Can we still reconstruct the function inside the circle from the values on the circumference?

This raises the idea of an isolated singularity.

Definition 3.9. An *isolated singularity* is a point a where f is holomorphic in a region containing the punctured disk $\{0 < |z-a| < \delta\}$.

We want to understand what can happen at a. Can we make f holomorphic at a?

Theorem 3.10 (Riemann's Theorem on Removable Singularities)**.** *If* $\lim_{z\to a} f(z)(z-a) = 0$ *then we can remove the singularity at* a*: There is a unique holomorphic function on* $\{|z-a| < \delta\}$ *which agrees with* f *on the punctured disk.*

Proof. Consider a circle centered at a. We will prove that a version of Cauchy's formula still holds: $z \neq a$ implies that $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$. Imagine that we can prove this. We can follow the process of differentiating with respect to z. Then the right hand side is holomorphic for all z inside the circle C, so we should use this to define $f(a)$.

We proved Cauchy's integral formula in the case when f is holomorphic inside the circle C, and here we do not know this. We examine our proof of Cauchy's formula, in which we used a keyhole contour. Here, we shall use two keyhole contours, one to avoid the point z and the other to avoid the point a.

Now, we claim that

$$\int_{\text{double keyhole contour}} \frac{f(w)}{w-z}\,dw = 0.$$

This is because f is holomorphic inside this double keyhole region, so we should be able to construct a primitive. To do this, we need to find a way to go from anywhere to anywhere else using horizontal or vertical lines; we ought to be able to do this using six segments, and this shouldn't be too hard.

Now, we shrink both keyhole contours, so that each keyhole corridor cancels out. This leaves us with:

$$\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_{\text{small circle around } a} \frac{f(w)}{w-z}dw + \frac{1}{2\pi i}\int_{\text{small circle around } z} \frac{f(w)}{w-z}dw = f(z).$$

The second integral is $f(z)$ by Cauchy's formula, and the first integral goes to zero by hypothesis, and we are done. □

That is the nicest type of singularity that we can have; it's not really a singularity at all. We will discuss two more types of singularities.

4. 10/6

Suppose we have a region Ω, with a function f that is holomorphic there except at some $z_0 \in \Omega$. In the case when $\lim_{z\to z_0} f(z)(z-z_0) = 0$, we have a removable singularity. What if this condition fails?

There are two cases:

(1) $|f(z)| \to \infty$ as $z \to z_0$. In this case, we say that f has a *pole* at z_0.
(2) not (1), and also not a removable singularity. In this case, we say that f has an *essential singularity* at z_0.

This gives a classification of isolated singularities.

Example 4.1. This is an example of an essential singularity. Consider $f(z) = e^{1/z}$, which is holomorphic on $\mathbb{C} \setminus \{0\}$. If $z \in \mathbb{R}$, then as $z \to 0^+$, $f(z) \to \infty$, while as $z \to 0^-$, $f(z) \to 0$. And if $z \in i\mathbb{R}$, $|f(z)| = 1$. There are lots of different types of behaviors.

Now, let's consider the case of poles. Let $g(z) = \frac{1}{f(z)}$. In some neighborhood of z_0, $f(z) \neq 0$, so g is holomorphic in a punctured neighborhood of z_0. But g is also bounded here, since $f(z) \to \infty$ as $z \to z_0$; in fact, g has a zero at z_0. So we have that $\lim_{z \to z_0} g(z)(z - z_0) = 0$. This implies that g has a removable singularity at z_0. Therefore, g is holomorphic in a neighborhood at z_0 and $g(z_0) = 0$. We can now write $g(z) = a_m(z - z_0)^m(1 + h(z))$ where h is holomorphic, $h(z_0) = 0$, and $a_m \neq 0$. This shows us that

$$f(z) = \frac{1}{a_m(z - z_0)^m}\left(\frac{1}{1 + h(z)}\right) = \frac{b_{-m}}{(z - z_0)^m} + \frac{b_{-m+1}}{(z - z_0)^{m-1}} + \cdots + b_0 + b_1(z - z_0) + \cdots .$$

In this case, we say that f has a pole of order m at z_0. We say that

$$\frac{b_{-m}}{(z - z_0)^m} + \frac{b_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{b_{-1}}{(z - z_0)}$$

is the *principal part* of f at z, and $f -$ principal part of f is holomorphic at z_0.

Remark. This is again different from real analysis, where there are functions like $\frac{1}{\sqrt{x}}$; this is a situation that cannot occur in complex analysis.

Definition 4.2. We say that a function f is *meromorphic* on a region Ω if it is holomorphic on $\Omega \setminus \{z_1, z_2, z_3, \ldots\}$, where it has at most polar singularities. Also, the points $z_1, z_2, z_3, \ldots$ are isolated and have no limit point in Ω.

The way to construct meromorphic functions is to consider functions f and g that are holomorphic on Ω; then f/g is meromorphic. We also want to show the converse (which would characterize all meromorphic functions), but this is not obvious. We will come back to this later.

It is sometimes convenient to think not just about points in the plane, but about the *extended complex plane*, which amounts to adding a point at infinity. Suppose that f is holomorphic at all points z with $|z|$ sufficiently large. Then we can look at $g(z) = f(1/z)$. Then g would be holomorphic in a punctured neighborhood of 0. What kind of singularity can g have at the origin?

Another way of thinking about this is to compactify the plane. This is the *Riemann sphere*, which is the one point compactification of $\mathbb{C}$. Take the complex plane, and consider a sphere centered at $(0, 0, 1/2)$ with south pole on the plane at $(0, 0, 0)$. From the north pole $(0, 0, 1)$, draw a line intersecting the plane at $(x, y, 0)$; this passes through the sphere at some point (X, Y, Z). This line is represented by $(0, 0, 1) + \frac{1}{1-Z}(X, Y, Z - 1)$, so we have that

$$x = \frac{X}{1 - Z}, \qquad y = \frac{Y}{1 - Z},$$

allowing us to go from any point on the sphere to a point on the plane. The line is also represented by $(0,0,1)+\lambda(x,y,-1)$ where $(\lambda x)^2+(\lambda y)^2+(\frac{1}{2}-\lambda)^2=\frac{1}{4}$, which means that $\lambda=\frac{1}{x^2+y^2+1}$. So we can go from any point on the plane to a point on the sphere, given by

$$X=\frac{x}{x^2+y^2+1},\quad Y=\frac{y}{x^2+y^2+1},\quad Z=\frac{x^2+y^2}{x^2+y^2+1}.$$

This is a correspondence between points on the sphere and points on the plane, except for that as $|x+iy|\to\infty$, the correspondence gives us the north pole.

Theorem 4.3. *Any holomorphic function on the extended complex plane is a constant.*

A meromorphic function on the extended complex plane is just a rational function.

Proof. The first fact is the same as Liouville's theorem.

If we have a meromorphic function f, it can only have finitely many poles because the extended complex plane is compact. Suppose the poles are at $z_1,\dots,z_k$ with multiplicity $m_1,\dots,m_k$ respectively. Then we see that

$$F(z)=f(z)\prod_{j=1}^{k}(z-z_j)^{m_j}$$

has at most a polar singularity at ∞. Then $|F(z)|\le C(|z|)^N$ if $|z|$ is large enough. We now argue as in the proof of Liouville's theorem. By Cauchy's formula, $F^{(n+2)}(z)=0$ for all z, so therefore $F(z)$ must be a polynomial, and hence $f(z)$ must be a rational function. $\square$

If we have a function that is meromorphic in a region instead of holomorphic, what can we say about the contour integrals? Suppose that f is meromorphic in Ω and C is a circle whose interior is in Ω. We want to understand $\int_C f(z)\,dz$. Inside a circle, we can only have a finite number of poles because the poles are isolated and the circle is a compact region. Suppose the poles are at $z_1,\dots,z_k$, with order $m_1,\dots,m_k$. At these points, suppose that the principal parts of f are $P_1(z),\dots,P_k(z)$. Write

$$P_j(z)=\sum_{l=1}^{m_j}\frac{b_j(-l)}{(z-z_j)^l}.$$

Note that $f-\sum_{j=1}^{k}P_j(z)$ is holomorphic. Therefore, we see that

$$\int_C f(z)\,dz=\sum_{j=1}^{k}\int_C P_j(z)\,dz.$$

We now need to consider

$$\int_C\sum_{l=1}^{m_j}\frac{b_j(-l)}{(z-z_j)^l}\,dz.$$

For each term with $\lambda\ge 2$, we can find a primitive for $\frac{1}{(z-z_j)^l}$ in $\mathbb{C}\setminus\{z_j\}$, and hence the integral of that term is zero. Therefore, we see that

$$\int_C\sum_{l=1}^{m_j}\frac{b_j(-l)}{(z-z_j)^l}\,dz=\int_C\frac{b_j(-1)}{(z-z_j)}\,dz=2\pi i b_j(-1).$$

The *residue* $\mathrm{Res}_{z_j} f$ of f at z_j is the coefficient of $\frac{1}{z-z_j}$, i.e. $b_j(-1)$.

What we've proved is the Residue Theorem:

Theorem 4.4 (Residue Theorem)**.** *If f is meromorphic on Ω and C is a circle whose interior is in Ω, (and if f has no poles on the circle C) then*

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{\text{poles } z_j} (\mathrm{Res}_{z_j} f).$$

Example 4.5. As a special case, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\,dz = \mathrm{Res}_w \frac{f(z)}{z-w} = f(w),$$

which is Cauchy's formula.

Example 4.6. Suppose that f is meromorphic and f has no poles on the circle C. Consider the function $\frac{f'}{f}$. This is also a meromorphic function that has no poles on C.

Using the residue theorem, we have that

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \sum_{\text{poles of } \frac{f'}{f}} \mathrm{Res}(f'/f).$$

If f has a zero of order m_j at z_j, then locally near z_j, we have $f(z) = a_j(z-z_j)^{m_j}(1+\cdots)$. Then

$$\frac{f'}{f}(z) = \frac{m_j}{z-z_j} + \text{holomorphic},$$

and hence the residue of f'/f is simply the order of the pole: $\mathrm{Res}(f'/f) = -m_j$. This gives us a useful result:

Corollary 4.7.

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \#\text{ zeros} - \#\text{ poles}.$$

As a nice application, we will prove Rouche's Theorem, also known as the "dog on a leash theorem".

Theorem 4.8 (Rouche's Theorem)**.** *Suppose that f and g are holomorphic in a region Ω containing C has its interior. Suppose that $|f(z)| > |g(z)|$ on C. Then f and $f+g$ have the same number of zeros inside C.*

Remark. Let $|f(z)|$ be the distance between you and the fire hydrant, and let $|g(z)|$ be the length of the leash. Then when you walk a dog around a fire hydrant, you and the dog go around the same number of times.

Proof. For any t such that $0 \le t \le 1$, look at $f(z)+tg(z)$. These are holomorphic, and never zero on C. Then

$$\frac{1}{2\pi i}\int_C \frac{f_t'}{f_t}(z)\,dz = \#\text{ of zeros of } f_t = n_t\,(\ge 0, \in \mathbb{Z}).$$

As a function of t, this is continuous. So n_t is continuous in t, but it also takes on integer values. Therefore, n_t must be constant. Then $n_0 = n_1$, so f and $f+g$ has the same number of zeros. □

Now, we can understand what happens to a holomorphic function locally. Suppose that f is holomorphic in some region Ω, and take some point $z_0 \in \Omega$. Suppose that

$$f(z) = w_0 + a_k(z - z_0)^k + \cdots$$

where $a_k \neq 0$, so that $f(z_0) = w_0$. There is a neighborhood of z_0 such that if $|w - w_0|$ is small enough, then $f(z) = w$ has exactly k solutions for z in this neighborhood of z_0. The point is that this function behaves exactly as simply as if the higher order terms did not exist.

We will prove this next time, and first we will state a corollary.

Corollary 4.9 (Open Mapping Theorem). *If f is holomorphic and not constant, then the image of an open set is open.*

Another consequence of this is the Maximum Modulus Principle:

Theorem 4.10 (Maximum Modulus Principle). *The absolute value of a holomorphic function in a region Ω has no maximum.*

5. 10/11

We will first make a digression to say something about essential singularities:

Theorem 5.1 (Casorati-Weierstrass Theorem). *If f has an essential singularity at a point z_0 then in a punctured neighborhood of z_0, the set of values of f is dense in $\mathbb{C}$.*

Proof. Suppose not. Then we can find a w with $|f(z) - w| > \delta$ for all z in a punctured neighborhood of z_0. Then we can look at $g(z) = \frac{1}{f(z)-w}$. This is holomorphic in this punctured neighborhood of z_0, but it is also bounded by $\frac{1}{\delta}$. Then g has a removable singularity at z_0. Therefore, as $z \to z_0$, we can define a number $g(z_0)$ such that $g(z) \to g(z_0)$.

We know that $\frac{1}{f(z)-w} \to g(z_0)$ as $z \to z_0$. This is a problem because we assumed that f is very badly behaved near z_0. There are two cases. If $g(z_0) \neq 0$ then $f(z) - w \to \frac{1}{g(z_0)}$, which would mean that f has a removable singularity at z_0, which we are assuming is not the case. In the other case, $g(z_0) = 0$, which means that $|f(z) - w| \to \infty$ as $z \to z_0$. But this means that f has a pole as $z \to z_0$, which is also impossible. That proves our result. $\square$

There is a much stronger version of this result, which we will try to come back to at the end of the class. This is a theorem of Picard:

Theorem 5.2 (Picard's Big Theorem). *If f has an essential singularity at z_0 then in any neighborhood of z_0, it takes on all complex values, except for maybe one value.*

Example 5.3. We need the exception of one value. For example, consider $e^{\frac{1}{z}}$ in a neighborhood of 0. This takes on every value except 0.

Here is a variant of Picard's theorem.

Corollary 5.4. *If f is holomorphic and not constant in $\mathbb{C}$, then it takes on every value except for maybe one.*

Proof. The reason is that we should think of f as a function in the extended complex plane. Either it can have an essential singularity or a pole at infinity. If the function has a pole at infinity, then it must be a polynomial, and by the fundamental theorem of algebra, it takes on every value. Otherwise, it has an essential singularity, and Picard's theorem applies. $\square$

We return to our discussion from last time. We had this consequence of Rouche's Theorem.

Proposition 5.5. *If $f(z) - w_0$ has a zero of order k at z_0, then there is a neighborhood of $|z - z_0| < \delta$ such that if $|w - w_0|$ is small enough then $f(z) = w$ for exactly k points in this neighborhood.*

Proof. We are looking for a circle C of radius δ around z_0. We want to count the number of zeros of $f(z) - w$ inside C.

We know that $f(z) = w_0 + a_k(z - z_0)^k +$ (higher order terms). In order to apply Rouche's Theorem 4.8, we can write $f(z) - w = (f(z) - w_0) + (w_0 - w)$. Here, let $F(z) = f(z) - w_0$ and $G(z) = w_0 - w$; then $F + G = f(z) - w$. If we can check that $|F(z)| > |G(z)| = |w - w_0|$ on C then we are done.

We claim that if $|z - z_0|$ is small enough, then we can choose δ so small that

$$|f(z) - w_0| > \frac{|a_k|}{2}|z - z_0|^k.$$

Now, if $|w - w_0|^k < \frac{|a_k|}{2}\delta^k$ then we have that $|F(z)| > |G(z)|$, and Rouche's theorem concludes the argument. □

As we mentioned last time, there are some nice consequences of this fact.

Theorem 5.6 (Open Mapping Theorem)**.** *If f is holomorphic and not constant, the image of an open set is open.*

Proof. Take a neighborhood of z_0; this maps to a neighborhood of w_0, and we have in fact just shown that this is a k-to-1 mapping. □

Theorem 5.7 (Maximum Modulus Principle)**.** *If f is holomorphic and not constant then $|f(z)|$ has no maximum on Ω.*

We'll give a couple of proofs of this.

Proof 1. Suppose that z_0 is a maximum for $|f|$. Then a neighborhood of z_0 is sent to a neighborhood of w_0, but that would mean that we can always find a point in a neighborhood of w_0 that has larger absolute value, and that is a contradiction. □

Proof 2. This is based on the Cauchy formulas. We can find a circle of sufficiently small radius centered at z_0 and contained in Ω. Then we can write

$$f(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{z - z_0}\,dz \le \frac{\left(\sup_{|z-z_0|=r} |f(z)|\right)}{2\pi r}(2\pi r).$$

This means that there is a point on the circle which is at least as large as the point z_0. This is also a strict inequality unless $f(z)$ is constant on $|z - z_0| = r$, which means that f is constant (for example, by the Cauchy-Riemann equations). □

Now we will consider general curves γ (not just circles). We'll start with a particular example which is already quite important.

Pick a point $a \notin \Omega$, and consider the holomorphic function $\frac{1}{z-a}$. What can we say about

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z - a}\,dz = W_\gamma(a)$$

for some closed curve γ in Ω? This is called the *winding number* of γ around a or the *index* of γ around a.

Example 5.8. We have already seen that if γ is a circle, we have

$$W_\gamma(a) = \begin{cases} 1 & a \text{ inside circle} \\ 0 & a \text{ outside circle.} \end{cases}$$

What we have in mind is if we have some sort of crazy curve, then given some point a, we should be able to compute how many times the curve goes around that point.

Proposition 5.9. *This winding number $W_\gamma(a)$ is always an integer.*

Proof. Consider some closed curve γ parametrized via $\gamma : [0, 1] \to \gamma(t)$. Then we have

$$\frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t) - a}\,dt.$$

If we had a notion of log, it would be clear how to proceed. But we don't have such a notion, so we do this by computation. Define

$$G(y) = \frac{1}{2\pi i}\int_0^y \frac{\gamma'(t)}{\gamma(t) - a}\,dt.$$

This satisfies $G'(y) = \frac{1}{2\pi i}\frac{\gamma'(y)}{\gamma(y)-a}$. Then let

$$H(y) = \exp(-2\pi i G(y))(\gamma(y) - a),$$

and we have

$$H'(y) = (\gamma'(y) - 2\pi i G'(y)(\gamma(y) - a))\exp(-2\pi i G(y)).$$

Basically, we're thinking about logs without explicitly saying it. Then $H(1) = H(0)$, which means that $\exp(-2\pi i G(1)) = 1$. This means that $G(1)$ is an integer. $\square$

Proposition 5.10. *$W_\gamma(a)$ is locally constant. This means that if $a \notin \gamma$ then we can find a small neighborhood of a where $W_\gamma(a)$ is constant.*

Proof. We have

$$W_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - a}.$$

View this as a function of a. This is clearly a continuous function in a, but it also takes integer values, so it must be constant in a neighborhood of a. $\square$

This yields the following property of the winding number.

Proposition 5.11. *The complement of γ is a region in $\mathbb{C}$ which splits into many connected components. The winding number $W_\gamma(a)$ is constant on each connected component, and exactly one connected component is unbounded. The winding number in that connected component is* 0.

Proposition 5.12. *Suppose that f is meromorphic in a disc Ω, and suppose that γ is a closed curve in Ω. Then γ is inside a compact disk inside Ω, and we only consider poles inside this compact disk.*

$$\frac{1}{2\pi i}\int_\gamma f\,dz = \sum_{z_j \text{ poles}} W_\gamma(z_j)(\operatorname{Res}_{z_j} f).$$

Proof. The proof is precisely the same as what we did before with the Residue Theorem. It's just that in what we did before, the winding number of a circle around a point in the circle was always equal to one. $\square$

We can also think about the argument principle. Look at a closed curve $f(\gamma)$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt = \frac{1}{2\pi i}\int \frac{(f(\gamma(t)))'}{f(\gamma(t))}\,dt = W_{f(\gamma)}(0).$$

In this setting, we can give another proof of Rouche's theorem.

Proof of Rouche's Theorem 4.8. Take a circle C. We look at the function $F = \frac{f+g}{f}$, and consider the curve $F(C)$. We want to know how many times it winds around 0. That will give us that # poles of f − # zeros of $f = 0$, which will be Rouche's theorem.

We know that $\frac{f+g}{f} = 1 + \frac{g}{f}$, and on the circle C, we have $\frac{g}{f}$ is less than 1 in size. This curve therefore avoids 0, and we can show that the point 0 belongs to the unbounded component, and hence the winding number is zero. $\square$

Now we return to our discussion of Cauchy's theorem for general contours.

Definition 5.13. Suppose that we are in a region Ω, and we want to consider two curves γ_0 and γ_1 with the same endpoints. We say that these are *homotopic* in Ω if there is a continuous family of curves $\gamma_s : [0,1] \to \mathbb{C}$ in Ω with $0 < s < 1$, satisfying

(1) γ_0 is the starting curve and γ_1 is the ending curve.
(2) $\gamma_s(t)$ is continuous is both s and t.

The point is that we can continuously deform γ_0 to get γ_1.

Theorem 5.14. *If f is holomorphic in Ω and if γ_0 and γ_1 are homotopic in Ω then we have $\int_{\gamma_0} f = \int_{\gamma_1} f$.*

We will prove this later. We first have an important definition:

Definition 5.15. A region Ω is called *simply connected* if any two curves with the same endpoints are homotopic in Ω.

We can state a corollary of the preceding (yet to be proved) theorem.

Corollary 5.16. *For any closed curve γ and simply connected region Ω, we have $\int_\gamma f\,dz = 0$ for any holomorphic function f.*

Proof. The curve γ consists of two homotopic curves with opposite directions, so by the theorem, the integral over the two curves will cancel out. $\square$

Corollary 5.17. *Every holomorphic function has a primitive.*

Corollary 5.18. *By considering $f(z) = \frac{1}{z-a}$ for $a \notin \Omega$, we have $W_\gamma(a) = 0$ for all $\gamma \subset \Omega$ and $a \notin \Omega$.*

6. 10/13

We need to prove the following theorem:

Theorem 6.1. *If γ_0 and γ_1 are homotopic and f is holomorphic on a simply connected region Ω, then $\int_{\gamma_0} f = \int_{\gamma_1} f$.*

Recall that we have the following corollary that is a generalized version of Cauchy's theorem:

Corollary 6.2. *If Ω is simply connected then $\int_\gamma f\,dz = 0$ for every closed curve γ in Ω.*

Proof of Theorem 6.1. Consider $K = \{\gamma_s(t) : 0 \le s \le 1, a \le t \le b\}$. This is a closed and bounded set in Ω. Find ε small enough so that every circle of radius ε, centered at a point in K is a subset of Ω.

Also, observe that $\gamma_s(t)$ is uniformly continuous, so we can find a small number δ such that $0 \le s_1 - s_2 \le \delta$ implies that $|\gamma(s_1)(t) - \gamma(s_2)(t)| \le \frac{\varepsilon}{10}$. We want to show that

$$\int_{\gamma_{s_1}} (t) = \int_{\gamma_{s_2}} f.$$

This would suffice to prove the theorem.

Draw a finite number of small discs, all contained in Ω, covering all of γ_{s_1} and γ_{s_2}. Let these disks be $D_0, D_1, \dots, D_n$. Pick points z_j on γ_{s_1} and w_j in γ_{s_2} such that $z_j, w_j \in D_{j-1} \cap D_j$.

On D_j, we know that f has a primitive F_j. A priori, the primitives might differ on different D_j. However, on the intersections $D_j \cap D_{j+1}$, the primitives must differ only by a constant. That means that $F_{j+1}(z_{j+1}) - F_j(z_{j+1}) = F_{j+1}(w_{j+1}) - F_j(w_{j+1})$.

Now, consider

$$\int_{z_j}^{z_{j+1}} f\,dz = F_j(z_{j+1}) - F_j(z_j)$$
$$\int_{w_j}^{w_{j+1}} f\,dz = F_j(w_{j+1}) - F_j(w_j).$$

Then everything telescopes:

$$\begin{aligned}\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{j=0}^{n-1} \left(\int_{z_j}^{z_{j+1}} f - \int_{w_j}^{w_{j+1}} f \right) \\ &= \sum_{j=0}^{n-1} (F_j(z_{j+1}) - F_j(z_j) - F_j(w_{j+1}) + F_j(w_j)) \\ &= -F_0(z_0) + F_0(w_0) + F_{n-1}(z_n) - F_{n-1}(w_n) = 0\end{aligned}$$

because $z_0 = w_0$ and $z_n = w_n$ (because they are just the endpoints). $\square$

So now we have a somewhat general form of Cauchy's theorem. We should probably give some examples of simply connected regions.

Example 6.3. The following regions are simply connected:

- Any convex region.

- Any star-like region. A star-like region is a region such that there exists $z_0 \in \Omega$ such that the line joining any point to z_0 is in Ω.
- The unit disk.
- Any strip.

The punctured disk is not simply connected.

Proposition 6.4. *If f is holomorphic and nonzero on a simply connected region Ω, then we can write $f(z) = \exp(g(z))$ for a holomorphic function g. (We also have that $f(z) = h(z)^2$ for some holomorphic function h.*

Proof. Observe that $\frac{f'}{f}$ is a holomorphic function on Ω. Then we can pick a point z_0 and look at the function

$$F(z) = \int_{z_0 \text{ to } z} \frac{f'}{f}(w)\, dw.$$

This is well-defined because the integral is independent of path. Then note that $F'(z) = \frac{f'}{f}(z)$. Observe that

$$(f(z)\exp(-F(z)))' = 0,$$

which implies that $f(z) = \exp(F(z))$ (up to a factor of a constant).

To define square roots, write $f(x) = \exp(g(z))$ and look at $h(z) = \exp(g(z)/2)$, which is the desired square root. $\square$

Another nice region that is simply connected is the slit complex plane $\mathbb{C} \setminus [-\infty, 0]$. To see this, take the point 1; then any straight line from 1 to any point is in the region.

In the slit complex plane, we can define a holomorphic function $\log z$. This construction gives us the *principal branch of* log, defined where $-\pi < \operatorname{Im} \log z < \pi$. Similarly, on the same region, we can define $\sqrt{z}$.

To imagine the logarithm, it is helpful to think of infinitely many copies of this slit complex plane. This is a Riemann surface, and we will discuss this later.

Proposition 6.5. *The following are equivalent characterizations of simply connected regions:*

1. *Any two curves with the same endpoints are homotopic.*
2. *If f is holomorphic and γ is a closed curve in Ω then $\int_\gamma f\, dz = 0$.*
3. *If $a \notin \Omega$ then the winding number $W_\gamma(a) = 0$.*
4. *The complement of Ω in the extended complex plane is connected.*

It is clear that (1) $\implies$ (2) $\implies$ (3). We won't be able to see that anything implies (1) for awhile. Let's think about the result (4).

Example 6.6. Consider some infinite strip in $\mathbb{C}$. This seems to divide the plane into two portions, but the complement is actually connected by the point at infinity.

Consider the unit disk, and let Ω be the outside of the unit disk. The complement of this $\mathbb{C}$ seems to be connected, but it is actually missing the point at infinity. Therefore, the complement of Ω is actually not simply connected.

Proof. We will prove that (4) $\implies$ (3). Suppose we have a region Ω with connected complement in the extended complex plane. Then we have $\gamma \subset \Omega$. Note that the winding number is constant in Ω^c, and the winding number around infinity is 0. $\square$

We will now extend integrals $\int_\gamma$ over curves to integrals over chains and cycles.

Definition 6.7. A *chain* is a finite sum of curves with coefficients in $\mathbb{Z}$.

Then we can define $\int_{3\gamma_1 - 2\gamma_2} = 3\int_{\gamma_1} - 2\int_{\gamma_2}$. If we add two curves that share an endpoint, the sum of the curves can be thought as simply one longer curve.

Definition 6.8. A *cycle* is a finite sum of curves with coefficients in $\mathbb{Z}$ such that the curves together are a union of closed curves.

Definition 6.9. A cycle γ is *homologous* to 0 in Ω if $W_\gamma(a) = 0$ for all $a \notin \Omega$. If this is the case, we write $\gamma \sim 0$.

Example 6.10. For example, if Ω is an annulus, a cycle that is homologous to zero would be a closed curve that does not go around the hole.

We want to prove that (3) $\implies$ (2) in proposition 6.5.

Theorem 6.11. *If f is holomorphic in Ω and $\gamma \sim 0$ in Ω then $\int_\gamma f\,dz = 0$.*

We also want to show that (3) $\implies$ (4) in 6.5

Theorem 6.12. *If for every cycle γ in Ω we have $\gamma \sim 0$ in Ω then the complement of Ω in the extended plane is connected.*

Proof. Assuming that Ω^c is not connected in the extended complex plane. We want to show that there exists a cycle γ with $a \notin \Omega$ such that $W_\gamma(a) \neq 0$.

Write $\Omega = A \cup B$ with A and B as closed, nonempty, disjoint sets. Suppose that A contains ∞ and B is bounded. Draw a small grid of very small side length δ. Here, δ should be so small that the distance between A and B should be $\geq 3\delta$. We can do this because A and B are closed sets.

We want to use a curve between A and B. Consider the union of all squares having at least one point in B. Let a be some point in B in the middle of some such square. Orient each square counterclockwise. Our cycle will be the union of the boundaries of all of these squares, oriented counterclockwise.

We are inside a bounded region, so there is only a finite number of squares to consider. If we have two squares that share an edge, that edge would cancel, and no internal edges remain. We are left with an outer polygon. Maybe B has some holes, and there are some small interior polygons too, but in any case, there is a finite number.

Therefore, we have a cycle composed of finitely many polygons. What is the winding number of the cycle around the point a? It is the sum of winding numbers of each square around a, but this is equal to 1 for the square containing a, and equal to 0 for all other squares. Here, we are assuming that $\frac{1}{2\pi i}\int_\square \frac{dz}{z-a} = 1$, which is true because the square is contained in a circle and we can apply Cauchy's theorem. $\square$

What we've proved is actually a bit more general. If Ω^c has n connected components $A_1, \ldots, A_n$, where A_n contains ∞, then we can find $n-1$ cycles having winding number 1 around points in $A_1, \ldots, A_{n-1}$.

What can we do with Theorems 6.11 and 6.12?

Consider Ω as a region that is an annulus within an annulus. Then the complement Ω^c has an unbounded component and two bounded components. Let the bounded components be A_1 and A_2. Then we get a cycle γ_1 with winding number 1 on γ_1 and 0 on γ_2, and a cycle γ_2 with winding number 1 on γ_2 and 0 on γ_1.

Take any cycle γ. It has winding number n_1 on A_1 and n_2 on A_2. Consider $\gamma - n_1\gamma_1 - n_2\gamma_2$. This has winding numbers 0 on A_1 and 0 on A_2, so therefore $\gamma - n_1\gamma_1 - n_2\gamma_2 \sim 0$. Hence

$$\int_\gamma f = n_1 \int_{\gamma_1} f + n_2 \int_{\gamma_2} f,$$

and it suffices to understand only two integrals. The number of integrals that we need to study is equal to the number of bounded connected components. We say that $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$ are the *periods* of the function f.

7. 10/18

Recall from last time that we were interested in equivalent characterizations of simply connected regions 6.5, namely

(1) Any two curves with the same endpoints are homotopic.
(2) If $a \notin \Omega$ then the winding number $W_\gamma(a) = 0$.
(3) If f is holomorphic and γ is a closed curve in Ω then $\int_\gamma f\,dz = 0$.
(4) The complement of Ω in the extended complex plane is connected.

It remains to prove that (2) $\implies$ (3).[1] This is similar to a proof from last time.

Proof. Suppose that γ is a cycle that is homologous in some region to 0, which means that $W_\gamma(a) = 0$ for every $a \notin \Omega$. We want to show that $\int_\gamma f\,dz = 0$ for all holomorphic f.

Consider the region Ω and a cycle γ. Divide the region into squares of small length δ. Let Ω_δ be the union of all squares strictly contained in Ω. Choose δ small enough that $\gamma \subset \Omega_\delta$. Let Γ_δ denote the boundary of Ω_δ, which is some sort of polygonal curve. This is a cycle, and it might have several (finite, as there are only finitely many squares) parts. Suppose that $z \in \Omega_\delta$. Then $f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(w)}{w-z}\,dw$. This is a sum of integrals over squares, and exactly one square contains z.

Now, we want to evaluate

$$\int_\gamma f(z)\,dz = \int_\gamma \left(\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(w)}{w-z}\,dw\right)\,dz.$$

Note that the cycle γ is bounded from Γ_δ, so we can interchange the integrals to get

$$\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_{\Gamma_\delta} f(w)\left(\int_\gamma \frac{dz}{w-z}\right)\,dw = -\int_{\Gamma_\delta} f(w)W_\gamma(w)\,dw.$$

It therefore suffices to show that $W_\gamma(w)$. By definition of Γ_δ, the square containing w also contains points outside of Ω, but the assumption was that $W_\gamma(a) = 0$ for all $a \notin \Omega$. Therefore $W_\gamma(w) = W_\gamma(a) = 0$. $\square$

We will come back to finish showing 6.5 once we have considered the Riemann mapping theorem.

Theorem 7.1 (Riemann Mapping Theorem)**.** *Every simply connected region is either the entire complex plane or there is a holomorphic bijection of it to the unit disk.*

[1]See theorem 15 in Ahlfors, in the relevant chapter.

Remark. Given this, the inverse function is also holomorphic, and we know that any two curves are homotopic. We should of course be careful not to use this in the proof. In fact, all we need is that any nonvanishing holomorphic function on this region has a logarithm or a square root. That only requires that the integral of any holomorphic function around any cycle is zero.

Here is an analog of Weierstrass's theorem in real variables regarding approximating any function by a polynomial:

Theorem 7.2 (Runge's theorem)**.** *If f is holomorphic in a region Ω containing a compact set K then we can find a sequence of rational functions R_n all of whose poles lie outside K such that $R_n \to f$ uniformly on K.*

Remark. In fact, we can arrange for the poles of R_n to lie in any region which meets each connected component of the complement K^c of K in the extended complex plane.

In particular, if the complement of K is connected, we can make all of the poles be at ∞, so the rational functions R_n are actually polynomials.

This means that given a holomorphic function on a compact set inside a simply connected region, we can approximate that function by polynomials.

Proof. Consider some region, and once again, divide it into small squares so that $K \subseteq \Omega_\delta$, where Ω_δ is the union of all squares contained in Ω. Then we have a cycle Γ_δ composed of finitely many polygons in Ω_δ, keeping away from the finite set K. For any $z \in \Omega_\delta$, we have that

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(w)}{w - z}\, dw.$$

Restrict z to be inside our compact set K. Then this is the integral of a nice continuous function because $w - z$ is bounded away from 0. Replace the integral using a Riemann sum approximation. This is just a rational function of z, with poles on the cycle Γ_δ outside of K. Thus, we get a rational function all of whose poles lie in K^c. ∎

Example 7.3. Suppose that the complement K^c is connected. Take a point $z_0 \in K^c$ and consider the holomorphic function $\frac{1}{z-z_0}$, which has a pole at z_0. We want to move this pole from z_0 to any other point z_1, so we approximate this function by functions whose only poles lie at z_1, i.e. sums of $\frac{1}{(z-z_1)^n}$.

Take a point w very close to z_0, and choose w such that $|z_0 - w| < \frac{1}{2}|z - w|$ for all $z \in K$.

$$\frac{1}{z - z_0} = \frac{1}{z - w + (w - z_0)} = \frac{1}{(z - w)\left(1 + \frac{w - z_0}{z - w}\right)} = \frac{1}{z - w} \sum_{n=0}^{\infty} \left(\frac{z_0 - w}{z - w}\right)^n.$$

Now, we can do this finitely many times to push a pole at z_0 to a pole at z_1.

We can check that when K^c is connected, we can push poles to ∞ and get polynomials. In particular, if $z \in K$ and w is large, we can write $\frac{1}{z-w} = -\frac{1}{w(1-z/w)} = -\frac{1}{w} \sum (\frac{z}{w})^n$.

This gives yet another property of a simply connected region: every holomorphic function can be approximated by polynomials.

Next, we will examine properties of holomorphic functions, especially entire functions, which are holomorphic in the whole complex plane.

There is a loose end from our earlier discussion. Our first goal will be to show that any meromorphic function can be written as a quotient of two holomorphic functions. We will prove Weierstrass's theorem on functions with prescribed zeros, and we will consider the growth of entire functions, and the effect that this growth has on the zeros of the function.

Proposition 7.4. *Suppose that f_n is a sequence of holomorphic functions on Ω, and $f_n \to f$ uniformly on a compact set $K \subseteq \Omega$. Then f is also holomorphic, and $f_n' \to f'$ uniformly on compact sets.*

Remark. This is not true in real analysis for continuous functions or differentiable functions, but it is true here.

Proof. Let C be a circle contained in Ω and let z be inside the circle. Then

$$\varphi_n(z) = \frac{1}{2\pi i} \int_C \frac{f_n(w)}{w - z}\, dw.$$

We can take limits on both sides to get

$$\varphi(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z}\, dw,$$

and from this expression, it is clear that f can be differentiated and hence f is holomorphic. □

Example 7.5. Here is an example of what we can use this to do. Define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$ and $\sigma > 1$. This is an absolutely convergent series, so the preceding result tells us that $\zeta(s)$ is holomorphic in this half-plane. This also means that we can differentiate $\zeta(s)$ term-by-term:

$$\zeta'(s) = \sum_{n=1}^{\infty} \frac{-\log n}{n^s}.$$

Our immediate goal is to construct functions with zeros at some specified points. Given a region Ω and a sequence of points $z_n \in \Omega$ with multiplicities m_n, we want to find a function f having zeros of order m_n at z_n, and no other zeros. (Note that this is only possible if z_n has no limit points in Ω.)

We might try to do this by constructing a product of $z - z_n$, so we have to discuss infinite products.

Consider an *infinite product* $\prod_{n=1}^{\infty}(1 + a_n)$ for $a_n \in \mathbb{C}$. We say that this converges if the partial products $\prod_{n=1}^{N}(1 + a_n)$ converges. For this to converge, a_n needs to become small, so we can approximate $(1 + a_n) \approx e^{a_n}$, which relates the convergence of $\prod_{n=1}^{\infty}(1 + a_n)$ to the convergence of $\sum a_n$.

Definition 7.6. If $\sum |a_n|$ converges then we say that $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely.

Example 7.7. $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n}\right)$ "converges" to 0, but not absolutely.

If a product converges absolutely, then the product is nonzero, unless one of the terms is zero.

Theorem 7.8 (Weierstrass's theorem for $\mathbb{C}$)**.** *Consider a sequence of points $z_n \in \mathbb{C}$ (maybe repeated finitely many times) with $|z_n| \to \infty$ (i.e. no limit points). Then there is a holomorphic function with zeros at these points and nowhere else.*

Proof. We want a function vanishing at each z_n, so we want to consider the product $\prod(z-z_n)$, and we want to make this convergent. An equivalent thing to look at is $\prod(1-\frac{z}{z_n})$. There is a slight problem here if $z_n = 0$ for some n, so let's assume that $z_n \neq 0$; if there were some zeros at 0, simply multiply by z^m at the end. Again, however, we don't know that this product converges. We want to multiply it by something without any zeros, and the best way to do this is to take the exponential of something.

Define Weierstrass factors

$$E_k(z) = (1-z) \cdot \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right).$$

The function that we want is then

$$f(z) = z^{m_0} \prod_{n=1}^{\infty} E_n\left(\frac{z}{z_n}\right).$$

Clearly, this has zeros at z_n and nowhere else. The question is: Does this converge? Since $|z_n| \to \infty$, the numbers $\frac{z}{z_n}$ are small, so we really only care about $E_k(w)$ where $|w|$ is small.

When $|w| < 1$, we have

$$\log(1-w) = -\sum_{n=1}^{\infty} \frac{w^n}{n}.$$

Therefore,

$$E_k(w) = \exp\left(-\sum_{k+1}^{\infty} \frac{w^n}{n}\right),$$

and hence $|E_k(w) - 1| \leq C|w|^{k+1}$. This constant C does not depend on k. Now, we are done because

$$\sum_{n \text{ large}} \left|E_n\left(\frac{z}{z_n}\right) - 1\right| \leq C \sum_{n \text{ large}} \left|\frac{z}{z_n}\right|^{n+1}$$

converges absolutely. $\square$

This was the theorem for $\mathbb{C}$; next time, we'll prove this for a general region Ω.

8. 10/20

Last time, we proved Weierstrass's theorem for $\mathbb{C}$. Now, we want to prove something similar for a region in $\mathbb{C}$.

Theorem 8.1 (Weierstrass's theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be some region, and suppose that $z_1, z_2, \ldots$ be a sequence of points with no limit point in Ω. We want to find function f with zeros at these points and nowhere else.*

Once we prove this, we will have shown that the meromorphic functions on a region Ω is simply a quotient of holomorphic, since we can multiply by a function f with zeros in the right places.

Proof. Pick a point a in it which is not one of the zeros. Pick δ small enough so that $|z-a|<\delta$ is in Ω and has no points z_n. Let $Tz=\frac{1}{z-a}$, which sends $\Omega \to T\Omega$ containing all z with $|z|>\frac{1}{\delta}$ and such that all $\frac{1}{z_n-a}$ lie inside $|z|<\frac{1}{\delta}$.

Suppose we find f which is holomorphic on $T\Omega$ with zeros at $\frac{1}{z_n-a}$ and nowhere else, and $f(z)\to 1$ as $z\to\infty$. Then we can take $g(z)=f(\frac{1}{z-a})$. This will have the property that g will have zeros at the points z_n, so g is holomorphic on $\Omega\setminus\{a\}$ with zeros at z_n. The singularity at a is removable.

It is now sufficient to look at regions $\Omega\neq\mathbb{C}$ which contain all $|z|>R$ for some R and a sequence $z_1,z_2,\dots$ of points all satisfying $|z_n|<R$.

Note that $\mathbb{C}\setminus\Omega$ is closed and bounded, and hence compact. For each n, we can therefore find a point $w_n\in\mathbb{C}\setminus\Omega$ with $|z_n-w_n|=\min_{w\in\mathbb{C}\setminus\Omega}|z_n-w|$. As $n\to\infty$, we then have $|z_n-w_n|\to 0$ because z_n has no limit points in Ω, so they must get close to the complement $\mathbb{C}\setminus\Omega$, now, define

$$\prod_n E_n\left(\frac{z_n-w_n}{z-w_n}\right).$$

This has zeros at $z=z_n$. Why does this converge? Suppose that $z\in K\subseteq\Omega$ for some compact K. On this compact set, as $n\to\infty$, note that $|z-w_n|\geq$ distance between K and $\mathbb{C}\setminus\Omega$. Then once $|\frac{z_n-w_n}{z-w_n}|<\frac{1}{2}$, we can use

$$\left|E_n\left(\frac{z_n-w_n}{z-w_n}\right)-1\right|\leq C\left|\frac{z_n-w_n}{z-w_n}\right|^{n+1}.$$

Lastly, as $z\to\infty$, $\left|\frac{z_n-w_n}{z-w_n}\right|\to 0$ for each n, so the product goes to 1. That complete the proof of Weierstrass's theorem. This was the same construction as for $\mathbb{C}$, but we had to do something at the beginning. $\square$

There is a nicer version of this due to Hadamard. It isn't as general as this, and gives information of a different kind. These are called *entire functions of finite order*. The main idea here is to make some kind of growth assumption on f.

Definition 8.2. We say that f has order at most ρ if for all z we have $|f(z)|\leq C\exp(|z|^\rho)$. The order of f is ρ means that $\rho=\inf_{\rho_1}(f$ has order $\leq\rho_1)$. This means that

$$|f(z)|\leq C_\varepsilon\exp\left(|z|^{\rho+\varepsilon}\right)$$

for any $\varepsilon>0$.

Example 8.3. e^z has order 1, and $\sum_{n=0}^\infty\frac{z^n}{(2n)!}$ has order $\frac{1}{2}$.

Theorem 8.4 (Hadamard's theorem)**.** *Suppose that f is entire and has order $k\leq\rho<k+1$. Then*

$$f(z)=z^m\prod_{n=1}^\infty E_k\left(\frac{z}{z_n}\right)\exp(g(z))$$

where $g(z)$ is a polynomial of degree k.

Remark. Note that instead of having a different E_n for each n, we just have a single function E_k.

Corollary 8.5. *If f is entire of finite order and f has no zeros, then f is the exponential of a polynomial.*

The main thing is: What extra information can be extracted from the fact that f has finite order? This is a nice general principle: The growth of f controls its zeros. Here is the general result.

Proposition 8.6 (Jensen's formula)**.** *Suppose that f is holomorphic in a region containing the disk of radius R centered at 0 (and $f(0) \neq 0$). Suppose that $z_1, \ldots, z_N$ are the zeros of f in $|z| < R$, and there are no zeros on $|z| = R$. Then*

$$\log|f(0)| + \sum_{n=1}^{N} \log \frac{R}{|z_n|} = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta.$$

This is saying that some weighted count of zeros is dominated by the size of f.

Proof. Note that both sides of the formula are additive, so if this is true for f_1 and f_2 then it is also true for $f_1 \cdot f_2$.

Suppose first that f has no zeros. Then we can write $f(z) = \exp(g(z))$. Then we want to check that

$$\operatorname{Re} g(0) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} g(e^{i\theta})\, d\theta.$$

Note that

$$g(0) = \frac{1}{2\pi i} \int_{|z|=R} g(z) \frac{dz)}{z} = \frac{1}{2\pi i} \int_0^{2\pi} g(e^{i\theta})\, d\theta,$$

so this case is done.

Now, suppose that $f(z) = z - z_1$ for some $|z_1| < R$. We want to show that

$$\log|z_1| + \log \frac{R}{|z_1|} = \frac{1}{2\pi} \int_0^{2\pi} \log|Re^{i\theta} - z_1|\, d\theta = \log R + \frac{1}{2\pi} \int_0^{2\pi} \log\left|1 - \frac{z_1}{Re^{i\theta}}\right|\, d\theta.$$

Note that

$$\log\left|1 - \frac{z_1}{Re^{i\theta}}\right| = -\operatorname{Re} \sum_{k=1}^{\infty} \frac{1}{k} \frac{z_1^k}{R^k e^{ik\theta}},$$

and this checks Jensen's formula because $\int_0^{2\pi} e^{-ik\theta} = 0$. $\square$

Let $n_f(r)$ denote the number of zeros of f with $|z| < r$. Then

$$\int_0^R n_f(r) \frac{dr}{r} = \sum_{z_n} \int_{|z_n|}^R \frac{dr}{r} = \sum_{z_n} \log \frac{R}{|z_n|}.$$

Now, suppose that f has finite order $\leq \rho$ and $f(0) \neq 0$, so that $|f(z)| \leq C \exp(|z|)^\rho$. Then

$$\sum_{|z_n| \leq R} \log \frac{R}{|z_n|} \leq C_1 + R^\rho$$

for all R and some constant C_1. Therefore,

$$\int_0^R n_f(r) \frac{dr}{r} \geq n\left(\frac{R}{2}\right) \int_{R/2}^R \frac{dr}{r} = (\log 2) n(R/2).$$

Corollary 8.7. *$n(R) \le C_1 R^\rho + C_2$ for all $R > 0$.*

Corollary 8.8. *If f has order ρ then $n(R) \le C_1 R^{\rho+\varepsilon}$ for all $R \ge 1$.*
If z_n are the zeros of f and $\sum \frac{1}{|z_n|^s} < \infty$ provided that $s > \rho$.

Proof. Consider the zeros in an annulus $2^j \le |z_n| < 2^{j+1}$. The number of zeros in the annulus is $\le C(2^{j+1})^{\rho+\varepsilon}$. Then

$$\frac{1}{|z_n|^s} \le \sum_j \frac{1}{2^{js}} C_1 \left(2^{j+1}\right)^{\rho+\varepsilon}$$

if $s > \rho$ and ε is small enough. $\square$

Suppose that $k \le \rho < k+1$. Consider $E(z) = z^m \prod_{n=1}^\infty E_k(\frac{z}{z_n})$. We claim that this actually converges uniformly on compact sets and hence is some holomorphic function. To do this, consider

$$\left| E_k\left(\frac{z}{z_n}\right) - 1 \right| \le c \left|\frac{z}{z_n}\right|^{k+1},$$

and so $\sum |E_k(z/z_n) - 1|$ converges since $\sum \frac{1}{|z_n|^{k+1}}$ converges.

Exercise 8.9. Check that the order of E is ρ.

At the moment, we can say that any function f which is entire and has finite order ρ with zeros $\{z_n\}$ looks like $E(z) \cdot \exp(g(z))$. We still have to show that g is a polynomial. Look at the function

$$\frac{f(z)}{E(z)} = \exp(g(z)).$$

We know that the numerator f has growth of finite order. We want to bound the growth of $\frac{f(z)}{E(z)}$. This is very delicate, and we want lower bounds on the denominator, but the denominator does vanish occasionally. We will do this on a special class of circles $|z| = r_n$ with certain values $r_n \to \infty$.

There will be two steps to do this. First, on this special sequence $|z| = r_n$, we can produce a bound

$$\left|\frac{f(z)}{E(z)}\right| \le C \exp\left(r_n^{\rho+\varepsilon}\right).$$

We then want to deduce that g is a polynomial.

We understand where the zeros of $E(z)$ are, and we know that there are not too many zeros. We have a bound on the number of bad radii containing zeros, and there are not too many of them. We can therefore pick a radius which is a bit away from each of the bad radii. We will choose the r_n so that they are bounded away from $|z_n|$. This can be done using the Pigeonhole principle, and we can find a point that is $R^{-\rho-\varepsilon}$ away from each bad radius.

9. 10/25

We were working towards Hadamard's theorem, which deal with functions that are entire of finite order ρ. We are looking for a factorization formula for writing f in terms of its zeros, sort of like polynomial factorizations. Let $k \le \rho < k+1$ for $k \in \mathbb{N}$.

What we showed was that a function of finite order does not have too many zeros; the number of zeros only grows polynomially.

Recall that we have
$$E(z) = z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{z_n}\right),$$
where we know that for z_n large we have
$$\left|E_k\left(\frac{z}{z_n}\right) - 1\right| \leq C_1 \left|\frac{z}{z_n}\right|^{k+1}.$$
We want to check that this has order ρ.

Proposition 9.1. *$E(z)$ has order ρ.*

Proof. Suppose $|z| = R$. Then
$$\left|\prod_{|z_n|>2R}\left(E_k\left(\frac{z}{z_n}\right)\right)\right| \leq C_1 \exp\left(C \sum_{|z_n|>2R}\left|\frac{z}{z_n}\right|^{k+1}\right) \leq C\exp\left(C_1 \sum_{|z_n|>2R}\left|\frac{z}{z_n}\right|^{s}\left|\frac{z}{z_n}\right|^{k+1-s}\right)$$
$$\leq C\exp(C_1|z|^s)$$
for any $s > \rho$. This is because each of the higher order terms is $\leq \frac{1}{2}$, so we can ignore them. Next, we must consider
$$\left|\prod_{|z_n|<2R}\left(E_k\left(\frac{z}{z_n}\right)\right)\right| \leq \prod_{|z_n|<2R}\left|1-\frac{z}{z_n}\right| \exp\left(\sum_{j=1}^{k}\left|\frac{z}{z_n}\right|^j \frac{1}{j}\right)$$
$$\leq \exp(C(2R)^{\rho+\varepsilon}) \exp\left(\sum_{j=1}^{k}\frac{1}{j}\sum_{|z_n|\leq 2R}\left|\frac{z}{z_n}\right|^j\right)$$
where we note that $|1-\frac{z}{z_n}| \leq C$ for some constant. Now,
$$\sum_{|z_n|<2R}\left|\frac{z}{z_n}\right|^j \leq \sum_{|z_n|<2R}\left|\frac{z}{z_n}\right|^s \left|\frac{z_n}{z}\right|^{s-j} \leq 2^{s-j}|z|^s\left(\sum \frac{1}{|z_n|^s}\right).$$
Therefore, the contribution from the small zeros is also bounded by $\exp(C|z|^s)$ for any $s > \rho$. Therefore, we have shown that the order of $E(z)$ is $\leq \rho$. In this case, we will not prove that $E(z) = \rho$. $\square$

Remark. The point is that we made two estimates, of the large zeros and the small zeros. Both were bounded by $CR^{\rho+\varepsilon}$.
$$\sum_{|z_n|>2R}\left|\frac{z}{z_n}\right|^{k+1} \leq CR^{\rho+\varepsilon}$$
$$\sum_{|z_n|<2R}\left|\frac{z}{z_n}\right|^{j} \leq CR^{\rho+\varepsilon}.$$

Therefore, we have shown that $E(z)$ is an entire function of order $\leq \rho$ and having zeros at z_n. We have that $\frac{f(z)}{E(z)}$ is holomorphic and nowhere zero, so it is $\exp(g(z))$ for some holomorphic $g(z)$. We hope to show that g is a polynomial of degree $< k$. The problem is that in general $E(z)$ is very small, and we worry that $\frac{f(z)}{E(z)}$ grows too quickly to be of finite order.

Proposition 9.2. *There is a sequence $r_n \to \infty$ of good radii such that if $|z| = r_n$ then $|E(z)| \geq \exp(-C|z|^{\rho+\varepsilon})$ for any $\varepsilon > 0$. This implies that*

$$\left|\frac{f(z)}{E(z)}\right| \leq \exp(cr_n^{\rho+\varepsilon}).$$

Proof. Pick some big number R, and suppose that $|z| < 2R$. There are at most $\leq CR^{\rho+\varepsilon}$ zeros of f or E. In the interval $[R, R+1]$, we can pick a radius r such that $|r - |z_n|| \geq \frac{c}{R^{k+1}}$ for all zeros z_n. We want to show that this r is a good radius.

This is like analyzing bounds for $E(z)$, except we want lower bounds where we had upper bounds. We split into two cases:

$$\prod_{|z_n|>2R} \left|E_k\left(\frac{z}{z_n}\right)\right| \geq c\exp\left(-C\sum_{|z_n|>2R}\left|\frac{z}{z_n}\right|^{k+1}\right) \geq c\exp\left(-CR^{\rho+\varepsilon}\right).$$

Now consider small zeros:

$$\prod_{|z_n|<2R} \left|E_k\left(\frac{z}{z_n}\right)\right| \geq \prod_{|z_n|<2R}\left|1-\frac{z}{z_n}\right|\exp\left(-c\sum_{j=1}^{k}\frac{1}{j}\left|\frac{z}{z_n}\right|^{j}\right) \geq \exp\left(-cR^{\rho+\varepsilon}\right)\prod_{|z_n|\leq 2R}\left(\frac{c}{R^{k+2}}\right)$$

Multiplying all of these together gives the desired result. □

Therefore, we know that on a good radius $r = r_n$, we have $|\exp(g(z))| \leq \exp(cr^{\rho+\varepsilon})$ for $|z| = r$. Hence on $|z| = r$, we have $\operatorname{Re} g(z) < Cr^{\rho+\varepsilon}$. Holomorphic functions are quite special in that controlling the real part actually controls the function. We want to use this to deduce that g is actually a polynomial of degree k.

Since g is entire, we know that $g(z) = \sum_{n=0}^{\infty} a_n z^n$. This converges for all values of z, which means that the a_n decreases rapidly. We want to show that $a_n = 0$ for $n \geq k+1$.

Write $z = re^{i\theta}$ where r is good, and write

$$g(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}.$$

Think of this as a Fourier series, and think of $a_n r^n$ as Fourier coefficients. Then for $n \geq 0$,

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})e^{-in\theta}\,d\theta.$$

We would like expressions of this type with $\operatorname{Re} g(z)$ instead of $g(z)$ because that's what we have information about. Then

$$\operatorname{Re}(g(re^{i\theta})) = \frac{1}{2}\left(\sum_{m=0}^{\infty} a_m r^m e^{im\theta} + \sum_{m=0}^{\infty}\overline{a_m} r^m e^{-im\theta}\right).$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}(\operatorname{Re} g(re^{i\theta}))e^{-in\theta}\,d\theta = \frac{a_n r^n}{2} + \begin{cases} 0 & n > 0 \\ \frac{\overline{a_0} r^0}{2} & n = 0. \end{cases}$$

This gives us

$$\operatorname{Re} a_0 = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re} g(re^{i\theta})\,d\theta$$

and taking $A = \max_\theta \operatorname{Re} g(re^{i\theta})$, or alternatively $A = Cr^{\rho+\varepsilon}$ to get

$$\frac{a_n r^n}{2} = \frac{1}{2\pi}\int_0^{2\pi} (\operatorname{Re} g(re^{i\theta}))e^{-in\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} (\operatorname{Re} g(re^{i\theta}) - A)e^{-in\theta}\,d\theta,$$

so that

$$\left|\frac{a_n r^n}{2}\right| \le \frac{1}{2\pi}\int_0^{2\pi} \left|A - \operatorname{Re} g(re^{i\theta})\right|\,d\theta \le \frac{1}{2\pi}\int_0^{2\pi} (A - \operatorname{Re} g(re^{i\theta}))\,d\theta = A - \operatorname{Re} a_0 \le Cr^{\rho+\varepsilon}.$$

Now we are done because we know that $|a_n r^n| \le Cr^{\rho+\varepsilon}$ is true for every good radius r, but we have a sequence of good radii $r_n \to \infty$. If $n > \rho + \varepsilon$, we therefore have $a_n = 0$. Recall that $k \le \rho < k+1$, which shows that $a_n = 0$ for all $n \ge k+1$. This completes the proof of Hadamard's Theorem.

Example 9.3. Here is an example of Hadamard's Theorem in action. Recall

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

This is entire of order 1 and its zeros are at all of the integers. We can now look at

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{n}\right)\exp\left(\frac{z}{n}\right)\cdot\exp(A + Bz),$$

and we can also determine the values of A and B. Alternatively, we can rewrite this as

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\exp(A + Bz).$$

Plugging in $z = 0$ shows that $A = 0$. Notice that $\frac{\sin(\pi z)}{\pi z}$ is even, which shows that in fact $B = 0$ too. This is a nice formula that we get for free from Hadamard's theorem.

Let's jump ahead and look at the Schwarz Lemma, which is a relative of the maximum modulus principle.

Lemma 9.4 (Schwarz Lemma)**.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ and f is holomorphic. Assume that $f(0) = 0$. Then we claim that $|f(z)| \le |z|$ for all $z \in \mathbb{D}$. Also, if $|f(z)| = |z|$ for some z then $f(z) = e^{i\theta}z$. Lastly, $|f'(0)| \le 1$ and if $|f'(0)| = 1$ then $f(z) = e^{i\theta}z$.*

Proof. We have $f(z) = z$, so therefore $\frac{f(z)}{z}$ is a holomorphic function. Therefore, we can apply the Maximum Modulus Principle to this. to get

$$\left|\frac{f(z)}{z}\right| \le \max_{|w|=r}\left|\frac{f(w)}{w}\right| \le 1$$

by letting $r \to 1^-$.

The second part follows easily since if $|\frac{f(z)}{z}| = 1$ then $\frac{f(z)}{z}$ needs to be a constant of size 1.

The last part follows from

$$f'(0) = \lim_{z\to 0}\frac{f(z) - f(0)}{z}. \qquad \square$$

Remark. The lesson here is that given a function with $f(0) = 0$, it is often nice to consider $f(z)/z$ and try to apply the Maximum Modulus Principle.

Now, consider f holomorphic on a region containing $|z| \leq R$. Assume for the moment that $f(0) = 0$; we can get rid of this later. Suppose that

$$\mathrm{Re}_{|z|=R} f(z) \leq A.$$

We will deduce from this that there is a bound on $|f(z)|$ for all z with $|z| = r < R$. The idea is that we can look at

$$g(z) = \frac{f(z)}{z(2A - f(z))}.$$

First, we claim that g is holomorphic inside $|z| \leq R$. To see this, we need see that $2A - f(z) \neq 0$. We can use Rouche's theorem 4.8 to do this. Now, use the Maximum Modulus Principle for this quantity. Then

$$\max_{|z|=r} \left| \frac{f(z)}{z(2A - f(z))} \right| \leq \max_{|z|=R} \left| \frac{f(z)}{z(2A - f(z))} \right| \leq \frac{1}{R}$$

because $\mathrm{Re}\, f(z) \leq A$. Therefore, we see that on $|z| = r$, we have

$$\frac{|f(z)|}{|2A - f(z)|} = \frac{r}{R},$$

and so

$$|f(z)| \leq \frac{r}{R} \left(2A + |f(z)|\right),$$

which implies that

$$|f(z)| \leq \frac{2Ar}{R} \cdot \frac{1}{1 - \frac{r}{R}} = \frac{2Ar}{R - r}.$$

This result is kind of extraordinary, and it is called the *Borel-Caratheodory Theorem.*

10. 10/27

We proved Hadamard's theorem, and this is quite pretty and useful. For example, one homework problem is to prove a baby version of Picard's theorem.

Now, we shall study some classical functions.

The Γ-function interpolates between values of factorial. There are a number of ways to define it, and here is one of them.

Definition 10.1.

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}.$$

In order for this integral to converge at 0, we need to require that $\mathrm{Re}\, s > 0$, and this is a holomorphic function in that region. To be pedantic, we can approximate this by a sequence of integrals converging uniformly on compact sets.

Integration by parts gives us

$$\Gamma(s+1) = \int_0^\infty t^s d(-e^{-t}) = t^s \times (-e^{-t})\Big|_0^\infty + \int_0^\infty s t^{s-1} e^{-t}\, dt = s\Gamma(s)$$

in the region $\mathrm{Re}\, s > 0$. This allows us to produce a meromorphic continuation of Γ, by defining $\Gamma(s) = \frac{\Gamma(s+1)}{s}$. This is a meromorphic continuation to $\mathrm{Re}\, s > -1$, and it is holomorphic except for a simple pole at $s = 0$. Also, $\operatorname{Res}_{s=0} \Gamma(s) = \Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$. In addition, from the functional equation, we see that $\Gamma(n) = (n+1)!$ for $n \in \mathbb{N}$.

We can continue this process and write

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{G(s+2)}{s(s+1)}$$

to get a meromorphic continuation to $\operatorname{Re} s > -2$, with another pole at $s = -1$, where it has residue $\operatorname{Res}_{s=-1} \Gamma(s) = \frac{\Gamma(1)}{-1} = -1$.

We can continue doing this to obtain a meromorphic continuation of Γ to $\mathbb{C}$ and simple poles at $s = 0, -1, -2, \dots$. The residue at $s = -n$ is then $\frac{(-1)^n}{n!}$.

Another remarkable fact about $\Gamma(s)$ is that it has no zeros: $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$. This is because of the following formula.

Proposition 10.2.

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

This is plausible because the poles and the residues match on each side.

Proof. We'll sketch the proof here; this is just a calculation. We can multiply through to get $\sin(\pi s)\Gamma(s)\Gamma(1-s) = \pi$.

The left hand side is now holomorphic, and we want to show that it is constant. So it suffices to show that the relation holds for $0 < s < 1$. Then using the definition via integral, and writing $u = tw$, we have

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty t^s e^{-t} \frac{dt}{t} \int_0^\infty u^{1-s} e^{-u} \frac{du}{u}. = \int_0^\infty t^s e^{-t} \int_0^\infty (tw)^{1-s} e^{-tw} \frac{dw}{w} \frac{dt}{t} \\
&= \int_0^\infty w^{1-s} \left(\int_0^\infty e^{-(w+1)t}\, dt \right) \frac{dw}{w} = \int_0^\infty \frac{w^{1-s}}{w+1}\, dw = \frac{\pi}{\sin \pi s},
\end{aligned}$$

where the final equality is achieved by substituting $w = e^x$ to obtain

$$\int_{-\infty}^\infty \frac{e^{x(1-s)}}{1+e^x}\, dx,$$

and then integrating along a rectangular contour bounded by $x = \pm R$ and $y = 0$ and $y = 2\pi i$. As $R \to \infty$, the integral along the vertical lines will disappear, with the integral along the horizontal lines are related. We can also compute the residue of the pole at πi to evaluate our integral. ∎

The right hand side is never zero, so $\Gamma(s)$ can have a zero only where $\Gamma(1-s)$ has a pole, but this can't work. From this we can see that $\Gamma(s)\Gamma(1-s)$ has no zeros.

Now, we see that $\frac{1}{\Gamma(s)}$ is a holomorphic function with zeros at $0, -1, -2, -3, \dots$.

Proposition 10.3. *$\frac{1}{\Gamma(s)}$ has order 1.*

Proof. From the relation we computed,

$$\frac{1}{\Gamma(s)} = \left(\frac{\sin \pi s}{\pi} \right) \Gamma(1-s).$$

Let us revisit the analytic continuation of Γ and do the continuation in a slightly different way. That is, break $\Gamma(s)$ into two parts:

$$\Gamma(s) = \int_0^1 e^{-t} t^s \frac{dt}{t} + \int_1^\infty e^{-t} t^s \frac{dt}{t}.$$

The second integral is holomorphic for all $s \in \mathbb{C}$; the only issues are near $s = 0$. For the first integral, we can write down the series for the exponential. That is,

$$\int_0^1 e^{-t} t^s \frac{dt}{t} = \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^{n+s-1}\, dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}.$$

This makes it easy to compute residues.

Now, let's use this expression to see that

$$\Gamma(s) = \left(\frac{\sin \pi s}{\pi}\right) \left(\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)} + \int_1^\infty e^{-t} t^{-s}\, dt\right).$$

The sum is now

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\frac{\sin \pi s}{\pi(n+1-s)}\right).$$

We claim that this is still a holomorphic function of s. This is certainly $\leq C \exp(\pi|s|)$, and the denominator doesn't cause us any trouble, so this sum is an entire function of order 1.

We would now like to show that the integral is also an entire function of order 1. We have $\int_1^\infty e^{-t} t^{-s}\, dt$ is bounded by $\int_1^\infty e^{-t} t^{-r}\, dt$ where $\operatorname{Re} s = r$. We can split this into two cases: $t < |r|$ or $t > |r|$. This is just some calculus. It increases for awhile, and then it decreases. Certainly, this is $\leq \Gamma(|r|+1)$, which is therefore bounded as $\leq \exp(C|r| \log |r|)$. This is still entire of order 1. $\square$

From this, we can now use Hadamard's theorem 8.4. The function $\frac{1}{\Gamma(s)}$ has a zero at zero, so we can write

$$\frac{1}{\Gamma(s)} = e^{A+Bs} \cdot s \cdot \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n}.$$

We can also determine the constants. Plug in $s \approx 0$. Then

$$\frac{1}{s\Gamma(s)} \to 1 = e^A \prod_{n=1}^\infty \left(1 + \frac{s}{n}\right) e^{-s/n}.$$

The product $\to 1$ as $s \to 0$, which shows that $A = 0$. The last thing is to determine B. Take s to be a large integer. Then we have

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= e^{Bs} \cdot s \cdot \prod_{n=1}^{N} \left(\frac{s+n}{n}\right) e^{-s/n} = \frac{e^{Bs} s}{s!} (N+1)\cdots(N+s) \exp\left(-s\sum_{n=1}^{N}\frac{1}{n}\right) \\
&= \frac{e^{Bs}}{(s-1)!}(N+1)\cdots(N+s)\exp\left(-s\sum_{n=1}^{N}\frac{1}{n}\right) \\
&= \frac{1}{(s-1)!} e^{Bs}\left(N^s \exp(-s(\log N + \gamma + \text{small error}))\right) \\
&= \frac{1}{(s-1)!} e^{Bs}\left(\exp(-s(\gamma + \text{small error}))\right)
\end{aligned}$$

as $N \to \infty$, which shows that $B = \gamma$ and

$$\frac{1}{\Gamma(s)} = e^{\gamma s} \cdot s \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}.$$

There is one more property of γ which is quite nice, but we will not prove it.

Theorem 10.4 (Bohr-Mollerup)**.** *Γ is the unique continuous function f with the properties for $x > 0$ that $f(x+1) = xf(x)$, $f(1) = 1$, and $\log f(x)$ is convex, i.e. $(\log f)'' > 0$.*

Now, $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$. For $s = \frac{1}{2}$, we see that this implies $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

There is also another nice functional equation:

$$C \cdot \Gamma(s)\Gamma(s + 1/2) = \Gamma(2s),$$

which again makes sense if we consider poles. This called the *duplication formula* for Γ.

That's everything we want to say about $\Gamma(s)$, and now we will move on to the Riemann zeta function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This converges absolutely and defines a holomorphic function on $\operatorname{Re} s = \sigma > 1$. We can also write

$$\zeta(s) = \prod_p \left(1 - \frac{1}{s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$

which corresponds to unique factorization into primes. This is called an *Euler product.* We should show that this converges in the same region as the sum, but this is easy. We can also see from this Euler product that $\zeta(s) \neq 0$ if $\operatorname{Re} s > 1$ because an absolutely convergent product with no term equal to zero is nonzero.

We want to continue this function over the complex plane. Riemann gave a pretty way to do this by giving a functional equation. Let $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Some people use variants of this formula (i.e. without $s(s-1)$ or with an extra $\frac{1}{2}$), but this is convenient. We will show that $\xi(s)$ is an entire function of order 1, and it satisfies the functional equation $\xi(s) = \xi(1-s)$. From this, we will see that $\zeta(s)$ has a meromorphic continuation to $\mathbb{C}$. In fact, $\zeta(s)$ is holomorphic except at $s = 1$, where it has a simple pole. In addition, we have

$\zeta(-2) = \zeta(-4) = \cdots = 0$, which we call the *trivial zeros of* $\zeta(s)$. This leads to one of the most famous problems in math.

Hypothesis 10.5 (Riemann Hypothesis)**.** All other zeros of $\zeta(s)$ satisfy $\operatorname{Re} s = 1/2$.

We will not prove this, but we will show that there are no zeros of $\zeta(s)$ with $\operatorname{Re} s = 1$. This will imply the Prime Number Theorem, which states that $\pi(x) = \#\{\text{primes } p \leq x\} \sim \frac{x}{\log x}$.

Now, we will prove the functional equation.

Proposition 10.6. *Let* $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$*. Then* $\xi(s) = \xi(1-s)$*.*

Proof. We have (letting $y = \pi n^2 t$)

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \pi^{-s/2}e^{-y}y^{s/2}\sum_{n=1}^\infty \frac{1}{n^s}\frac{dy}{y} = \sum_{n=1}^\infty \int_0^\infty e^{-y}\left(\frac{y}{\pi n^2}\right)^{s/2}\frac{dy}{y}$$
$$= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t} = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right)t^{s/2}\frac{dt}{t}.$$

To understand this, we wish to understand

$$\psi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t},$$

and it is useful to study a slightly more symmetric function

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\psi(t).$$

This is closely related to the theta function from last week's homework.

There is a remarkable relation connecting $\theta(t)$ and $\theta(1/t)$. We see that $\theta(t) \to 1$ as $t \to \infty$. As $t \to 0$, we see that only $|n| \leq \frac{1}{\sqrt{t}}$ are relevant.

The relation that we will prove is therefore that

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

This also gives a relation between $\psi(t)$ and $\psi(1/t)$, which will allow us to finish the proof. We will break our integral into two parts, $\int_0^1$ and $\int_1^\infty$, and applying our relation will give the result.

In the process, we will also need to use the Poisson summation formula, which is proved in Chapter 4 of the book. $\square$

11. 11/3

We were discussing the Riemann zeta function, and we would like to use this to prove the prime number theorem.

Define $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. We will prove the following theorem.

Theorem 11.1. $\xi(s) = \xi(1-s)$ *and* $\xi(s)$ *is an entire function of order 1.*

This will give a meromorphic continuation of the zeta function to the complex plane except for a simple pole at $s = 1$.

Proof. We started working toward a proof. For $\operatorname{Re} s > 1$, we have

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty e^{-y} y^{s/2} \pi^{-s/2} \sum_{n=1}^\infty \frac{1}{n^s} \frac{dy}{y} = \int_0^\infty z^{s/2} \sum_{n=1}^\infty e^{-\pi n^2 z} \frac{dz}{z}.$$

at $y = \pi n^2 z$. Then we let

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$$

and

$$\psi(t) = \frac{\theta(t) - 1}{2} = \sum_{n=1}^\infty e^{-\pi n^2 t}.$$

Therefore,

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty t^{s/2}\psi(t)\frac{dt}{t}.$$

This integral certainly makes sense at large values of t; the issue is with lower values of t. We were going to show:

Proposition 11.2.

$$\theta(t) = \frac{1}{\sqrt{t}}\theta(1/t)$$

for all $t > 0$.

From this, we will deduce our functional equation, and get analytic continuation.

Proof. We will use the Poisson Summation Formula to prove this.

Suppose that f is some nice function, e.g. C^∞ or Schwartz space or a holomorphic function restricted to the real axis. Then we have

$$\hat{f}(\xi) = \int_{-\infty}^\infty f(x) e^{-2\pi n x \xi}\, dx.$$

Proposition 11.3. *The summation formula is*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k).$$

Think of this as a Riemann sum approximating $\hat{f}(0) = \int_{-\infty}^\infty f(x)\, dx$. There are a number of nice proofs of this fact.

Proof. Suppose that f is a holomorphic function in a strip and decreases as $|\operatorname{Re} z| \to \infty$. Consider a box shaped region. Then from residues we have

$$\int_{\text{box}} \frac{f(z)}{e^{2\pi i z} - 1}\, dz = \sum_{n\in\mathbb{Z}} f(n).$$

We can compare this to the horizontal lines in the contour, given by $R - iy$ and $R + iy$. The lower contour gives

$$\int_{-\infty}^\infty \frac{x - iy}{e^{2\pi i x + 2\pi y} - 1}\, dx = \sum_{n=1}^\infty \int_{-\infty}^\infty f(x - iy) e^{-2\pi i n (x - iy)}\, dx = \sum_{n=1}^\infty \hat{f}(n)$$

because

$$\frac{1}{e^{2\pi ix+2\pi y}(1-e^{-2\pi ix-2\pi y})} = \sum_{n=1}^{\infty} e^{-2\pi inx-2\pi ny}.$$

The upper line gives something similar, and we get some form of the Poisson summation formula, and this is covered in Chapter 4 of the book. □

If a function is supported on a small area, the Fourier transform should be spread out over a large area, and conversely. This is sometimes called the uncertainty principle. A toy version of this is the following:

Proposition 11.4. *We can show that f and $\hat{f}$ cannot both be compactly supported.*

Proof. Suppose that f is compactly supported in $[-A, A]$. Then we have

$$\hat{f}(\xi) = \int_{-A}^{A} f(x)e^{-2\pi ix\xi}\,dx.$$

This integral makes sense for all complex values $\xi \in \mathbb{C}$, and it is a holomorphic function of ξ. The problem is that there are not so many holomorphic functions with compact support, so $\hat{f}$ cannot have compact support. □

This is a toy case, but we can see the general picture. All we had to do was to make sense of the integral.

We need one more thing about Fourier transforms, and that is the Fourier transform of the Gaussian $f(x) = e^{-\pi x^2}$. Then we have

$$\begin{aligned}\hat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-\pi x^2-2\pi ix\xi}\,dx = \int_{-\infty}^{\infty} e^{-\pi(x+i\xi)^2-\pi\xi^2}\,dx = e^{-\pi\xi^2}\int_{-\infty-i\xi}^{\infty+i\xi} e^{-\pi z^2}\,dz \\ &= e^{-\pi\xi^2}\left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right) = e^{-\pi\xi^2}.\end{aligned}$$

We actually need to do this $f_t(x) = e^{-\pi tx^2}$. Looking at it's Fourier transform, we have

$$\begin{aligned}\hat{f}_t(\xi) &= \int_{-\infty}^{\infty} e^{-\pi tx^2-2\pi ix\xi}\,dx = \int_{-\infty}^{\infty} e^{-\pi t(x+i\xi/t)^2-\pi\xi^2/t}\,dx \\ &= e^{-\pi\xi^2/t}\left(\int_{-\infty}^{\infty} e^{-\pi tx^2}\,dx\right) = \frac{1}{\sqrt{t}}e^{-\pi\xi^2/t}.\end{aligned}$$

Now let's use Poisson summation on this function. That is,

$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi tn^2} = \sum_{n\in\mathbb{Z}} f_t(n) = \sum_{k\in\mathbb{Z}} \hat{f}_t(k) = \frac{1}{\sqrt{t}}\theta(1/t),$$

which is the functional equation that we wanted. □

Now, we return to

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \left(\int_0^1 + \int_1^{\infty}\right)\left(t^{s/2}\frac{\theta(t)-1}{2}\frac{dt}{t}\right).$$

The integral $\int_1^\infty$ makes sense for all values of s. We need to analyze $\int_0^1$. That is,

$$\begin{aligned}\int_0^1 t^{s/2}\frac{\frac{1}{\sqrt{t}}\theta(1/\sqrt{t})-1}{2}\frac{dt}{t} &= \int_0^1 t^{s/2}\left(\frac{1}{\sqrt{t}}\frac{\theta(1/t)-1}{2}+\frac{1}{2\sqrt{t}}-\frac{1}{2}\right)\frac{dt}{t}\\ &= \frac{1}{s-1}-\frac{1}{s}+\int_0^1 t^{(s-1)/2}\left(\frac{\theta(1/t)-1}{2}\right)\frac{dt}{t}.\end{aligned}$$

Putting everything together, we have (writing $y=1/t$)

$$\begin{aligned}\pi^{-s/2}\Gamma(s/2)\zeta(s) &= \frac{1}{s-1}-\frac{1}{s}+\int_0^1 t^{(s-1)/2}\left(\frac{\theta(1/t)-1}{2}\right)\frac{dt}{t}+\int_1^\infty t^{s/2}\left(\frac{\theta(t)-1}{2}\right)\frac{dt}{t}\\ &= \frac{1}{s-1}-\frac{1}{s}+\int_1^\infty y^{(1-s)/2}\left(\frac{\theta(y)-1}{2}\right)\frac{dy}{y}+\int_1^\infty t^{s/2}\left(\frac{\theta(t)-1}{2}\right)\frac{dt}{t}\\ &= \frac{1}{s-1}-\frac{1}{s}+\int_1^\infty \left(\frac{\theta(t)-1}{2}\right)\left(t^{s/2}+t^{(1-s)/2}\right)\frac{dt}{t},\end{aligned}$$

which is invariant under $s\to 1-s$. Therefore, we have proved that $\xi(s)=\xi(1-s)$, and we have also produced a meromorphic continuation.

We have shown that $\xi(s)$ is entire for all $s\in\mathbb{C}$. This also implies that $\zeta(s)$ is entire except for a pole at $s=1$. We still need to show that $\xi(s)$ is an entire function of order 1. We need to give a growth rate. Since $\xi(s)=\xi(1-s)$, we only need to consider the growth rate where $\operatorname{Re} s>\frac{1}{2}$.

To do this, we need to bound $(s-1)\zeta(s)$ in the region $\operatorname{Re} s\geq\frac{1}{2}$. When $\operatorname{Re} s>1$, we can write

$$\zeta(s)=\sum_{n=1}^\infty\frac{1}{n^s}=\sum_{n=1}^N\frac{1}{n^s}+\int_{N+}^\infty\frac{1}{y^s}d[y].$$

We will integrate this last integral by parts, and write $[y]=y-\{y\}$ to get

$$\int_{N+}^\infty\frac{1}{y^s}d[y]=\frac{1}{y^s}[y]\bigg|_{N+}^\infty+s\int_{N+}^\infty\frac{[y]}{y^{s+1}}\,dy=\frac{-N}{N^s}+s\int_N^\infty\frac{dy}{y^s}-s\int_N^\infty\frac{\{y\}}{y^{s+1}}\,dy.$$

Therefore, we can write

$$\zeta(s)=\sum_{n=1}^N\frac{1}{n^s}+\frac{N^{1-s}}{1-s}-s\int_N^\infty\frac{\{y\}\,dy}{y^{s+1}}.$$

This integral is holomorphic for $\operatorname{Re} s>0$. We have acquired yet another meromorphic continuation of ζ, and we are free to choose N to be any integer we want. We can bounded this integral trivially as

$$|s|\int_N^\infty\frac{dy}{y^{\sigma+1}}\leq\frac{C|s|}{N^\sigma}.$$

Then this part of the integral gets smaller as $N\to\infty$, and this can be used to evaluate $\zeta(s)$ on a computer.

We can also use this to bound $|\zeta(s)||s-1| \ll |s|^A$ for $\operatorname{Re} s \geq \frac{1}{2}$. Suppose that $|s-1| > \delta$, and $\operatorname{Re} s \geq \frac{1}{2}$. Then we can write that

$$|\zeta(s)| \leq \sum_{n=1}^{N} \frac{1}{n^\sigma} + CN^{1-\sigma} + \frac{|s|}{N^\sigma} \leq C + CN^{1-\sigma} + \frac{|s|}{N^\sigma} \leq C + C|s|^{1-\sigma}$$

where we used $N = \lceil |s| \rceil$. Here, if $\frac{1}{2} < \sigma < 1$, we get some polynomial bound.

We can now conclude that $\xi(s)$ is entire of order 1. □

Using Hadamard's theorem, since $\xi(s)$ has order 1, we can write

$$\xi(s) = e^{A+Bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

where ρ runs over all zeros of $\xi(s)$, which are the nontrivial zeros of $\zeta(s)$.

Now, the question is: Where are the nontrivial zeros of $\zeta(s)$? Recall that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\operatorname{Re} s > 1$, so therefore $\zeta(s) \neq 0$ if $\operatorname{Re} s > 1$. Since $\xi(s) = \xi(1-s)$, we can conclude that $0 \leq \operatorname{Re} \rho \leq 1$.

The main theorem that we will prove next time is a small but important refinement of this: The inequalities are strict. That is, $0 < \operatorname{Re} \rho < 1$, i.e. $\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$.

We could take a big box $0 \to 1 \to 1 + iT \to 0 + iT$, and we could count the number of zeros of the zeta function inside this box. If we count the number of zeros in this box, then the argument principle gives us a very nice proof (assuming that we know Stirling's formula and something about the growth of $\Gamma(s)$) that the box contains

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + (\text{small error})$$

zeros.

We can also show that

$$\xi(\frac{1}{2} + it) = \xi(\frac{1}{2} - it) = \overline{\xi(\frac{1}{2} + it)},$$

which means that $\xi(\frac{1}{2} + it) \in \mathbb{R}$.

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We defined $\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$, and we showed that this is entire of order 1. Then $\zeta(s)$ is meromorphic in $\mathbb{C}$ with a simple pole at $s = 1$.

Starting with $\operatorname{Re} s > 1$, we could approximate

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + \int_N^\infty \frac{\{y\}}{y^{s+1}}\, dy,$$

where the integral is holomorphic in $\operatorname{Re} s > 0$. This also gives an meromorphic continuation of $\zeta(s)$ to $\operatorname{Re} s > 0$.

There is one more interesting thing that we can notice. We can write

$$\zeta(s) = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} \right).$$

Thinking about $s \to 1$, we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}-1}{1-s} + s\int_N^\infty \frac{\{y\}}{y^{s+1}}\,dy.$$

Therefore, near $s = 1$ we have the Laurent expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + \gamma_1(s-1) + \dots.$$

In the proof that ξ is entire of order 1, we had to give some bounds for $\zeta(s)$. To give such a bound, we could choose an appropriate value of N. If s is not close to 1 and $s = \sigma + it$ for $\sigma \geq \frac{1}{2}$ then

$$|\zeta(s)| \leq \sum_{n\leq N} \frac{1}{n^\sigma} + \frac{N^{1-\sigma}}{|s-1|} + |s|\int_N^\infty \frac{dy}{y^{\sigma+1}} \leq C(N^{1-\sigma}+1)\log N + C\frac{|s|}{N^\sigma} \leq C|s|^{1-\sigma+\varepsilon} + C,$$

where we chose $N = [|s|] + 10$.

It is a homework problem for this week to show that $|\zeta(1+it)| \leq C\log|t|$.

Qualitatively, in the region $\sigma > 1-\varepsilon$ and $|t| \geq 1$ we have a bound like $|\zeta(\sigma+it)| \leq C|t|^\varepsilon$.

Define the prime counting function

$$X(x) = \sum_{p\leq x} 1.$$

We can write ζ as a product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where $\operatorname{Re} s > 1$. Then, using the series for $\log(1-x)^{-1} = \sum \frac{x^k}{k}$, we have

$$\log\zeta(s) = \sum_p \log\left(1-\frac{1}{p^s}\right)^{-1} = \sum_p \sum_{k=1}^\infty \frac{1}{kp^{ks}},$$

which is absolutely convergent if $\operatorname{Re} s > 1$. This is now correct as written down; the two sides agree on the real line, so they agree everywhere by analytic continuation. Note that the most important term in the inner sum is $k = 1$; the other terms are much smaller. We can clean this up a bit by considering

$$\frac{\zeta'}{\zeta}(s) = \sum_{p^k} \frac{1}{kp^{ks}}(-k\log p) = -\sum_{p^k} \frac{\log p}{p^{ks}} = -\sum_n \frac{\Lambda(n)}{n^s},$$

where

$$\Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & n \neq p^k. \end{cases}$$

Instead of studying primes themselves, it is more natural to consider

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$

which is called the von Mangoldt function.

We are hoping to prove the prime number theorem, conjectured by Legendre and Gauss:

Theorem 12.1 (Prime Number Theorem)**.** $\pi(x) \sim \frac{x}{\log x}$.

In fact, Gauss conjectured the more accurate $\pi(x) \sim \int_2^x \frac{dt}{\log t}$. The Prime Number Theorem is equivalent to showing that $\psi(x) \sim x$.

Note that

$$\psi(x) = \sum_{p \le x} \log p \sum_{p^k \le x} 1 = \sum_{p \le x} (\log p) \left[\frac{\log x}{\log p} \right].$$

Equivalently, $\psi(x) = \log(\text{lcm of numbers from 1 to } x)$.

Certainly, we have that

$$\psi(x) \le (\log x)\pi(x).$$

For example, this means that

$$\liminf \frac{\psi(x)}{x} = 1 \le \liminf \frac{\pi(x)}{x/\log x}.$$

We clearly have

$$\psi(x) \ge \sum_{p \le x} \log p \ge \sum_{x^{1-\varepsilon} \le p \le x} (1-\varepsilon) \log n \ge (1-\varepsilon)(\log x)\pi(x) - (1-\varepsilon)(\log x) x^{1-\varepsilon}.$$

Therefore, we've shown that

$$\pi(x) \le \frac{\psi(x)}{(1-\varepsilon)\log x} + x^{1-\varepsilon},$$

so therefore

$$\limsup \frac{\pi(x)}{x/\log x} \le \limsup \frac{\psi(x)}{x}.$$

All we are saying is that we can pass information about sums like $\pi(x) = \sum_{p \le x} 1$ to information about slightly smoothed sums like $\sum_{p \le x} \log p = \psi(x)$. This is a general principle called *partial summation*.

Instead of looking at $\psi(x)$, we look at an average of $\psi(x)$:

$$\psi_1(x) = \int_1^x \psi(t)\, dt.$$

Then $\psi(x) \sim x$ is equivalent to $\psi_1(x) \sim \frac{x^2}{2}$. We'll show one direction of the equivalence; the other follows easily from the above integral.

Take

$$\int_x^{x(1+\varepsilon)} \psi(t)\, dt = \psi_1(x(1+\varepsilon)) - \psi_1(x) \sim \left(\frac{x(1+\varepsilon)^2}{2} - \frac{x^2}{2} \right) \sim \varepsilon x.$$

Now,

$$\psi(x)(\varepsilon x) \le \int_x^{x(1+\varepsilon)} \psi(t)\, dt \le \varepsilon x \psi((1+\varepsilon)x).$$

This shows that $\psi(x) \le (1+o(1))x$ and $\psi((1+\varepsilon)x) \ge (1+o(1))x$, which shows the desired equivalence.

Now, observe that ψ_1 is the type of sum that we are interested in, except smoothed a bit.

$$\psi_1(x) = \int_1^x \psi(t)\,dt = \int_1^x \left(\sum_{p\le t} \log p\right)\,dt = \sum_{p\le x} (\log p)(x-p).$$

In both ψ and ψ_1, we are counting primes with smoothed weights.

We will now use Mellin inversion. Suppose that $y > 0$ and $C > 0$. Then

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{y^s}{s(s+1)}\,dy.$$

This is a nice integral that we can compute by shifting contours and computing poles.

In the case $y \le 1$, choose C to be very large because we can move this line to be anywhere we want. Our integral is then bounded by

$$\text{Integral} \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{|C+it||C+1+it|} \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{C^2+t^2} \le \frac{\text{constant}}{C}.$$

Letting $C \to \infty$ yields that the integral is zero.

Now, consider the case $y \ge 1$. Choose C to be large and negative. There are poles at 0 and -1, where the residues are $1 - y^{-1}$. We can also show that the remaining integral

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{y^s}{s(s+1)}\,dy \to 0$$

as $C \to -\infty$. Therefore we have

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{y^s}{s(s+1)}\,dy = \begin{cases} 0 & y \le 1 \\ 1-\frac{1}{y} & y \ge 1. \end{cases}$$

What would have been easier to do is

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{y^s}{s}\,ds = \begin{cases} 1 & y > 1 \\ \frac{1}{2} & y = 1 \\ 0 & 0 < y < 1. \end{cases}$$

This integral is not absolutely convergent though, so we have to integrate on a symmetric interval.

Another example is

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \Gamma(s)x^s\,ds = e^{-1/x}$$

for $x > 0$. Think of this as an exercise.

We can write

$$-\frac{\zeta'}{\zeta}(s) = \sum \frac{\Lambda(n)}{n^s}.$$

We will use the Mellin transform as a detector for whether $\frac{x}{n} \le 1$ or $\frac{x}{n} > 1$.

We want to look at ($c = \operatorname{Re} s > 1$)

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right) ds = \sum_n \Lambda(n)\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{(x/n)^s}{s(s+1)}\, ds$$
$$= \sum_{n\le x} \Lambda(n)\left(1-\frac{n}{x}\right) = \frac{\psi_1(x)}{x}.$$

We now try to evaluate this by studying singularities of the integrand. We have a pole at 0, a pole at -1, and poles of ζ'/z, which include the poles of $\zeta(s)$. The most important contribution turns out to be the pole of $\zeta(s)$ at $s=1$. Then for s near 1, $\zeta(s) = \frac{1}{s-1}+\gamma+\cdots$, so therefore

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \cdots .$$

Therefore,

$$\operatorname{Res}_{s=1} \frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right) = \frac{x}{2}.$$

The main term in the prime number theorem comes from this pole. Now, we must show that this integral really picks out this pole, and everything else is just some kind of error term.

We now need to understand the other singularities of $\frac{\zeta'}{\zeta}(s)$. The point is that if we had another zero $1+it$ on the line $\operatorname{Re} s = 1$, i.e. $\zeta(1+it) = 0$ for some t then the size of the contribution from this pole could be of size

$$-\frac{x^{1+it}}{(1+it)(2+it)},$$

which is roughly of size x. This would interfere with what we think is our main term. The key point is to rule out this case:

Lemma 12.2. *$\zeta(1+it) \neq 0$ for all $t \in \mathbb{R}$.*

Proof. Suppose $\zeta(1+it) = 0$. Then the idea is to show that $\zeta(1+2it)$ has a pole.

The whole proof amounts to the fact that

$$3 + 4\cos\theta + \cos 2\theta = 3 + 4\cos\theta + 2\cos^2\theta - 1 = 2(1+\cos\theta)^2 \ge 0.$$

Suppose that $\sigma > 1$. Then Recall that

$$\log\zeta(s) = \sum_{p^k} \frac{1}{kp^{ks}},$$

and

$$3\zeta(\sigma) + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|$$
$$= \sum_{p^k}\frac{1}{k}\left(\frac{3}{p^{k\sigma}} + \frac{4\cos(kt\log p)}{p^{k\sigma}} + \frac{\cos(k(2t)\log p)}{p^{k\sigma}}\right) \ge 0.$$

Now, exponentiating gives

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \ge 1$$

for $\sigma > 1$. Suppose that $\zeta(1+it) = 0$. Then as $\sigma \to 1^+$, we have $|\zeta(\sigma+it)| = C(\sigma-1)+\cdots$, and $\zeta(s) = \frac{1}{\sigma-1} + \gamma + \cdots$.

Therefore, the left hand side becomes

$$\left(\frac{1}{\sigma-1}\right)^3 (C(\sigma-1))^4|\zeta(\sigma+2it)| \to C(\sigma-1)|\zeta(\sigma+2it)| \geq 1,$$

and therefore $|\zeta(\sigma+2it)| \to \infty$ as $\sigma \to 1^+$. $\square$

Now, let us return to the proof of the Prime Number Theorem.

We are looking at

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\,ds.$$

We can deform the contour to lie on the line $\operatorname{Re} s = 1$, except for a section between imaginary part T and $-T$ for some large T. (Draw a picture). This contour is $1-i\infty \to 1-it \to \sigma_T - iT \to \sigma_T + iT \to 1+iT \to 1+i\infty$. Here, we have σ_T is such that $\zeta(s) \neq 0$ in the region $\operatorname{Re} s > \sigma_T$ and $|\operatorname{Im} s| \leq T$. It could be very close to 1. Then

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\,ds = \frac{x}{2} + \frac{1}{2\pi i}\int_{\text{new contour}} \frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\,ds.$$

In this proof, think of T as a fixed large number, and we will eventually let $T \to \infty$.

First, we will bound

$$\int_{1+iT}^{1+i\infty} \left(\frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\right)\,ds$$

and the corresponding thing on the other side. We also need to bound

$$\int_{\text{indented middle piece}} \left(\frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\right)\,ds.$$

First, we show that this last piece is small compared to x. The integrand is bounded by $\leq x^\sigma C(T)$ where

$$C(T) = \max\left|\frac{\zeta'}{\zeta}(s)\right|\frac{1}{|s||s+1|}.$$

Then

$$\int_{\text{middle piece}} x^\sigma C(T) \leq C(T)x^{\sigma_T}(2T) + 2C(T)\int_{\sigma_T}^{1} x^\sigma\,d\sigma \leq C(T)x^{\sigma_T}(2T) + 2C(T)\frac{x}{\log x} \leq \varepsilon\cdot x$$

if x is large enough.

Now, we just need to show that the two tail integrals are small as $T\to\infty$, which we will have to delay until next time. We will have to prove that

$$\left|\frac{\zeta'}{\zeta}(1+it)\right| \leq C\sqrt{|t|}.$$

Assuming this bound, we have that

$$\int_{1+iT}^{1+i\infty} \left(\frac{x^s}{s(s+1)}\left(-\frac{\zeta'}{\zeta}(s)\right)\right)\,ds \leq \int_T^\infty \frac{x}{t^2}C\sqrt{t}\,dt \leq \frac{C}{\sqrt{T}}.$$

Therefore, showing that

$$\left|\frac{\zeta'}{\zeta}(1+it)\right| \leq C\sqrt{|t|}$$

will suffice to complete our proof.

We first summarize the proof of the prime number theorem so far. We are interested in getting an asymptotic formula for $\pi(x) = \sum_{p \le x} 1 \sim x/\log x$. This is the same as looking at prime powers weighted by a logarithm:

$$\psi(x) = \sum_{n \le x} A(n) = \sum_{p \le x} \log p \left[\frac{\log x}{\log p}\right] \sim x.$$

This is not too bad because we are basically counting primes with weight $\log x$.

Even this is hard to evaluate, so we look at

$$\psi_1(x) = \int_1^x \psi(t)\, dt = \sum_{n \le x} \Lambda(n)(x-n) \sim x^2/2.$$

This is just a small averaging.

We can detect the condition $n \le x$ by looking at a contour integral:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\, ds = \begin{cases} 0 & y \le 1 \\ 1 - \frac{1}{y} & y > 1. \end{cases}$$

The connection with the Riemann zeta function is that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for $\operatorname{Re} s > 1$. Then we could write

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s(s+1)}\, ds = \sum_{n \le x} \Lambda(n)\left(1 - \frac{n}{x}\right) = \frac{\psi_1(x)}{x}$$

for $c > 1$.

Now, we have to deform this contour. Choose a large number T, and deform the contour into segments $1 + iT \to \sigma_T + iT \to \sigma_T - iT \to 1 - iT$, where $\sigma_T < 1$ and this region has no zeros of $\zeta(s)$. This uses the crucial fact that $\zeta(1+it) \ne 0$.

Recall that the proof of this used $\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \ge 1$ for $\sigma > 1$ because of $3 + 4\cos\theta + \cos 2\theta \ge 0$. All of the zeros are isolated and hence bounded away from $\operatorname{Re} s = 1$, so that gives us σ_T.

Shifting the contour therefore does not change the integral except for a pole at 1, which gives us a residue of $\frac{x}{2}$. That's the main term that we want. Now, we have to estimate the integrals on the five segments of the contour.

Looking at the integral of the indented middle three segments, we showed that the integrals are bounded by

$$B(T)x^{\sigma_T} + B(T)\int_{\sigma_T}^1 x^\sigma\, d\sigma \le B(T)(x^{\sigma_T} + x/\log x) \le \varepsilon x$$

for x sufficiently large (for any fixed T).

Now, we need to show the same bound for the other two contours, for T sufficiently large.

We need to show the following:

Proposition 13.1.
$$\left|\frac{\zeta'}{\zeta}(1+it)\right| < C|t|^{\varepsilon}$$
if $|t| \geq 1$.

This would show that
$$\int_T^{\infty} \left|\frac{\zeta'}{\zeta}(1+it)\right| \frac{x}{|(1+it)(2+it)|}\, dt < C\frac{x}{T^{1-\varepsilon}}.$$

The last step is to prove this proposition. Recall that we have $|\zeta(\sigma+it)| \leq C(1+|t|^{1-\sigma+\varepsilon})$. Given a bound like this, we can also get a bound
$$|\zeta'(1+it)| \leq \left|\frac{1}{2\pi i}\int_{|z-1+it|=\varepsilon} \frac{\zeta(z)}{(z-1-it)^2}\, dt\right| \leq C|t|^{\varepsilon}.$$
In view of this, what we really want is that
$$\frac{1}{|\zeta(1+it)|} \leq C|t|^{\varepsilon},$$
which means that we want a lower bound
$$|\zeta(1+it)| \geq C|t|^{-\varepsilon}.$$
We shall now prove this inequality.

For $\sigma > 1$, using an earlier proved inequality, we have
$$|\zeta(\sigma+it)| \geq \zeta(\sigma)^{3/4}|\zeta(\sigma+2it)|^{-1/4} \geq c|t|^{-\varepsilon}(\sigma-1)^{3/4}.$$
The crucial fact here is that $3 < 4$. Therefore, we have
$$|\zeta(\sigma+it)| \leq |\zeta(1+it)| + |\zeta(\sigma+it) - \zeta(1+it)| \leq |\zeta(s+it)| + (\sigma-1)C|t|^{\varepsilon},$$
where the final quantity is a bound for $|\zeta'(\alpha+it)|$ for $1 \leq a \leq \sigma$. Therefore, we have
$$|\zeta(1+it)| \geq c|t|^{-\varepsilon}(\sigma-1)^{3/4} - (\sigma-1)C|t|^{\varepsilon}.$$
Now, take $\sigma - 1 = |t|^{-40\varepsilon}$; then clearly the first term is clearly larger than the second, which means that we have shown that $|\zeta(1+it)| \geq c|t|^{-31\varepsilon}$, so we are done.

Remark. This is in some ways not the best way to look at this problem. If we don't worry about smoothing, we can look at
$$\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\, ds = x - \sum_{\text{nontrivial zeros } \rho \text{ of } \zeta(s)} \frac{x^{\rho}}{\rho} - \text{poles at trivial zeros.}$$
This is called the *explicit formula* and relates the zeros of the zeta function to the primes.

We have to be careful because $\sum \frac{1}{|\rho|^{1+\varepsilon}} < \infty$ but $\sum \frac{1}{|\rho|}$ diverges. This is a much more illuminating formula. In this context, the Riemann Hypothesis is equivalent to the fact that
$$|\psi(x) - x| \leq Cx^{\frac{1}{2}+\varepsilon}.$$
That's all that we'll say about the Prime Number Theorem.

Now, we will move on to the Riemann Mapping Theorem. We stated this before but did not prove it. Recall that that we discussed equivalent characterizations of simple connectivity 6.5. We showed that any two curves are homotopic, and the complement in $\mathbb{C} \cup \{\infty\}$ is connected. Also, for any $\gamma \subseteq \Omega$, and $W_\gamma(z) = 0$ for all $z \notin \Omega$. What we did not do at the time is that the first condition imply either of the others. We were also able to define for f holomorphic on Ω and $f(z) \neq 0$ the functions $\log f(z)$ and $\sqrt{f(z)}$.

Theorem 13.2 (Riemann Mapping Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ with $\Omega \neq \mathbb{C}$. If Ω has the property that f is holomorphic and nonzero on Ω then $\log f$ and $\sqrt{f}$ are holomorphic on Ω and there is a bijective holomorphic map from Ω to $\mathbb{D} = \{|z| \leq 1\}$.*

If we have two sets U and V and a map $f : U \to V$ is bijective and holomorphic, then understanding the holomorphic functions on U is the same as understanding the holomorphic functions on V.

Proposition 13.3. *Suppose we have $f : U \to V$ bijective and holomorphic. Then the inverse function $g : V \to U$ is holomorphic.*

First, suppose that $f : U \to \mathbb{C}$ and f is bijective and holomorphic, then $f'(z) \neq 0$ for all $z \in U$. This is because if $f(z) = f(z_0) + a_k(z - z_0)^k + \cdots$ then f would be locally k-to-1 and hence not injective. This condition says that f is locally injective, but it might not be globally injective.

Example 13.4. Consider $f(z) = z^2$ on $\mathbb{C} \setminus \{0\}$. Then $f'(z) \neq 0$, and this is locally injective but not globally injective (since $f(z) = f(-z)$).

Now, we will prove our proposition that the inverse map is holomorphic.

Proof. It is obvious that the inverse map g is well-defined, and g is continuous by the Open Mapping Theorem 5.6.

Now, as $w \to z$, we have

$$\frac{g(z) - g(w)}{z - w} = \frac{g(z) - g(w)}{f(g(z)) - f(g(w))} = \frac{1}{\frac{f(g(z)) - f(g(w))}{g(z) - g(w)}} \to \frac{1}{f'(g(z))}.$$

This makes sense because we know that $f' \neq 0$. □

Definition 13.5. We say that U and V are *conformally equivalent* and U is *biholomorphic* to V is this is true.

Therefore, the condition that log is well-defined implies that Ω is equivalent to $\mathbb{D}$, which is simply connected, and this shows that Ω is also simply connected. This completes our earlier discussion of simple connectivity.

Our first goal is to describe bijective holomorphic maps $\mathbb{D} \to \mathbb{D}$.

We will need to use the Schwarz Lemma 9.4.

Lemma 13.6 (Schwarz Lemma)**.** *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ and if $|f(z)| = |z|$ for some z, then $f(z) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$ is a rotation. In addition, $|f'(0)| \leq 1$, and if $|f'(0)| = 1$ then $f(z) = e^{i\theta} z$.*

Proof. Consider $f(z)/z$, and apply the Maximum Modulus Principle 5.7. □

Using this, we want to describe the automorphisms of $\mathbb{D}$; these are bijective holomorphic maps $\mathbb{D} \to \mathbb{D}$.

Proposition 13.7. *Suppose we have a map $f \in \mathrm{Aut}(\mathbb{D})$ with $f(0) = 0$. Then $f(z) = e^{i\theta} z$ for some θ.*

Proof. f and f^{-1} are both automorphisms in $\mathrm{Aut}(\mathbb{D})$, and $f^{-1}(0) = 0$. Therefore $|f(z)| \leq |z|$ and $|f^{-1}(z)| \leq |z|$. Therefore f is a rotation. □

Example 13.8. Suppose $|\alpha| < 1$. Define

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

This is certainly holomorphic for $z \in \mathbb{D}$ (in fact for $|z| < \frac{1}{|\alpha|}$). This satisfies $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$. Then $\psi_\alpha \circ \psi_\alpha =$ identity, so ψ_α is certainly an automorphism.

Example 13.9. Now, suppose we have a general $f \in Aut(\mathbb{D})$. Suppose $f(0) = \alpha$. Then $(\psi_\alpha \circ f)(0) = 0$, which means that $\psi_\alpha \circ f$ is a rotation, and $f(z) = e^{i\theta}\psi_\alpha(z)$. This describes explicitly what happens in the case of the unit disc.

The other basic example is the upper half plane. This is the set $\mathbb{H} = \{x + iy : y > 0\}$. The Riemann Mapping Theorem says that $\mathbb{H}$ is biholomorphic to $\mathbb{D}$. Here, we can explicitly describe what the map looks like. For $z \in \mathbb{H}$, define

$$f(z) = \frac{z - i}{z + i},$$

and the size of the denominator is greater than the size of the numerator. This is therefore a map $f : \mathbb{H} \to \mathbb{D}$. Write $w = \frac{z-i}{z+i}$ and solve for z to see that this is bijective.

Here, f sends the point at infinity to 1. Otherwise, on the boundary of $\mathbb{H}$, we see that $f(x) = \frac{x-i}{x+i}$, which has size 1. This is nice because the boundary also gets mapped to the boundary in a nice continuous way.

We can now ask about the automorphisms of $\mathbb{H}$; these are just conjugate to the automorphisms of $\mathbb{D}$. There is any even nicer description though;

$$\mathrm{Aut}(\mathbb{H}) = PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\},$$

and for each element of $PSL_2(\mathbb{R})$ we have a map $\mathbb{H} \to \mathbb{H}$ given by.

$$z \to \frac{az + b}{cz + d}.$$

Example 13.10. Consider the map $z \to z^\alpha$. Then $\mathbb{H} = \{z = re^{i\theta}, 0 < \theta < 2\pi\} \to \{re^{i\theta}, 0 < \theta < \alpha\pi\}$. Therefore, for $0 < \alpha < 2$, we have a map between $\mathbb{H}$ and a sector.

Example 13.11. The map $z \to \log z$ sends $\mathbb{H} \to \{x + i\theta : x \in \mathbb{R}, 0 < \theta < \pi\}$, which is a strip.

Those are nice examples, but even for a polygon or a rectangle, it is not obvious how to produce a map.

14. 11/15

Today we should prove the Riemann Mapping Theorem 7.1.

Theorem 14.1. *Let Ω be a region such that any f holomorphic on Ω and $f(z) \neq 0$ has a log which is holomorphic on Ω. Then either $\Omega = \mathbb{C}$ or there is a holomorphic bijection from Ω to the unit disk $\mathbb{D}$.*

We pointed out that this condition is satisfied if Ω is simply connected. We also showed that if f is a holomorphic bijection from U to V then $f'(z) \neq 0$ and the inverse is holomorphic.

We began by studying the automorphisms of the unit disk $\mathrm{Aut}(\mathbb{D})$. They look like functions $\psi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$. When $|\alpha| < 1$, we have $\psi_\alpha \in \mathrm{Aut}(\mathbb{D})$ interchanges 0 and α and $\psi_\alpha \circ \psi_\alpha = id$. Any automorphism is a rotation times one of these. To prove this, we used the Schwarz Lemma 9.4: If $f(0) = 0$ and $f : \mathbb{D} \to \mathbb{D}$ then $|f(z)| \leq |z|$ for $z \in \mathbb{D}$. The other important example we did was the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$.

Now, consider the slit plane $\mathbb{C} \setminus [0, -\infty)$. Then the map $\sqrt{z}$ sends this to a half plane $\operatorname{Re} z > 0$, which is something we understand. This is interesting because there are two ways of approaching the boundary, from above and below; each will give half of the boundary of the half plane, and the boundary is a bit complicated. The way you approach the boundary is relevant here.

Now, let's consider the proof of the Riemann Mapping Theorem. This was discovered in the 1910s by Koebe and Caratheodory. The proof is a bit tricky.

Proof of Riemann Mapping Theorem. We want to find a bijection between Ω and the unit disc. The first step is to show that there are injective maps from Ω to the unit disc. Of course, if Ω is bounded then this is obvious, so we are really interested in the case when Ω is unbounded.

We might as well assume that $\Omega \neq \mathbb{C}$, so pick $\alpha \notin \Omega$. Then look at the function $f(z) = \sqrt{z - \alpha}$. This is a holomorphic function on Ω. First, it is clearly injective because $f(z_1) = f(z_2)$ implies that $z_1 = z_2$ via squaring. On the other hand, we also have that if $f(z_1) = -f(z_2)$ then $z_1 = z_2$ as well.

Now, take some point $z_0 \in \Omega$ and look at

$$\frac{1}{f(z) + f(z_0)} - \frac{1}{2f(z_0)}.$$

We know that $f(z) + f(z_0)$ is nonzero and $|f(z) + f(z_0)|$ is bounded away from zero by what we said before, and this first term is bounded. We subtract a constant, so the whole thing is bounded.

For small ε, we have that

$$g(z) = \varepsilon \left(\frac{1}{f(z) + f(z_0)} - \frac{1}{2f(z_0)} \right).$$

Then $|g(z)| < 1$ for $z \in \Omega$ and $g(z_0) = 0$. This is a nice map.

Now, we have that Ω is equivalent to $\Omega_0 \subseteq \mathbb{D}$ and $0 \in \Omega_0$. Now we might as well prove the Riemann Mapping theorem for this region Ω_0. We will show that Ω_0 is equivalent to $\mathbb{D}$. This is the tricky part.

Roughly, the idea is that if Ω_0 is not the unit disc then we will "expand" to a bigger region inside $\mathbb{D}$. We will try to use a maximality argument instead of iterating.

Proposition 14.2. *If $0 \in \Omega_0 \subseteq \mathbb{D}$ and $\Omega_0 \neq \mathbb{D}$, and if $f \neq 0$ on Ω implies that f has a square root, then there is a function $\kappa : \Omega \to \mathbb{D}$ with*

(1) κ injective
(2) $\kappa(0) = 0$
(3) If $z \neq 0 \in \Omega_0$ then $|\kappa(z)| > |z|$.

Also we have that $|\kappa'(0)| > 1$.

Proof. Take some $\alpha \notin \Omega_0$ and $\alpha \in \mathbb{D}$. Consider $\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}\zeta}$. This a map $\psi_\alpha : \mathbb{D} \to \mathbb{D}$, so it can be restricted to a holomorphic $\psi_\alpha : \Omega_0 \to \mathbb{D}$. In addition, notice that $0 \notin \psi_\alpha(\Omega_0)$. Therefore, we can take the square root of ψ_α to get $g = \sqrt{\psi_\alpha} : \Omega_0 \to \mathbb{D}$. Then $g(0) = \sqrt{\psi_\alpha(0)} = \sqrt{\alpha} = \beta$.

We want to construct a function sending 0 to 0, so we want a function sending β back to 0. This suggests that we should have

$$\kappa = \psi_\beta \circ g = \psi_\beta \circ \sqrt{\cdot} \circ \psi_\alpha.$$

This is clearly injective as it is two automorphisms and an injection, and $\kappa(0) = 0$.

Now, $\kappa : \Omega_0 \to \mathbb{D}$. It need not be a function $\mathbb{D} \to \mathbb{D}$. Nonetheless, we look at the inverse

$$\kappa^{-1} = \psi_\alpha^{-1} \circ (\cdot)^2 \circ \psi_\beta^{-1} = \psi_\alpha \circ (\cdot)^2 \circ \psi_\beta.$$

This is a function $\kappa^{-1} : \mathbb{D} \to \mathbb{D}$ and $\kappa^{-1}(0) = 0$. So by the Schwarz Lemma 9.4 we can say that $|\kappa^{-1}(z)| \leq |z|$ but we cannot have equality, since in that case κ^{-1} would be a rotation and hence an automorphism, but $(\cdot)^2$ is not an automorphism. This means that the inequality is strict: $|\kappa^{-1}(z)| < |z|$. This proves that result that we want. ☐

Let the inner radius of Ω_0 be the radius of the biggest (take supremum) disk centered at 0 which is contained in Ω_0. We claim that inner radius of $\Omega_0 \leq$ inner radius of $\kappa(\Omega_0)$. This says that the inner radius keeps expanding as we apply κ, which is good. We can draw a picture.

Now we return to the proof of the Riemann Mapping theorem, and we will use a maximality argument instead of iterating this procedure.

Let

$$\mathcal{F} = \{\text{injective functions } f : \Omega_0 \to \mathbb{D} \text{ with } f(0) = 0\}.$$

Pick $f \in \mathcal{F}$ with $|f'(0)|$ maximal. Alternatively, pick $z_0 \in \Omega$ with $z_0 \neq 0$, and pick $|f(z_0)|$ is maximal. We need to check that there actually exists a maximal f; this will be proved later and we will assume it for now.

Now, look at $f : \Omega_0 \to \mathrm{Im}(f)$. If $\mathrm{Im}(f) = \mathbb{D}$ then we are done; otherwise, take κ for $\mathrm{Im}(f)$ and argue for a contradiction. So construct an expanding map $\kappa : \mathrm{Im}_f(\Omega_0) \to \mathbb{D}$ with $|\kappa(z)| > |z|$ and $|\kappa'(0)| > 1$. Then $\kappa \circ f : \Omega_0 \to \mathbb{D}$ is injective.

Now, compute

$$(\kappa \circ f)'(0) = \kappa'(f(0)) \cdot f'(0) = \kappa'(0) \cdot f'(0),$$

so $|(\kappa \circ f)'(0)| > |f'(0)|$, contradicting maximality. Therefore, we see that $\mathrm{Im}(f) = \mathbb{D}$, which concludes the proof.

The last step of the proof is to show the existence of this maximal element. There are two parts to the proof of this. Suppose we have a sequence of functions $f_1, \ldots, f_n, \ldots$ injective from Ω_0 to $\mathbb{D}$, and $|f_n'(0)| \to \sup_{f \in \mathcal{F}} |f'(0)|$. We want to extract a convergent subsequence $f_{n_1}, f_{n_2}, \cdots \to f$; this f will be our maximal element. Except there is one more thing that we should check: f should be injective. These two parts go by different names. The first is called *Montel's theorem* and the second is called *Hurwitz's theorem*.

Let's start with the second one:

Theorem 14.3 (Hurwitz's Theorem)**.** *Let Ω be a region and let $f_1, f_2, \ldots, f_n, \ldots$ be nonzero on Ω. Suppose that $f_n \to f$ on Ω. Then either $f \equiv 0$ or f is never zero.*

This means that we don't pick up any extra zeros.

Proof. Suppose that f is not identically zero. Then it is holomorphic and its zeros are isolated.

Suppose that $f(z) = 0$. Then we can take a small circle C centered at z and see that

$$\text{multiplicity of zero at } f = \frac{1}{2\pi i}\int_C \frac{f'}{f}(w)\,dw = \lim_{n\to\infty}\frac{1}{2\pi i}\int_C \frac{f_n'}{f_n}(w)\,dw = 0,$$

so f has no zeros. ☐

Corollary 14.4. *If $f_n \to f$ and f_n are injective and f is either constant or injective.*

Proof. Suppose that $f(z_1) = f(z_2)$ and $z_1 \neq z_2$, and f is not constant. Then look at the function $f(z) - f(z_2)$. Consider this in a small neighborhood of z_1. This is a function that has a zero at z_1, but on the other hand, this is also the limit $\lim_{n\to\infty}(f_n(z) - f_n(z_0))$ and these functions are nonzero in this small neighborhood. That's a contradiction. ☐

The last thing to prove is that we can extract a convergent subsequence from f_n.

Definition 14.5. A family $\mathcal{F}$ of functions is called *normal* if from every sequence of functions in $\mathcal{F}$ we can extract a convergent subsequence.

Note that the convergent subsequence might not converge to something in the family.

Definition 14.6. A family $\mathcal{F}$ of holomorphic functions on a region Ω is *uniformly bounded* if for every compact $K \subseteq \Omega$, there is a bound $B(K)$ with $|f(z)| \leq B(K)$ for all $z \in K$ and for all $f \in \mathcal{F}$.

In our case, we have a bunch of functions $\Omega \to \mathbb{D}$, so they are all uniformly bounded.

Definition 14.7. We say that $\mathcal{F}$ is *equicontinuous* if for every compact $K \subseteq \Omega$, the following is true: For any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|z_1 - z_2| < \delta$ (with $z_1, z_2 \in K$) then $|f(z_1) - f(z_2)| < \varepsilon$ for all $f \in \mathcal{F}$.

(This means that the functions $f \in \mathcal{F}$ are all uniformly continuous with the same parameters.)

There are two parts to Montel's theorem.

Theorem 14.8 (Montel's Theorem)**.** *If $\mathcal{F}$ is uniformly bounded then $\mathcal{F}$ is equicontinuous. If $\mathcal{F}$ is equicontinuous then $\mathcal{F}$ is normal.*

Remark. The first part is true for holomorphic functions. The second part is often called the *Arzela-Ascoli theorem*, and it is true for any metric space.

There is another version of Montel's theorem, which is an analog of Picard's theorem:

Theorem 14.9. *If $\mathcal{F}$ is a family of holomorphic functions on Ω such that any $f \in \mathcal{F}$ omits the values 0 and 1. Then f is normal.*

We will now prove Montel's theorem:

Proof of Montel's theorem. Now, suppose we have some region Ω and a compact set $K \subseteq \Omega$. Take $z_1, z_2 \in \Omega$ and choose a small circle containing z_1 and z_2 and contained in Ω. Now we can just use the Cauchy formula:

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_C \left(\frac{f(w)}{w - z_1} - \frac{f(w)}{w - z_2}\right) dw = \frac{1}{2\pi i}\int_C \frac{f(w)(z_1 - z_2)}{(w - z_1)(w - z_2)}\, dw$$

We have that $f(w)$ is uniformly bounded. We can choose the circle to make the denominator bounded too, so f is equicontinuous. This proves the first part of Montel's theorem.

Now, we wish to prove that f is equicontinuous implies that f is normal. Suppose that we have a compact set $K \subseteq \Omega$. Then we can extract a subsequence converging uniformly on K.

Let our equicontinuous functions be $f_1, f_2, \dots$, and choose $w_1, w_2, \dots \in K$ which are dense. First, extract a sequence $f_{1,1}, f_{2,1}, f_{3,1}, \dots$ converging at w_1. From this, extract a subsequence $f_{1,2}, f_{2,2}, f_{3,2}, \dots$ converging at w_2, and so on. We diagonalize this and look at $f_{n,n}$. This will be a subsequence converging at all of the w_n. Now we need to use compactness. Take δ-neighborhood of the w_n, which is an open cover. Then we can find an finite subcover, and we can work through the definition of equicontinuous to see that f converges at all points in K.

The last thing is that we must do this for every compact set. For Ω we can find a sequence $K_1 \subseteq K_2 \subseteq \cdots$ with $\Omega = \bigcap K_n$. From here extract $f_{1,1}, f_{2,1}, \dots$ converging on K_1, and a sequence $f_{1,2}, f_{2,2}, \dots$ converging on K_2, and so on. We can diagonalize again, and look at $f_{n,n}$ to finish the proof. □

This concludes the proof of the Riemann Mapping Theorem. □

15. 11/17

Last time, we proved the Riemann Mapping theorem. There were several things that went into the proof:

(1) Reduce to $\Omega \subseteq \mathbb{D}$
(2) Expansion maps: For $0 \in \Omega_0 \in \mathbb{D}$, $\Omega_0 \neq \mathbb{D}$, we found κ so that $|\kappa(z)| > |z|$ for all $\kappa \neq 0$, and $\kappa(0) = 0$ and κ injective, and $|\kappa'(0)| > 1$.
(3) We considered the class of all injective f from Ω_0 to $\mathbb{E}$ with $f(0) = 0$, and picked the f with maximal $|f'(0)|$. If $f(\Omega_0) \neq \mathbb{D}$, use κ for region $f(\Omega_0)$ and look at $\kappa \circ f$.
(4) For this proof of maximality, we needed Montel's theorem 14.8. We also needed Hurwitz's theorem 14.3, which says that a limit of injective functions is either constant or injective.

Today we will discuss what happens when Ω is a polygon. In general, we cannot say what the maps from Ω to $\mathbb{D}$ look like, but in this case we can write down a formula.

We will also consider the following question: In the case $f : \mathbb{D} \to \Omega$ or $f^{-1} : \Omega \to \mathbb{D}$, does f extend to a homeomorphism from $\partial(\mathbb{D})$ to $\partial(\Omega)$? This is not always the case; consider the slit complex plane mapped to the right half plane by $\sqrt{z}$; in this case, approaching the slit from different sides gives bad behavior, and the map does not extend to the boundary. On the other hand, there is a general theorem due to Caratheodory:

Theorem 15.1 (Caratheodory). *If $\partial(\Omega)$ is a simple closed curve (Jordan curve) then the map $f : \Omega \to \mathbb{D}$ extends to the boundary.*

We will only prove the case for polygons. Consider a map F from the unit disc $\mathbb{D}$ to the interior of some polygon $\mathbb{P}$.

Theorem 15.2. *F extends continuously to the boundary.*

Proof. First, we will show that if $z_0 \in \partial(\mathbb{D})$, the limit $\lim_{z\to z_0} F(z)$ exists. If not, we would have two sequences of points inside the disc, approaching the same point z_0, but being sent to different points in the polygon. Connect the image of the points to two curves in the polygon, and send those curves back to the disc.

Look at a circle of radius r (small) around z_0. Then we can find two points z_1 and z_2 in the unit disc with $|F(z_1) - F(z_2)| > c$ for some $c > 0$. Write

$$\left| \int_{z_1}^{z_2} F'(w)\, dw \right| = |F(z_1) - F(z_2)| > c.$$

We can write $w = z_0 + re^{i\theta}$ for $\theta_1 \leq \theta \leq \theta_2$. Then

$$r \int_{\theta_1}^{\theta_2} |F'(z + re^{i\theta})|\, d\theta \geq c.$$

By Cauchy-Schwarz, this implies that

$$(\theta_2 - \theta_1) \int_{\theta_1}^{\theta_2} |F'(z_0 + re^{i\theta})a|^2\, d\theta = \left(\int_{\theta_1}^{\theta_2} |F'(z_0 + re^{i\theta})|\, d\theta \right)^2.$$

Therefore,

$$r \int_{\theta_1}^{\theta_2} |F'(z_0 + re^{i\theta})|^2\, d\theta \geq \left(\frac{c}{r}\right)^2 \cdot r = \frac{c^2}{r}.$$

If f is a conformal map, then we have

$$\text{area}(f(U)) = \int_{f(U)} dx\, dy = \int_U |f'(z)|^2.$$

Then

$$\infty > \int \int |F'(z_0 + re^{i\theta}|^2 r\, dr\, d\theta = \text{area of image of } f \text{ near } z_0 \geq \int \frac{c^2}{r}\, dr,$$

which is impossible. Hence, our limit $\lim_{z\to z_0} F(z)$ exists. Therefore we can define this to be $F(z_0)$, and we can check that this is continuous on the unit circle.

We can do the same argument with the inverse map G from a polygon to $\mathbb{D}$. The same proof works to give an extension of G to a continuous map on the boundary.

Now, since $F \circ G = id$ on the interior of the polygon, we see that $F \circ G$ continues to be the identity on the boundary. □

Now, we want to use this to describe how to produce maps from $\mathbb{D}$ to the polygon $\mathbb{P}$. In fact, we will actually look at maps $\mathbb{H} \to \mathbb{P}$. Keep in mind that the point at infinity is getting mapped to a point on the boundary of $\mathbb{P}$.

Now, the vertices of $\mathbb{P}$ will be on the boundary of $\mathbb{H}$ or at infinity. In the simplest case, we have the points $A_1, A_2, \ldots, A_n$ on the line, and F sends them to points $F(A_1), F(A_2), \ldots, F(A_n)$ in the plane. In general, we might have an extra point $F(\infty)$ in the plane.

A nice description of how these maps look is given by *Schwarz-Christoffel integrals.* Take numbers $0 < \beta_1 < \cdots < \beta_n = 1$. Take $\mathbb{C}$ and remove slits corresponding to $A_j - iy$ for $y > 0$. This is still simply connected. Consider an integral

$$S(w) = \int_0^w \frac{dz}{(z - A_1)^{\beta_1} \cdots (z - A_n)^{\beta_n}},$$

where $w \in \mathbb{H}$. This is holomorphic, so we can integrate along any contour that we like. We can even try to think about what happens as w goes to the real axis. We can integrate through singularities at A_j because $\beta_j < 1$. Now, as $w \to \infty$, we have some integral that could potentially grow. However, if $\beta_1 + \cdots + \beta_j > 1$ then the integral remains bounded as $w \to \infty$.

Now, consider the special case where $\beta_1 + \cdots + \beta_n = 2$. (If the sum is < 2 then we will get an extra vertex corresponding to ∞.)

Example 15.3. This is the case of an elliptic integral. These are integrals involving square roots of third or fourth degree polynomials. In the simplest case, we have

$$I(z) = \int_0^z \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}},$$

corresponding to four points $-1/k, -1, 1, 1/k$ with $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1/2$. What happens when z varies from $-\infty$ to ∞?

Note that the point 0 is mapped to 0. As $z \to 1$, we are just integrating positive quantities. Suppose

$$k = \int_0^1 \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}}$$

to get that 1 is mapped to k. Now, as we increase z further, we move in a purely imaginary direction to reach $k + ik'$ where k' is a function of k. Then we move in a negative direction. The same thing happens on the other side. The effect is that we get a rectangle $0 \to k \to k + ik' \to -k + ik' \to -k \to 0$. Why do we get the same answer on each side? Because the integrand is holomorphic. Then the boundary of the upper half plane gets mapped to the boundary of the rectangle.

It is actually true that this also maps the upper half plane into the interior of the rectangle, so this is a conformal map.

Once we've seen this example, the general case is similar. Let's say we have $A_1 < A_2 < \cdots < A_n$. Take $x > A_n$. Then

$$S(x) = S(A_n) + \int_{A_n}^x \frac{dt}{(t - A_1)^{\beta_1} \cdots (t - A_n)^{\beta_n}}.$$

Here, everything is positive, so we start from $a_n = S(A_n)$ and proceed in the positive real direction.

Now, suppose $A_{n-1} < x < A_n$. Then

$$S(x) = S(A_{n-1}) + \int_{A_{n-1}}^x \frac{dt}{(t - A_1)^{\beta_1} \cdots (t - A_n)^{\beta_n}}.$$

Here, all but the last term in the denominator is real, and $(t - A_n)^{\beta_n} = |t - A_n|^{\beta_n} \cdot e^{i\pi\beta_n}$. Therefore, we are adding a quantity whose argument is $-i\pi\beta_n$. Therefore, we go from a_n and go to some point $a_{n-1} = S(A_{n-1})$ along a straight line, with angle $\pi\beta_n$.

We do this again, and pick up an additional angle of $\pi\beta_{n-1}$, and so on. This gives rise to a polygon with interior angles $\pi\alpha_n = \pi(1 - \beta_n)$.

What happens at the end? We get down to $a_1 = S(A_1)$. For $x < A_1$, in the case $\beta_1 + \cdots + \beta_n = 2$ then this is the same line as the $x > A_n$ case. This is because $\sum \pi\alpha_j = \pi(n-2)$ for an n-gon, and this is the same condition as $\beta_1 + \cdots + \beta_n = 2$. Alternatively, we always get an $(n+1)$-gon, but in this case one of the angles is $180°$. This picture is a bit misleading, because we could get a polygon that crossed itself. This is not great, but it could happen.

Theorem 15.4. *Given a polygon (n-gon) we can find points $A_1, \ldots, A_n$ real such that this Schwarz-Christoffel mapping gives a conformal map from $\mathbb{H}$ to the polygon $\mathbb{P}$.*

Let's describe why this is true. We must allow that one of the A_k might be the point at infinity. We'll say for now that there is a map where all points are finite.

We know by the Riemann Mapping Theorem that there exists a map $F : \mathbb{H} \to \mathbb{R}$. Moreover, we know that F extends to the boundary. We make now the assumption that each vertex of $\mathbb{P}$ corresponds to a point in $\mathbb{R}$. By rearranging, we have n points $A_1, A_2, \ldots, A_n$ that are mapped to a polygon $F(A_1), F(A_2), \ldots, F(A_n)$. We want to prove that F is given by an integral of this type.

The rough idea is that we want F to be an integral from 0 to z, so we should find F' and integrate back. To find F', we will first find its logarithmic derivative $\frac{F''}{F'}$ and integrate to get F'.

Let's say we have three points A_{k-1}, A_k, A_{k+1} getting mapped to points a_{k-1}, a_k, a_{k+1}. Suppose that they form an angle $\pi\alpha_k$.

Take $h_k(F(z) - A_k)^{1/\alpha_k}$. The effect here is that we are widening a sector into a straight line. Computing a logarithmic derivative yields

$$\frac{h_k'(z)}{h_k(z)} = \frac{1}{\alpha_k}\frac{F'(z)}{F(z) - A_k}.$$

Extend h_k to the lower half strip also. Why can we do this? We invoke the Schwarz reflection principle. We need to know one more thing: Also, $h_k'(z) \neq 0$ inside the whole strip.

Now we know that we can look at

$$F'(z) = \alpha\frac{h_k'(z)}{h_k(z)}(F(z) - A_k) = \alpha h_k'(z)h(z)^{\alpha_k - 1} = \alpha h_k'(z)h(z)^{\beta_k}.$$

From this we get that

$$\frac{F''}{F'}(z) = \frac{h_k''}{h_k'}(z) - \frac{\beta_k h_k'(z)}{h_k(z)}.$$

The point is that what we have demonstrated is that the first part is analytic and the second part only has a polar singularity at A_k. Therefore, we have

$$\frac{F''}{F'}(z) = \frac{-\beta_k}{z - A_k} + \text{holomorphic in strip}.$$

We can do this for each strip. Adding all of these up, we see that

$$\frac{F''}{F'}(z) = \sum \frac{\beta_j}{z - A_j} + \text{holomorphic in } \mathbb{C}.$$

Finally, we should check that this is last part goes to zero as $z \to \infty$, and then we can use Liouville's theorem to say that this last piece is constantly zero.

Morally, if we integrate we can find that

$$F'(z) = \frac{C}{(z - A_1)^{\beta_1} \cdots (z - A_n)^{\beta_n}}.$$

Integrating this gives us that $F(z)$ is given by the Schwarz-Christoffel integral.

16. 11/29

Last time we discussed maps from $\mathbb{H}$ or $\mathbb{D}$ to simply connected regions Ω, and we discussed what happens when Ω is a polygon; we got that these maps extended to homeomorphisms of boundaries. More explicitly, we had Schwarz-Christoffel maps with $A_1, \ldots, A_n \in \mathbb{R}$ and $0 < \beta_1, \ldots, \beta_n < 1$. If $\sum \beta_j = 2$, we have

$$S(z) = \int \frac{dw}{(w - A_1)^{\beta_1} \cdots (w - A_n)^{\beta_n}}$$

maps to a polygon. When $\sum \beta_j < 2$, we get an $(n+1)$-gon with an extra vertex corresponding to $A_{n+1} = \infty$.

First, we showed that this maps the real axis to the boundary of the polygon. Also, given an n-gon $\mathbb{P}$, the Riemann map from $\mathbb{H}$ to the the polygon $\mathbb{P}$ is given by such a Schwarz-Christoffel map. This was messy. If F is the Riemann map, the idea is to look at F' and consider the singularities of its logarithmic derivative F''/F'.

We will discuss a bit what happens in the case of the rectangle. In this case, we get elliptic integrals, which are

$$\int_0^z \frac{dw}{\sqrt{\text{cubic or biquadratic}}}.$$

For example, we could use points $-1/k, -1, 1, 1/k$ on the real axis, and get the integral

$$I(z) = \int_0^z \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}}.$$

The boundary maps $\mathbb{H}$ to a rectangle with vertices $k, k + ik', -k + ik', -k$, where

$$k = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}, \qquad k' = \int_1^{1/k} \frac{dx}{\sqrt{(x^2 - 1)(1 - k^2 x^2)}}.$$

Now, think about the inverse map from the interior of the rectangle to the upper half plane. Imagine that we reflect this rectangle across the right vertical edge to get another rectangle. The inverse function is real-valued on the boundary, so we can use the Schwarz reflection principle to define the holomorphic function in the new rectangle, which complex conjugate values in symmetric points. We can also reflect across a horizontal edge, and apply the Schwarz reflection principle to get the same behavior. We can keep doing this, and reflect repeatedly. Note that if we do this operation of complex conjugation twice, we get back what we started with, so the inverse function is periodic, repeating every two rectangles.

Therefore, the inverse function is periodic with period $4k$ and also with period $2ik'$. This is almost a holomorphic function; the only issue is at the midpoint of the upper edge ik', where the value is infinite.

The inverse function is therefore meromorphic in $\mathbb{C}$ and is doubly periodic. These are called *elliptic functions* and come up a lot in number theory. We will describe these in further detail, and this will tell us what the Riemann map is for a rectangle.

Now, we will think of meromorphic functions $f : \mathbb{C} \to \mathbb{C}$ with two periods ω_1 and ω_2, so that $f(z+\omega_1) = f(z) = f(z+\omega_2)$. This means that $f(z) = f(z + m\omega_1 + n\omega_2)$ for all $m, n \in \mathbb{Z}$. The set $m\omega_1 + n\omega_2$ is called a lattice: $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$.

Suppose we look at the case $\frac{\omega_2}{\omega_1} \in \mathbb{Q}$. In this case, we can find a common period, and f is just periodic with one period, which is not especially interesting. If $\frac{\omega_2}{\omega_1} \in \mathbb{R} \setminus \mathbb{Q}$ then f is just constant, because we can find an arbitrarily small period. We only get something interesting with $\frac{\omega_2}{\omega_1} \in \mathbb{C} \setminus \mathbb{R}$.

We can also scale to $g(z) = f(\omega_1 z)$, which will be period with periods 1 and ω_2/ω_1. So we might just assume that one of the periods is 1. We will now assume that the periods are 1 and $\tau = \omega_2/\omega_1 \in \mathbb{H}$.

Drawing a picture of our lattice, it looks like a tiling by parallelograms. We are really interested in functions that live on a fundamental parallelogram; we could then extend by periodicity to the entire plane. For example, we could take $\{x + y\tau : 0 \le x, y < 1\}$. We could of course take any other parallelogram as well; there are many choices of periods and parallelograms. The basis for the lattice is not unique; for example, we might have $(a + b\tau, c + d\tau)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. We want $f(z) = f(z+\lambda)$ for $\lambda \in \Lambda$.

There are no non-constant holomorphic f which are doubly periodic. The proof is easy: such a function would be bounded on each compact parallelogram, and hence it would be bounded and entire, and hence constant by Liouville's Theorem 3.3. Therefore, f must have singularities.

We can also try to integrate f around the boundary of one of these period paralleograms. By Cauchy's theorem, we have

$$\int_P f\,dz = 2\pi i \sum (\operatorname{Res} f),$$

but by periodicity, the integral should be zero. Therefore, the residues must cancel out. This means that it should have at least two poles (counted with multiplicity) or a pole of order 2.

We can also integrate f'/f, which is also meromorphic and doubly periodic. Then as before,

$$0 = \int_P \frac{f'}{f}\,dz = 2\pi i(\#\text{ poles} - \#\text{ zeros}).$$

Therefore, the number of zeros of f equals the number of poles of f. Also, $f(z) - c$ is also doubly periodic, so

$$\#\text{ poles of } f = \#\text{ zeros of } f = \#\text{ times any value is attained in a period parallelogram.}$$

There is a nice way to write down these elliptic functions. This is called the Weierstrass $\wp$-function.

The idea is to take a nice function f and look at $\sum_{\lambda \in \Lambda} f(z+\lambda) = F(z)$. We need to choose a nice holomorphic function f so that this series converges; then F would be doubly periodic.

Definition 16.1. Let
$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0 \in \Lambda} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

We want to prove that this is absolutely convergent and gives a doubly periodic function. First, we should show absolute convergence.

Proof. Here,
$$\sum_{0 \neq \lambda \in \Lambda} \frac{1}{|\lambda|^r} < \infty$$
if $r > 2$. To see this, split the lattice into big annuli. The number of lattice points in a big disc should be $\sim cR^2$ for some constant c, so
$$\sum_{R < |\lambda| \leq 2R} \frac{1}{|\lambda|^r} \leq \frac{cR^2}{R^\lambda}.$$
Partitioning the sum into dyadic blocks $2^k \leq |\lambda| \leq 2^{k+1}$ gives us a convergent series for $r > 2$, and it does not converge for $r = 2$.

If $|\lambda|$ is large then we have
$$\left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2} \right) = \left| \frac{-2\lambda z - z^2}{\lambda^2 (z+\lambda)^2} \right| \leq \frac{C(1+|z|)}{|\lambda|^3},$$
so the series for $\wp$ converges absolutely. Therefore, the series for $\wp$ converges absolutely, but $\wp$ has double poles at all lattice points. $\square$

We should also show that $\wp$ is doubly-periodic.

Proof. First, we have
$$\wp'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z+\lambda)^3}$$
converges absolutely, and this is clearly doubly periodic with period 1 and τ. Now, consider
$$\wp(z+1) - \wp(z) = \int_z^{z+1} \wp'(w)\, dw = a$$
for some constant a independent of z because $\wp'$ is known to be periodic with period 1. Similarly, $\wp(z+\tau) - \wp(z) = b$ for some constant b. Observe that $\wp$ was defined to be an even function. Plugging in $z = 1/2$, we have $\wp(1/2) - \wp(-1/2) = a = 0$ and $\wp(\tau/2) - \wp(-\tau/2) = b = 0$. So therefore $\wp(z)$ is periodic with periods 1 and τ. $\square$

Observe that $\wp'$ is an odd function by definition; everything is defined using cubes instead of squares, and $\wp'(z) = -\wp'(-z)$. Therefore $\wp'(1/2) = -\wp'(-1/2) = 0$, and $\wp'(\tau/2) = \wp'((1+\tau)/2) = 0$. On each fundamental parallelogram, $\wp'$ has zeros at $1/2, \tau/2, (1+\tau)/2$. Then we have
$$\wp'(z)^2 = 4(\wp(z) - \wp(1/2))(\wp(z) - \wp(\tau/2))(\wp(z) - \wp((1+\tau)/2)).$$

Proof. This makes sense because the two sides have matching zeros. There are no other zeros because the number of zeros equals the number of poles. Observe that the poles at 0 also match. Then

$$\frac{\wp'(z)^2}{(\wp(z)-\wp(1/2))(\wp(z)-\wp(\tau/2))(\wp(z)-\wp((1+\tau)/2))}$$

is a holomorphic doubly-periodic function without singularities, which means that it is constant. We can then work out what the constant should be. In fact, $\wp'$ looks like $-2/z^3$, and that squared would give 4. □

These functions $\wp$ and $\wp'$ are fundamental because every elliptic function is a rational function of $\wp$ and $\wp'$.

Proof. If we have an even elliptic function F with zeros at $a_1, -a_1, a_2, -a_2, \ldots, a_n, -a_n$ then we could write

$$(\wp(z)-\wp(a_1))(\wp(z)-\wp(a_2))\cdots(\wp(z)-\wp(a_n)),$$

and if F had poles at $b_1, -b_1, \ldots, b_n, -b_n$, then could look at

$$\frac{1}{(\wp(z)-\wp(b_1))\cdots(\wp(z)-\wp(b_n))}.$$

We must think a little if $a_1 = -a_1$, etc, but in any case, we can write down some expression of this type to cancel out the zeros and poles of F, so the quotient just becomes a constant. Therefore, every even elliptic function is just a rational function of $\wp$. Now, if F were an odd elliptic function, then $F/\wp'$ is an even elliptic function. □

Now, consider the case where $\tau \in i\mathbb{R}$, and construct the Weierstrass $\wp$ function with these two periods. Recall that when we did the reflection argument, the period got doubled because we had to reflect (complex conjugate) twice. So we actually want smaller rectangles, scaled down by a factor of 2. We want to show that $\wp$ is real-valued on the boundaries of the small rectangles. There are four small rectangles in each fundamental rectangle. Then $\wp$ will map two of the small rectangles to the upper half plane and two to the lower half plane.

The picture is that there are three points A_1, A_2, A_3 on the real axis with another point of infinity, corresponding to $\wp(1/2)$, $\wp(\tau/2)$, $\wp((1+\tau)/2)$.

We can try to think about the Laurent exapansion of $\wp$ around 0. We could write down

$$\frac{1}{(z+\lambda)^2} = \frac{1}{\lambda^2}\frac{1}{(1+\frac{z}{\lambda})^2} = \frac{1}{\lambda^2}\left(1 - 2\frac{z}{\lambda} + 3\left(\frac{z}{\lambda}\right)^2 - \cdots\right).$$

Then

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\lambda\neq 0}\left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right) = \frac{1}{z^2} + \sum_{\lambda\neq 0}\sum_{k=1}^{\infty}(-1)^k(k+1)\frac{z^k}{\lambda^{k+2}} \\ &= \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)z^{2k}\sum_{\lambda\neq 0}\frac{1}{\lambda^{2k+2}}.\end{aligned}$$

Now let

$$E_k(\tau) = \sum_{0\neq\lambda\in\Lambda}\frac{1}{\lambda^k}.$$

(Note that $E_k(\tau) = 0$ if k is odd.) These come up naturally in the Laurent expansion of $\wp(z)$, but they are interesting in their own right. These are called *Eisenstein series* of weight k and are the first example of *modular forms*. They satisfy a periodicity property $E_k(\tau+1) = E_k(\tau)$ and $E_k(-1/\tau) = \tau^k E_k(\tau)$. This is in $\mathbb{H}$, so we have some map

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

This is where we leave the book. We will discuss three topics over the next three lectures.

(1) Dirichlet's Problem
(2) Picard's Theorem
(3) Analytic continuation and monodromy theorem.

17. 12/1

Today we will discuss harmonic functions and the Dirichlet problem. This is another approach to the Riemann Mapping Theorem.

Definition 17.1. A C^2 function $u(x, y)$ is *harmonic* if it satisfies

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

on Ω.

Locally, a harmonic function is just the real part of a holomorphic function. But this need not be true globally.

Example 17.2. On $\mathbb{C} \setminus \{0\}$, consider the function $\log(x^2 + y^2) = \log |z|^2$. This is harmonic, but we can check that this is not globally the real part of a holomorphic function.

If, on the other hand, the region Ω is simply connected, then the harmonic functions on Ω are simply $u = \mathrm{Re}(f)$ for a holomorphic f.

Here is Dirichlet's problem:

Problem 17.3 (Dirichlet's Problem)**.** Suppose that we are given a region Ω and a continuous function f on the boundary of Ω. We want to find a function u harmonic on Ω such that for any a on the boundary, $\lim_{z \to a} u(z) = f(a)$.

Remark. Riemann's original argument for wanting to solve this is as follows. Given a map f from a simply connected region Ω into the unit disk, sending $z_0 \to 0$, then write $f(z) = (z - z_0)g(z)$. Then g is not 0 on the region, so $f(z) = (z - z_0)\exp(h(z))$.

As $z \to \partial\Omega$, we have $|f(z)| \to 1$, which means that $\mathrm{Re}\, h(z) \to -\log(z - z_0)$. This gives a form for the Riemann map.

The problem was that at the time, there was no proof that the Dirichlet problem could always be solved. Riemann had a limiting argument that he took for granted, but Weierstrass proved that it wasn't always true. Hilbert eventually proved that the Dirichlet problem could be solved.

We begin with some preliminaries about harmonic functions. Since they are the real parts of holomorphic functions, we already know some things about them.

Theorem 17.4 (Mean Value Property)**.** *If $a \in \Omega$ and we have a disk with radius r around a contained in Ω, then*

$$u(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta})\, d\theta.$$

Proof. Take real parts of Cauchy's formula

$$f(a) = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(a+z)}{z}\, dz. \qquad \square$$

This always means that u satisfies the maximum and minimum principle:

Theorem 17.5. *u has no local maximum or minimum unless u is constant.*

If u is harmonic on $\mathbb{D} \setminus \{0\}$ and it is bounded near the origin, then we can remove the singularity at the origin. Otherwise, u either grows as $c \log |z|$ as $z \to 0$ (corresponding to the poles of holomorphic functions) or we have some sort of essential singularity. We haven't actually given a proof for this, but it is similar to the holomorphic case.

This means that we can't always solve the Dirichlet problem.

Example 17.6. Suppose we have $\mathbb{D} \setminus \{0\}$ with boundary $\{0\} \cup \{|z| = 1\}$. Consider boundary conditions $f(0) = 1$ and $f(z) = 0$ if $|z| = 1$. We cannot solve this because we could remove the singularity at 0 to get a solution that is harmonic in $\mathbb{D}$. Then the boundary conditions violate the maximum principle, so therefore the Dirichlet problem has no solution.

The point is that we can solve the Dirichlet problem as long as the complement of the region has no connected components that shrink to a point (e.g. simply connected). Otherwise, we run into cases like this one.

Now, we consider Dirichlet's problem for a disk. It suffices to consider the unit disk $\mathbb{D} = \{|z| < 1\}$. Suppose that f is continuous on $|z| = 1$. We want a harmonic function matching f on the boundary. To do this, we will use the Poisson kernel.

Definition 17.7. The *Poisson kernel* is

$$P_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta} = \frac{1}{1 - re^{i\theta}} + \frac{re^{-in\theta}}{1 - re^{-i\theta}} = \frac{1 - re^{-i\theta} + re^{i\theta} - r^2}{|1 - re^{i\theta}|^2} = \frac{1 - r^2}{|1 - re^{i\theta}|^2}$$

for $\theta \in \mathbb{R}$ and $0 \le r < 1$.

From this, it is clear that $P_r(\theta) \ge 0$, and we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1$$

by integrating term by term. Lastly, as $r \to 1$ and if $|\theta| \ge \delta$, we have that $P_r(\theta) \to 0$ uniformly, for any fixed $\delta > 0$.

If $z = re^{i\theta} \in \mathbb{D}$, then consider the convolution

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi) f(e^{i\varphi})\, d\varphi.$$

We need to show that this satisfies the Dirichlet problem, i.e. it is harmonic and has the right behavior as we approach the boundary.

We can also write

$$P_r(e^{i\theta}) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

In other words, we can write

$$u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{1+ze^{-i\varphi}}{1-ze^{-i\varphi}}\right) f(e^{i\varphi})\,d\varphi = \operatorname{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1+ze^{-i\varphi}}{1-ze^{-i\varphi}}\right) f(e^{i\varphi})\,d\varphi\right).$$

This is the real part of a holomorphic function, and hence it is harmonic.

Now, let $z \to e^{i\theta}$. We need to show that $u(z) \to f(e^{i\theta})$. Note that this kernel is positive, integrates to 1, and goes to zero for $|\theta| \geq \delta$. Therefore, when $\theta - \varphi$ is fairly large, this will go to zero. The only interesting part of the integral is when φ is close to θ, but then we can use the continuity of f, so $f(e^{i\varphi}) \approx f(e^{i\theta})$ if $\varphi \approx \theta \bmod 2\pi$. This is a standard result.

We will use this formula again, and we can say more about what harmonic functions do inside the unit disc. First, we can translate and dilate to solve the Dirichlet problem for any disc. Let's stick to the unit disc for simplicity. In this case, we can obtain

$$|u(z)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\varphi)|f(e^{i\varphi})|\,d\varphi \leq M$$

if $|f(e^{i\varphi})| = M$ is bounded. The most useful inequalities are when f is nonnegative, so we assume $f \geq 0$ on the unit disk. In this case, we can take out the maximum of the Poisson kernel:

$$(\min|P_r(\theta-\varphi)|)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\,d\varphi \leq u(z) \leq (\max|P_r(\theta-\varphi)|)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\,d\varphi,$$

and here, the integrals are simply $u(0)$. In addition, we have

$$\frac{1-r^2}{(1+r)^2} \leq \frac{1-r^2}{|1-re^{i\theta}|^2} \leq \frac{1-r^2}{(1-r)^2}.$$

Therefore, we have

$$\frac{1-r}{1+r}u(0) \leq u(z) \leq \frac{1+r}{1-r}u(0). \tag{1}$$

This is a nice inequality satisfied by harmonic functions on the unit disk, and it is known as *Harnack's inequality.*

Suppose we take a compact set K inside the unit disk. If we take two points $z_1, z_2 \in K$, this tells us that there is some constant $c > 0$ such that

$$\frac{u(z_2)}{C} \leq u(z_1) \leq Cu(z_2).$$

This means that the values of u are more or less of the same size on any compact set. This implies a convergence theorem which is like Montel's theorem 14.8.

Theorem 17.8 (Harnack's theorem)**.** *If Ω is some region and we have an increasing sequence $u_1 \leq u_2 \leq \cdots$ of harmonic functions on Ω, then either $u_n \to \infty$ for all $z \in \Omega$ or $u_n \to u$ is harmonic for all $z \in \Omega$. In both cases, the convergence is uniform on all compact sets.*

Proof. Note that $u_n - u_1$ is a sequence of nonnegative harmonic functions. Suppose that there is a point z_0 where $u_n(z_0)$ converges to some finite value. Fix any compact set K with $z_0 \in K$ and apply Harnack's inequality 1 to see that

$$\frac{1}{C}(u_n(z_0) - u_1(z_0)) \leq u_n(z) - u_1(z) \leq C(u_n(z_0) - u_1(z_0)).$$

Therefore, $u_n(z) - u_1(z)$ is now uniformly bounded. Looking at $m > n$, we have the same inequalities

$$\frac{1}{C}(u_m(z_0) - u_n(z_0)) \leq u_m(z) - u_n(z) \leq C(u_m(z_0) - u_n(z_0)).$$

Arguing by Cauchy sequences yields that every point converges to some finite value. If, on the other hand, $u_n(z_0) \to \infty$, then these inequalities also give the desired result. $\square$

We showed that harmonic functions satisfy the mean value property. In fact, the converse is also true.

Theorem 17.9. *If φ satisfies the mean value property 17.4*

$$\varphi(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta$$

then φ is harmonic.

Proof. Suppose we have a point a and a disk around a satisfying the mean value property. Construct φ' to be the same as φ on the circle, and defined using the Poisson kernel inside the circle. Now φ' satisfies the mean value property.

Now, consider $\varphi - \varphi'$. This satisfies the mean value property on the disc and is 0 on the boundary. By the maximum principle, this means that $\varphi - \varphi' \leq 0$. By the minimum principle, $\varphi - \varphi' \geq 0$, so therefore $\varphi = \varphi'$ and hence φ is indeed harmonic. $\square$

Perron defined a richer class of functions called *subharmonic* functions.

Definition 17.10. A continuous function φ on Ω is *subharmonic* if

$$\varphi(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta$$

whenever $|z - a| \leq r$ is contained in Ω. The function φ is *superharmonic* if the reverse inequality holds.

Example 17.11. Take any harmonic function f. Then for any $\alpha > 0$, the function $|f(z)|^\alpha$ is subharmonic. This can be done using Holder's inequality.

Here are some properties of subharmonic functions:

(1) If φ is subharmonic then $-\varphi$ is superharmonic.
(2) If φ_1 and φ_2 are subharmonic then $\varphi_1 + \varphi_2$ is subharmonic.
(3) If φ_1 and φ_2 are subharmonic then $\max(\varphi_1, \varphi_2)$ is also subharmonic.

Proposition 17.12. *Consider some Ω and suppose that Δ is some disk contained in Ω. Suppose that φ is subharmonic on Ω. Look at the function*

$$\varphi' = \begin{cases} \varphi \text{ in } \Omega \setminus \Delta \\ \text{Poisson kernel construction in } \Delta. \end{cases}$$

This means that φ' is subharmonic in $\Omega \setminus \overline{\Delta}$ and is harmonic in Δ. We claim that in fact φ' is subharmonic in Ω.

For this, we need a different version of what it means to be subharmonic.

Proposition 17.13. *If φ is subharmonic on Ω then for every $\Omega' \subseteq \Omega$ and every harmonic u on Ω', we have that $\varphi - u$ satisfies the maximum principle, and conversely.*

Proof. Note that φ is subharmonic (but not necessarily superharmonic). Therefore, it is obvious that $\varphi - u$ is subharmonic and satisfies the maximum principle. For the second part, we have a point a and a disk around a. We want to bound $\varphi(a)$ by an average around the circle. Again, we construct φ' to be harmonic inside the disk, and matching with φ on the boundary.

Then $\varphi - \varphi'$ satisfies the maximum principle, and it equals zero on the boundary. Then $\varphi(a) - \varphi'(a) \le 0$, so

$$\varphi(a) \le \varphi'(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi'(a + re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta$$

because f' satisfies the mean value property, and hence we are done. ☐

Now we are ready to finish the proof of the proposition above.

Proof of Proposition 17.12. Take any $\Omega' \subseteq \Omega$ and take any harmonic u on Ω'. We want to show that $\varphi' - u$ satisfies the maximum principle. The only thing that can go wrong is that $\varphi' - u$ has some maximum on the boundary of Δ. But that's impossible because on the boundary of Δ, we had $\varphi' = \varphi$, so $\varphi - u$ would have a maximum at that point, which is a contradiction. ☐

Now we can see how the proof will go. The idea is to look at a class of subharmonic functions called the *Perron family*. Consider f continuous on the boundary of Ω, and f bounded by $|f| \le M$ on the boundary of Ω. Then take the class of all subharmonic functions v with

$$\limsup_{z \to a} v(z) \le f(a).$$

for a on the boundary of Ω. (For example, any constant $< -M$ is in this family.) Then we define a new function

$$u(z) = \sup_{v \text{ in Perron family}} v(z).$$

Then we will have two propositions:

Proposition 17.14. *This $u(z)$ is harmonic on Ω.*

Proposition 17.15. *For many nice regions, $u(z) \to f(a)$ as $z \to a$ for boundary points a.*

Remark. This last proposition can't be true for every region, as we demonstrated earlier. Here, "nice region" will be regions for which one can construct a "barrier".

18. 12/6

Last time, we were talking about harmonic functions. We have boundary values on $\partial\Omega$, and we want a harmonic function inside Ω with those boundary values.

For the disk, we solved the Dirichlet problem explicitly. For this we used the Poisson kernel

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}.$$

The idea is that given a function $u(e^{i\varphi})$ for $\varphi \in (-\pi, \pi]$ on the boundary, we can form a convolution

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi) u(e^{i\varphi})\, d\varphi.$$

As $z \to e^{i\varphi}$, we showed that $u(z) \to u(e^{i\varphi})$. This is one of the main tools that we have in the general form of Dirichlet's problem.

We also proved that the mean value property also implies that the function is harmonic. The reason is that we can use the Poisson kernel to solve the Dirichlet problem in any disk. If we have a function φ satisfying the mean value property, it also satisfies the maximum principle. Then the difference between φ and the Poisson formula solution is zero on the boundary and satisfies the minimum property, so it is identically zero.

One of the facts that we need about harmonic functions is Harnack's principle and inequalities for non-negative harmonic functions. For any compact subset $K \subseteq \Omega$, we have that

$$M^{-1} \leq \frac{u(z_2)}{u(z_1)} \leq M$$

for some $M = M(K)$. This allowed us to prove Harnack's theorem: If we have an increasing sequence of harmonic functions $u_1 \leq u_2 \leq \cdots$ on Ω then either $u_n \to \infty$ for all $z \in \Omega$ or $u_n \to u$ is harmonic for all $z \in \Omega$. The point of the proof is that the values of u are approximately the same size on any compact set, so convergence at some point implies convergence at every point.

We also discussed subharmonic functions, which are functions such that

$$\varphi(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta$$

satisfying the mean value property as an upper bound. Another way of characterizing this is that φ is subharmonic on Ω if for any harmonic u on some region $\Omega' \subseteq \Omega$ we have that $\varphi - u$ satisfies the maximum property.

Now, choose φ to be subharmonic on Ω, and choose some disk Δ inside Ω. Define φ' to be equal to φ outside Δ and in the boundary of Δ, but satisfies the Poisson construction inside Δ. Then φ' is subharmonic. That's essentially of all of the last lecture.

Why did we go to these subharmonic functions instead of working with harmonic functions? There is one additional nice property about subharmonic functions.

Proposition 18.1. *If φ_1 and φ_2 are subharmonic, then $\max(\varphi_1, \varphi_2)$ is also subharmonic.*

Note that this is not necessarily true for harmonic functions; the maximum of harmonic functions might only be subharmonic.

Now, suppose that Ω is bounded and f is continuous on $\partial\Omega$. Suppose that $|f(a)| \leq M$ on the boundary.

Definition 18.2. The *Perron family* $P(f)$ is defined as

$$P(f) = \{\varphi \text{ subharmonic} : \limsup_{z \to a} \varphi(z) \leq \varphi(a)\}.$$

Note that all constants $< -M$ are in this family. Then we can define a new function

$$u(z) = \sup_{\varphi \in P(f)} \varphi(z).$$

Theorem 18.3. *u is harmonic on Ω.*

Proof. By the maximum principle, we know that $|\varphi(z)| \leq M$ for all $z \in \Omega$ and for all $\varphi \in P(f)$. Take a point $z_0 \in \Omega$. Then there is some disk Δ around z_0 contained in Ω. We know that there is a sequence of subharmonic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that $\varphi_n(z_0) \to u(z_0)$. We don't know what this does at any other point. We want to replace these by harmonic functions to apply Harnack's principle.

Let $\Phi_n = \max(\varphi_1, \ldots, \varphi_n)$; then we can replace φ_n by Φ_n and get that $\Phi_n(z_0) \to u(z_0)$. This is a legitimate thing to do because the maximum of subharmonic functions is still subharmonic. In addition, we also know that $\Phi_1 \leq \Phi_2 \leq \ldots$ is increasing.

Now, replace Φ_n by

$$\Phi'_n = \begin{cases} \Phi_n \text{ on the boundary of } \Delta \\ \text{Poisson kernel construction inside } \Delta, \end{cases}$$

which is harmonic in Δ. We know that $\Phi_n - \Phi'_n$ satisfies the maximum principle by one of the characterizations of subharmonic functions. Therefore, $\Phi_n \leq \Phi'_n$ in Δ and $\Phi'_{n+1} \geq \Phi'_n$. This last fact is true because $\Phi'_{n+1} - \Phi'_n$ is harmonic and nonnegative on the boundary of Ω, so it is nonnegative inside Ω.

Therefore, $\Phi'_1 \leq \Phi'_2 \leq \ldots$ is an increasing sequence of harmonic functions belonging to the Perron family (and are hence bounded). By Harnack's theorem 17.8, they must converge to some harmonic function, so $\Phi'_n \to U$ harmonic in Δ. The only point where we can evaluate anything is z_0, and $U(z_0) = u(z_0)$. We have therefore proved the result at z_0. We want to show that $u(z)$ matches $U(z)$, which would show that u is harmonic.

Now, choose any other point $z_1 \in \Delta$. By the same process, we get $\psi_1, \psi_2, \ldots$ subharmonic with $\psi_n(z_1) \to u(z_1)$. First, replace ψ_n by $\Psi_n = \max(\Phi_n, \psi_1, \ldots, \psi_n)$, which is also subharmonic. Again, replace Ψ_n by

$$\Psi'_n = \begin{cases} \Psi_n \text{ on the boundary of } \Delta \\ \text{Poisson kernel construction inside } \Delta. \end{cases}$$

This is harmonic and increasing, and we know that $\Psi'_n \geq \Phi'_n$. So therefore we know by Harnack's principle again that $\Psi'_n \to V$ is harmonic, and $V \geq U$. In addition, $V(z_1) = u(z_1)$, and here, $U(z) - V(z)$ is some nonpositive function, and $V(z_0) = U(z_0) = u(z_0)$. Therefore, $U(z) - V(z)$ has a local maximum at z_0. By the maximum principle, this implies that $U(z) = V(z)$. Therefore, $U(z_1) = u(z_1)$, and so $U(z) = u(z)$ everywhere inside the disk, and hence $u(z)$ is harmonic. $\square$

We've produced a harmonic function $u(z)$ which we call the Perron function of f. Now, as $z \to a \in \partial\Omega$, does $u(z) \to u(a)$? Recall that this is not true for the punctured disk; we need to know something about the boundary for this to work. This requires the notion of a barrier at a boundary point a. It is also called a peaking function. This terminology is due to Lebesgue.

Definition 18.4. A *barrier* at a point a is a harmonic function v which is continuous up to the boundary, satisfying the property that at the boundary, v is strictly positive except at a, where $v(a) = 0$.

Theorem 18.5. *If there is a barrier at a then the limit of the Perron function u equals $f(a)$ as $z \to a$.*

Therefore, if we can construct a barrier then we can solve the Dirichlet problem for that region.

Example 18.6. As an example, suppose we take the unit circle. Then a barrier at 1 might be $1 - \operatorname{Re} z$, and a barrier at $e^{i\theta}$ must be $1 - \operatorname{Re}(ze^{-i\theta})$.

Example 18.7. Let Ω be some region and suppose that 0 is on the boundary. Suppose that the line segment $[-1, 0]$ is not in Ω. Then we can define $\operatorname{Re}(\sqrt{\frac{z}{z+1}})$ and check that this is a barrier at 0. Therefore, we can construct a barrier for any point which is the endpoint of a line segment not in Ω.

This doesn't quite work for a slit $(-\infty, 0]$, but a barrier can be found. For example, a barrier at 0 would be $-\operatorname{Re} z$, and a barrier at -1 might involve $\sqrt{z}$.

Let's prove the theorem:

Proof of Theorem 18.5. Suppose that $|f| \leq M$, and suppose that $v(z)$ is a barrier at a. Consider the function

$$W(z) = f(a) + \varepsilon + \frac{v(z)}{v_0}(M - f(a)).$$

This is a harmonic function. Note that $W(z) \geq f(a) + \varepsilon$.

We know that $v(a) = 0$ and $v(z)$ is strictly positive elsewhere. So remove a small region of the boundary near a. What remains is a compact set, where $v(z)$ attains a minimum. Define v_0 to be the minimum of $v(z)$ outside a small region near a.

If z goes to a boundary point not near a, then $v(z)/v_0 > 1$, so $W(z)$ will tend to something $> M + \varepsilon$. If $z \to b$ near a, then $W(z) \geq f(a) + \varepsilon \geq f(b)$.

Then the boundary values of W are always $\geq f$, which means that W is bigger than every element of the Perron family. This implies that $W(z) \geq u(z)$, where $u(z)$ is the Perron solution. Therefore, $u(a) \leq W(a) = f(a) + \varepsilon$.

The same argument gives that $u(a) \geq f(a) - \varepsilon$, which finishes the proof. ∎

That's the story with the Dirichlet problem. We showed last time that solving the Dirichlet problem is closely related to proving the Riemann Mapping Theorem, by considering the function $\log |z - z_0|$.

For simply connected regions, all the regions are the same because they can all be mapped to the disk. Consider the next best thing, with annuli bounded by $|z| = 1$ and $|z| = R > 1$.

Proposition 18.8. *Any two such annuli are* not *conformally equivalent.*

Proof. Suppose we have annuli with radii $1, R_1$ and $1, R_2$, and a map f between them. Then we can show with a continuity argument that the boundary of the disk at 1 must be mapped to the boundary of 1 or the boundary of R_2, and the same is true for the boundary at R_1.

Therefore, we can assume that as $|z| \to 1$, we have $|f(z)| \to 1$, and as $|z| \to R_1$, we have $|f(z)| \to R_2$. Then we can consider the harmonic function

$$\log|f(z)| - \frac{\log R_2}{\log R_1}\log|z|$$

equaling zero on the boundaries. Therefore, this is a constant, so

$$\log|f(z)| = (\log|z|)\frac{\log R_2}{\log R_1}.$$

Therefore, we see that $|f(z)| = |z|^\alpha$ for $\alpha = \log R_2/\log R_1$. This is only possible if $\alpha \in \mathbb{Z}$, and we have $f(z) = z^n$. But the only way for this to be injective is to have $n = 1$, so in this context, we have $f(z) = z$. Thus this is the only conformal map between two annuli. $\square$

We can use this solution to the Dirichlet problem to characterize a region with n connected components. In particular, it can be shown that any such region looks like annuli with slits removed.

19. 12/8

Today we will prove this remarkable theorem of Picard.

Theorem 19.1 (Picard's Little Theorem)**.** *Any nonconstant entire function f takes all values except possibly one.*

Example 19.2. The function $f(z) = e^z$ takes all values except zero.

We already had a homework problem where we proved this with the added assumption that f is entire of finite order, which was not too hard. There is also a stronger version of this theorem:

Theorem 19.3 (Picard's Big Theorem)**.** *If f is holomorphic in some region and has an essential singularity at 0, then f takes on all values except maybe for one.*

Remark. This is stronger than Casorati-Weierstrass 5.1, which said that the image of f is dense in a neighborhood of an essential singularity.

Example 19.4. Consider the function $e^{1/z}$, with an essential singularity at 0.

Note that Picard's Big Theorem implies Picard's Little Theorem for functions in the extended complex plane, by considering types of singularities at infinity.

There are many approaches to this theorem. We will use an approach due to Andre Bloch. First, we state a corollary of this method.

Corollary 19.5. *If f is entire then the image of f contains disks of arbitrarily large size.*

Let's assume this corollary for the moment, and see how to reach Picard's little theorem as a result.

Proof of Theorem 19.1. Suppose that f is entire that f omits two values. Without loss of generality, suppose these are 0 and 1, so that $f(z) \neq 0$ and $f(z) \neq 1$. Suppose further that f is not constant.

First, observe that $F(z) = \frac{1}{2\pi i} \log f(z)$ is entire. Note that $F(z) \neq 0$, and since $f(z) \neq 1$, the log also omits all integer multiples of $2\pi i$. Therefore, $F(z)$ omits all of the integers: $F(z) \neq n$ for all $n \in \mathbb{Z}$.

Now, consider $h(z) = \sqrt{F(z)} - \sqrt{F(z) - 1}$. This is entire, and $h(z) \neq 0$. That means that we can define an entire function $H(z) = \log h(z)$. What are the values that are omitted by $h(z)$ and $H(z)$?

Note that $h(z)$ omits values $\sqrt{n} - \sqrt{n-1}$ for all $n \geq 1$. Observe that $1/h(z) = \sqrt{F(z)} + \sqrt{F(z) - 1}$. Now,

$$\frac{h(z) - 1/h(z)}{2} = \sqrt{F(z)},$$

which omits all values of the form $\sqrt{n}$, so therefore h must also omit values $\sqrt{n} + \sqrt{n-1}$.

We see that $H(z)$ omits all values $\pm \log(\sqrt{n} + \sqrt{n-1}) + 2\pi i m$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. On the real axis, there is a sequence of omitted points, with difference tending to zero at $n \to \infty$. For each value of m, we get something like this, so we get a grid of these points, all with bounded differences between neighboring points. Therefore, $H(z)$ cannot contain a disk of radius 10, which contradicts Corollary 19.5. $\square$

Now, we need to show that the image of any entire function contains large disks. To show this, we need the following general result.

Theorem 19.6 (Bloch-Landau Theorem)**.** *Suppose that f is holomorphic on $\overline{\mathbb{D}}$ and f is normalized to have $f(0) = 0$ and $f'(0) = 1$. Then we have two results:*

(1) (Landau) The image of f contains a disk of radius $\geq 1/24$.
(2) (Bloch) There is a disk $S \subseteq \overline{\mathbb{D}}$ such that $f : S \to f(S)$ is bijective and $f(S)$ contains a disk of radius $\geq 1/72$.

In both of these results, it is an open problem to determine the largest constant for these to be true. In the Landau result, it is known that the best constant is $\geq 1/2$, and it is conjectured to be $L = \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} = 0.5432\ldots$. In the Bloch result, Ahlfors showed that the best constant is $\geq \sqrt{3}/4$, and it is conjectured that the best constant is

$$B = \frac{\Gamma(1/3)\Gamma(1/12)}{\sqrt{1+\sqrt{3}} \cdot \Gamma(1/4)}.$$

Remark. Bloch was an interesting guy. He served in World War I and was wounded in the war. He and his brother were trained to be mathematicians, and they were recovering from their wounds. One day Bloch murdered his brother and aunt and uncle and was sentenced to live in a mental institution. This theorem was proved eight years after he was in the asylum. He wrote letters to other mathematicians, and Hadamard wanted to invite him to dinner, but Bloch could never go. According to one story, Bloch thought that his brother and aunt and uncle were mentally ill.

Proof of Corollary 19.5. Suppose that we have this theorem. Then we can make an obvious change. By translating, suppose that $f(0) = 0$ and $f'(0) = \mu$. Let f be holomorphic by a disk of radius R centered at 0. Then $g(z) = f(Rz)/R\mu$ is defined on the unit disk satisfying

$g(0) = 0$ and $g'(0) = 1$. This then implies that the image of $g(z)$ contains a disk of some radius like $1/24$, and therefore the image of f contains a disk of radius $\geq R|\mu|/24$. From this, Corollary 19.5 is clear.

For this, we only needed the Landau part of the Bloch-Landau theorem. $\square$

Now, let's try to prove the Bloch-Landau theorem.

Proof of Landau part of Theorem 19.6.

Lemma 19.7. *First, suppose that $f(0) = 0$ and $f'(0) = 1$. Suppose that $|f(z)| \leq M$ (with $M \geq 1$) is bounded on $\overline{\mathbb{D}}$. Then we claim that the image of f contains a disk of radius $\geq 1/6M$. (In fact, we can take the disk to be centered at the origin). This is a weak step toward the proof; we ultimately want something without dependence on the bounds of f.*

Proof. To prove this claim, write $f(z) = z + a_2 z^2 + \cdots$, and note that

$$a_k = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} \leq M.$$

Now, look at $|z| = R$. This circle gets mapped to some loop, and we want to get a bound on $|f(z)|$. Clearly, we have

$$|f(z)| \geq |z| - |a_2||z|^2 - |a_3||z|^3 - \cdots \geq r - M(r^2 + r^3 + \cdots) = r - \frac{Mr^2}{1-r} = \frac{r - (M+1)r^2}{1-r}.$$

Then the image of f contains the disk $|z| \leq \frac{r}{1-r}(1 - (M+1)r)$. Choose $r = \frac{1}{2(M+1)}$ to get that the image of f contains the disk $|z| \leq \frac{1}{2(2M+1)} \leq \frac{1}{6M}$. $\square$

So if we assume that the size of f is bounded then we have some result. From the same thing, we can say the following.

Lemma 19.8. *Suppose that f is holomorphic in a disk of radius R centered at the origin, f is zero at the center, $f'(0) = \mu$, and $|f(z)| \leq M$. Then the image of f contains a disk of radius $\geq \frac{1}{6}\frac{(R\mu)^2}{M}$.*

Proof. Let $g(z) = f(Rz)/R\mu$, and $|g| \leq M/r\mu$, and $g'(0) = 1$. Then applying the previous result yields that the image of g contains a disk of radius $1/6 \cdot R\mu/M$. $\square$

Now, let's return to the original problem. Suppose that $f(0) = 0$ and $f'(0) = 1$. Suppose that f is holomorphic in $\mathbb{D}$. Pick a point $z_0 \in \mathbb{D}$, and suppose that $|z_0| = r_0$. Then f is holomorphic in a disk of radius $(1 - r_0)$ centered at z_0.

We would like $\mu = |f'(z_0)|$ to be big and $(1 - r_0)$ to be big; then we would be able to use the preceding results. Choose z_0 such that $|f'(z_0)|(1 - |z_0|)$ is a maximum. This just a continuous function in the unit disk, so it takes a maximum value. Note that the maximum value is $\geq |f'(0)|(1 - 0) = 1$, so this maximum is actually pretty big. Let $|z_0| = r_0$. We examine the disk of radius $(1 - r_0)/2$ centered at z_0. What we want is a bound for $|f(z) - f(z_0)|$ for z

inside this disk; this will be the parameter M. We now have

$$|f(z) - f(z_0)| = \left|\int_{z_0}^{z} f'(w)\,dw\right| \leq |z - z_0| \left(\max_{|w-z_0|<\frac{1-r_0}{2}} |f'(w)|\right)$$
$$\leq |z - z_0| \left(\max_{|w-z_0|<\frac{1-r_0}{2}} \frac{|f'(w)|(1-|w|)}{1-|w|}\right) \leq \frac{|z-z_0|}{(1-r_0)/2}\mu(1-r_0) = \mu(1-r_0)$$

since $1 - |w| \geq (1 - |z_0|) - |z_0 - w| \geq (1 - r_0)/2$.

Therefore, taking $R = (1 - r_0)/2$ and $M = \mu(1 - r_0)$, we see that the image of f contains a disk centered at $f(z_0)$ with radius $\geq \mu(1 - r_0)/24 \geq 1/24$. □

Remark. The first part of this proof tells us that if the function is bounded then we win, but otherwise we can do something with the derivative. We start out with something weak and get something strong.

We haven't proved Bloch's part of Theorem 19.6.

Sketch of proof of Bloch part of Theorem 19.6. To show injectivity, consider a disk around z_0 of radius $(1-r_0)/6$, and work with the derivative. In this region, we look at $f'(z)-f'(z_0)$, which is 0 at z_0, and by our previous work, we have a bound for $f'(z) - f'(z_0)$ of the form $|f'(z) - f'(z_0)| \leq |z - z_0|(\cdots)$. If this is done carefully, when we have

$$f'(z_1) - f'(z_0) = \int_{z_1}^{z_2} f'(w)\,dw = (f'(z_0))(z_2 - z_1) - \int_{z_1}^{z_2} (f'(z_0) - f'(w))\,dw,$$

which is positive in size unless $z_1 = z_2$. □

Putting together everything we've done so far, we have a proof of Picard's Little Theorem.

Now, we will discuss how to use the method to prove Picard's Big Theorem 19.3 and produce a strengthening of Montel's Theorem 14.8.

Theorem 19.9 (Schottky's Theorem)**.** *Suppose that $\mathcal{F}$ is a class of holomorphic functions on $\mathbb{D}$ with*

(1) $f \in \mathcal{F}$ always omits 0 and 1
(2) $|f(0)| \leq C$ for some constant C

then $\mathcal{F}$ is uniformly bounded on any compact set of $\mathbb{D}$.

Proof. Recall our proof of Picard's little theorem 19.1. We defined $F(z) = \frac{1}{2\pi i}\log f(z)$ and $h(z) = \sqrt{F(z)} - \sqrt{F(z) - 1}$ and $H(z) = \log h(z)$. We had to choose some branches of log to make |imaginary part| $\leq \pi$.

Also, assume that $|f(0)| \geq \frac{1}{2}$. If H' gets too large, we get a contradict to Bloch's theorem 19.6, so we have a bound for H' and hence a bound for H, which then gives a bound for f. Of course, the bound here depends on which compact set we use. □

Now, finally we can prove Picard's Big Theorem.

Proof of Theorem 19.3. Suppose that f has an essential singularity at 0. Also suppose that it is entire in a punctured neighborhood of the origin and omits the values 0 and 1. By Casorati-Weierstass 5.1, we can find a sequence $|z_n| \to 0$ with $|f(z_n)| \leq 1$. Now, define $g_n(w) = f(z_n e^{2\pi i w})$ for $|w| \leq 1$. This is a sequence of functions omitting values 0 and 1, and also $|g_n(0)| = |f(z_n)| \leq 1$. By Schottky 19.9, we see that $|g_n([-1/2, 1/2])| \leq B$ for some constant B. This implies that $|f(z)| \leq B$ when $|z| = |z_n|$.

Now, consider a picture with concentric circles of radius $|z_1|, |z_2|, \ldots$. Note that $|f(z)| \leq B$ is bounded on each of these, so it is bounded on each annulus by the maximum modulus principle, and hence $|f| \leq B$ on a punctured neighborhood of 0. This is a contradiction to the fact that f has an essential singularity. $\square$

A slightly more general version of Montel's theorem 14.8 is this:

Theorem 19.10. *If $\mathcal{F}$ is a class of holomorphic functions in a region Ω omitting 0 and 1, and $f_1, f_2, \ldots, f_n \in \mathcal{F}$ then either $f_n \to \infty$ uniformly on compact sets or there is a subsequence which converges.*

Finally, here is a nice result that is almost like a Putnam problem. We know that $f : \mathbb{C} \to \mathbb{C}$ need not have fixed points; for example, consider $f(z) = z + e^z$. However, we can show the following:

Theorem 19.11. *$f(f(z))$ has a fixed point unless $f(z) = z + c$.*

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