Problem Solving Heuristics

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How do people approach a problem that they don't know how to solve? There are a few heuristic strategies when looking for a method to attack a problem. For more information, see Pólya's classic book *How to Solve It*.

Before you begin:

- Be confident. Believe in yourself.
- Be creative.
- Be persistent.
- Be courageous: "Fearless courage is the foundation to all success."
- If you worry then you will fail. . . so don't worry!
- Be brave and try it.

First, understand the problem:

- What are we trying to find?
- What information are we given?
- What conditions do we have on the problem?
- Get your hands dirty: plug in numbers, draw a diagram, do some simple calculations.

Next, try to find a solution:

- Try some special cases: plug in simple numbers, extremal values, guess and check.
- Estimate. Order of magnitude estimate, draw a careful diagram.
- Choose convenient notation
- Pursue symmetry, in geometry, in algebraic expressions, and in notation.
- Find a pattern, make a hypothesis. Can you prove it?
- Work backwards, go back to the definitions, wishful thinking.
- Have you used all of the data and conditions? Can you do a more general problem?
- Change the data and the conditions. How much can they vary?
- Have you seen the problem before? Do you know of a related problem?
- Look at the problem in a different way: convert between different topics in math.
- Divide and conquer: Separate into easier subproblems and attack each separately.
- "If you can't solve a problem, then there is an easier problem you can solve: find it."

Once you've solved the problem: Check your answer.

- Check your work. Are you sure that every step is correct? Is every step valid?
- Estimate. Is the answer reasonable?
- Do you see another way to solve the problem? Does it give the same result?

> A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.
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> – George Pólya, *How to Solve It*

A Collection of Problems

These problems require minimal technical knowledge. They only require ingenuity to solve.

1. Find positive integers n and $a_1, a_2, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n = 1000$ and the product $a_1 a_2 \cdots a_n$ is as large as possible.
2. If x is a positive integer and $x(x+1)(x+2)(x+3)+1 = 379^2$, compute x. (1989 ARML I-1)
3. Compute the least solution to $\dfrac{x}{\lfloor x \rfloor} = \dfrac{2002}{2003}$. Write your answer in the form $\dfrac{a}{b}$, where a and b have no common factors. (2003 ARML Tiebreaker 2)
4. Let n be an integer. Of all fractions $\frac{1}{n}$, the fractional part of $\sqrt{123456789}$ is closest to one such fraction. Compute that value of n. (2003 ARML I-6)
5. Compute the four complex solutions of $(x-1)^4 + (x-5)^4 + 14 = 0$. (2008 ARML Local T-10)
6. When expanded as a decimal, the fraction $1/97$ has a repetend (the repeating part of the decimal) that begins right after the decimal point and is 96 digits long. If the last three digits of the repetend $A67$, compute the digit A. (1990 ARML T-3)
7. In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$. (2009 AIME-I 4).
8. Compute: $199919981997^2 - 2 \cdot 199919981994^2 + 199919981991^2$. (1998 ARML I-5)
9. Compute the smallest positive integer value of a such that the set $\{\sqrt{a}, \sqrt{a+1}, \sqrt{a+2}, \ldots, \sqrt{a+2008}\}$ contains exactly three integers. (2008 ARML Local I-8)
10. For $0 < x < 1$, let $f(x) = (1+x)(1+x^4)(1+x^{16})(1+x^{64})(1+x^{256})\cdots$. Compute $f^{-1}\left(\dfrac{8}{5f(\frac{3}{8})}\right)$. (2001 ARML I-8)
11. The solutions of $64x^3 - 96x^2 - 52x + 42 = 0$ form an arithmetic progression. Compute the difference between the largest and smallest of the three solutions. (2010 ARML Local I-6)
12. A circle with radius 5 and center (a, b) is tangent to the lines $y = 6$ and $y = 0.75x$. Compute the largest possible value of $a + b$. (2008 ARML Local I-7)
13. Compute $\sqrt{(111,111,111,111)(1,000,000,000,005)+1}$. (1992 ARML T-7)
14. Compute $\dfrac{(1990)^3 - (1000)^3 - (990)^3}{(1990)(1000)(990)}$. (1990 ARML I-1)
15. Compute all real numbers a such that the equation $x^3 - ax^2 - 2ax + a^2 - 1 = 0$ has *exactly* one real solution in x.
16. There are 5 computers, A, B, C, D, and E. For each pair of computers a coin is flipped. If it is heads, then a link is built between the two computers; if it is tails, there's no link between the two. Every message that a computer receives is sent to every computer to which it is linked. Compute the probability that every computer is able to receive messages from every other computer. (2006 ARML I-8)
17. Let a and b be real numbers such that $a^3 - 15a^2 + 20a - 50 = 0$ and $8b^3 - 60b^2 - 290b + 2575 = 0$. Compute $a + b$. (2009 ARML T-9)
18. Equilateral triangle $\triangle ABC$ has side lengths 1. Circle O is tangent to sides AB and BC, and is tangent to the perpendicular bisector of BC at M. If the cevian PC passes through M, compute the area of triangle $\triangle APC$. (2010 ARML Local Tiebreaker)
19. $ABCD$ is a trapezoid and P is the intersection of the diagonals. $\overline{AB}$ is parallel to $\overline{CD}$. E is drawn on $\overline{AD}$ such that $\overline{EP}$ is parallel to $\overline{AB}$. F and G are drawn on $\overline{CD}$ such that $\overline{EF}$ and $\overline{PG}$ are perpendicular to $\overline{CD}$. $\overline{AB} = 8$ and $\overline{CD} = 32$. If $EPGF$ is a square, compute the height of $ABCD$. (2005 ARML I-4)
20. A rabbit climbs out at its hole, and walks 1 mile in a straight line. Then, the rabbit repeatedly turns $\pi/3$ radians (clockwise) and walks half of the distance it just walked. How far away from the rabbit's hole is the point at which the rabbit's location converges?