QUINTIC SPECTRAHEDRA

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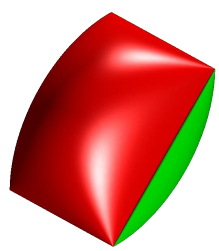
1. Background

Spectrahedra of degree n in $\mathbb{R}^k$ are convex bodies given by k-dimensional affine slices of the cone of $n \times n$ positive semidefinite matrices. They arise as feasible domains in semidefinite programming and each is described by a linear matrix inequality. We will work with spectrahedra in $\mathbb{R}^3$.

Definition 1. A *spectrahedron* of degree n in $\mathbb{R}^3$ is a convex body of the form

$$S = \{(x, y, z) \mid Ax + By + Cz + D \text{ is positive semidefinite}\}$$

where A, B, C, D are real symmetric $n \times n$ matrices.



The pillow: the spectrahedron $\begin{pmatrix} 1 & x & 0 & x \\ x & 1 & y & 0 \\ 0 & y & 1 & z \\ x & 0 & z & 1 \end{pmatrix} \succeq 0$. Here $k = 3$, $n = 4$.

Because the cost function of an SDP is linear, the optimal point always lies on the surface. One interesting and practically useful question is the likelihood for the result of an optimization to be a node, one of the corner points seen when visualizing the spectrahedron. A generic matrix represented by a point on the surface of the spectrahedron has rank $n - 1$, while the matrix at a node typically has rank $n - 2$. This low-rank property of nodes often translates to easier computation.

To better understand the nodal structure of spectrahedra, we work with the notion of a symmetroid.

Definition 2. The *symmetroid* S of degree n is a surface defined by

$$\det(Ax + By + Cz + D) = 0$$

where A, B, C, D are $n \times n$ matrices.

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A spectrahedron can be thought of as a component of the symmetroid. To compute the nodes of a spectrahedron, we compute the nodes of the symmetroid, and see how many lie on the spectrahedron.

Proposition 3. *Generically, a symmetroid S of degree n has $\binom{n+1}{3}$ nodes.*

Question. How many nodes can be real? How many nodes can lie on the spectrahedron?

These questions have been answered in the quartic case. [3] classifies the number of nodes that quartic spectrahedra can have.

Theorem 4. *There exists a quartic spectrahedron with σ nodes on its boundary and ρ real nodes in its symmetroid if and only if $0 \leq \sigma \leq \rho$, both are even, and $2 \leq \rho \leq 10$.*

Hence there are 20 types of quartic spectrahedra, each with its own (ρ, σ) node count.

2. The Quintic Case

We studied the nodal structure of quintic spectrahedra in hopes of obtaining an analogous result.

To carry this out, we generated random spectrahedra and computed the positions of nodes, while also running multiple optimization problems on each. Jacob Emmert-Aronson and Joe Kileel wrote much of the code to carry this out last semester. We had used Singular to determine locations of nodes through a Gröbner basis algorithm; due to apparent numeric instabilities, however, this misidentified nodes in certain edge cases. We have replaced this with the homotopy algorithms implemented in Bertini, which have produced more reliable results.

More specifically, here is what we did:

- Generate matrices A, B, C, D.
- Use Bertini to compute the 20 complex nodes.
- Count the number of real nodes.
- For each real node, check if it lies on the spectrahedron. (A node lies on the spectrahedron if its eigenvalues have the same sign.)

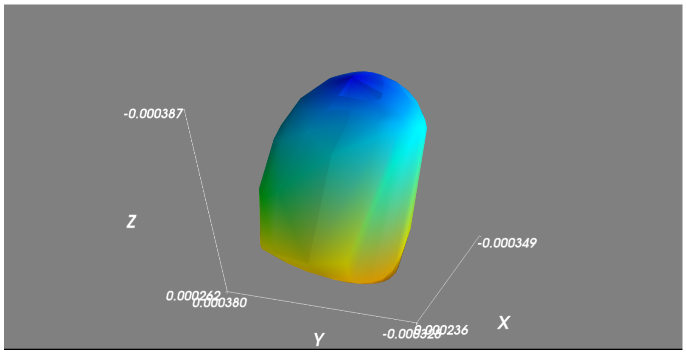
Initially, we generated matrices A, B, C by selecting their entries from a Gaussian distribution with mean 0 and standard deviation 1000, and choosing D as the identity matrix. This meant that our spectrahedra always contained the origin. This method of sampling, however, did not allow us to find spectrahedra with high numbers of nodes. Since then, we have used different sampling methods as well. For example, we can force nodes at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in order to skew the distribution of spectrahedra toward those with higher node counts.

So far, after generating more than 10000 random quintic spectrahedra, we have observed 45 types of spectrahedra. We conjecture that a result analogous to Theorem 4 holds in the quintic case, which would suggest that there should be 65 different types of quintic spectrahedra.

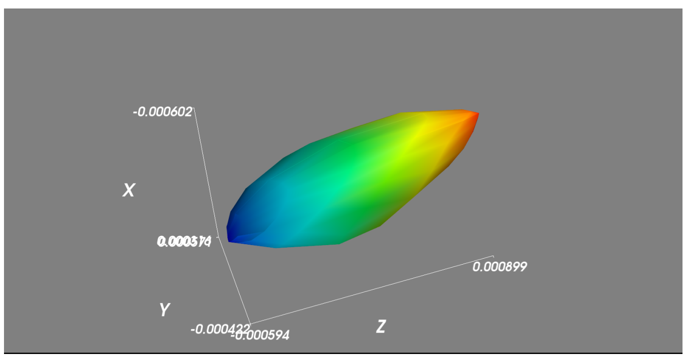
Conjecture 5. *There exists a quintic spectrahedron with σ nodes on its boundary and ρ real nodes in its symmetroid if and only if $0 \leq \sigma \leq \rho$, both are even, and $2 \leq \rho \leq 20$.*

3. Gallery of Spectrahedra

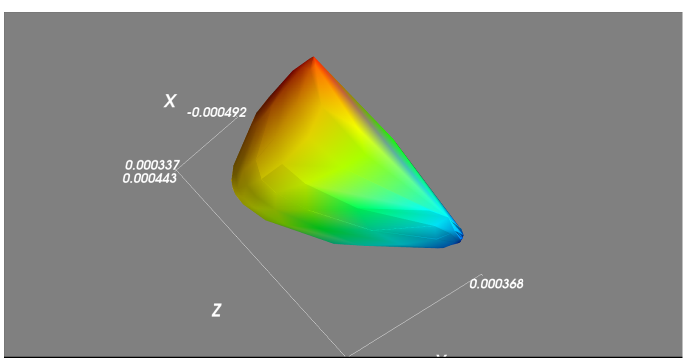
We have so far seen 45 types of spectrahedra out of the 65 types that are conjectured to exist. The missing types are all spectrahedra with at least 12 spectrahedral nodes, spectrahedra with 0 spectrahedral nodes and $\geq$ 12 symmetroid nodes, as well as types $(\rho, \sigma) = (8,8), (10,10), (12,12), (18,2), (20,2)$. We believe that these types are missing because they are rare, but examples do exist and can be found.



A spectrahedron of type $(\rho, \sigma) = (2, 0)$.



A spectrahedron of type $(\rho, \sigma) = (2, 2)$.



A spectrahedron of type $(\rho, \sigma) = (4, 4)$.

On the following pages, we provide an example of a spectrahedron from each of the 45 types that we have found so far, giving a zoo of spectrahedra.

$$(2,0):\begin{bmatrix}3138&-1780&822&-125&771\\-1780&-945&-1275&359&-367\\822&-1275&-1980&-1422&1094\\-125&359&-1422&-1088&1650\\771&-367&1094&1650&-1740\end{bmatrix}\begin{bmatrix}941&-1943&599&1951&18\\-1943&460&-1662&920&830\\599&-1662&671&723&215\\1951&920&723&1004&165\\18&830&215&165&-1574\end{bmatrix}\begin{bmatrix}-1590&345&49&-443&1136\\345&-1102&-364&-212&-936\\49&-364&181&-1124&-2491\\-443&-212&-1124&860&370\\1136&-936&-2491&370&-1570\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(2,2):\begin{bmatrix}-1555&110&92&-694&28\\110&-1188&95&-861&12\\92&95&-212&965&-2797\\-694&-861&965&-1206&1125\\28&12&-2797&1125&-1279\end{bmatrix}\begin{bmatrix}599&-905&-944&370&-365\\-905&-168&612&-273&-373\\-944&612&1785&-894&1426\\370&-273&-894&1536&464\\-365&-373&1426&464&-760\end{bmatrix}\begin{bmatrix}195&-62&-755&-349&962\\-62&-337&-963&549&-1033\\-755&-963&467&-556&-323\\-349&549&-556&-573&-32\\962&-1033&-323&-32&247\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(4,0):\begin{bmatrix}1206&-258&-198&1068&126\\-258&557&-1820&-1297&42\\-198&-1820&1638&1244&851\\1068&-1297&1244&561&444\\126&42&851&444&-1551\end{bmatrix}\begin{bmatrix}-178&421&-13&883&-464\\421&-1359&2074&2909&-334\\-13&2074&-1037&-692&-415\\883&2909&-692&1575&-824\\-464&-334&-415&-824&-653\end{bmatrix}\begin{bmatrix}1537&-369&-750&-2516&82\\-369&139&-229&-1371&92\\-750&-229&-459&-321&365\\-2516&-1371&-321&784&-314\\82&92&365&-314&874\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(4,2):\begin{bmatrix}-439&739&780&-1661&-1304\\739&-1740&-433&100&268\\780&-433&-54&-49&370\\-1661&100&-49&1197&-590\\-1304&268&370&-590&263\end{bmatrix}\begin{bmatrix}-1986&-628&1085&-1063&541\\-628&287&-1556&1466&-715\\1085&-1556&-270&-178&-942\\-1063&1466&-178&-1528&1253\\541&-715&-942&1253&-183\end{bmatrix}\begin{bmatrix}-852&-218&-633&958&796\\-218&823&-610&-1635&-1478\\-633&-610&-8&-1672&-1336\\958&-1635&-1672&-492&-15\\796&-1478&-1336&-15&-490\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(4,4):\begin{bmatrix}758&125&-1476&-291&-4\\125&-192&-231&229&1080\\-1476&-231&375&767&-855\\-291&229&767&-68&1068\\-4&1080&-855&1068&-1834\end{bmatrix}\begin{bmatrix}-388&-1353&-578&-314&-912\\-1353&-2163&-869&-193&60\\-578&-869&2664&-910&1022\\-314&-193&-910&1110&1171\\-912&60&1022&1171&-852\end{bmatrix}\begin{bmatrix}436&241&372&-1118&-485\\241&-831&-1076&872&566\\372&-1076&709&-781&-69\\-1118&872&-781&239&-903\\-485&566&-69&-903&-1500\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(6,0):\begin{bmatrix}1158&1110&530&-359&92\\1110&-1380&-1690&330&-310\\530&-1690&225&-1336&-990\\-359&330&-1336&-186&251\\92&-310&-990&251&194\end{bmatrix}\begin{bmatrix}268&-1104&-588&693&290\\-1104&-1367&76&230&-1433\\-588&76&341&2026&-828\\693&230&2026&550&-419\\290&-1433&-828&-419&535\end{bmatrix}\begin{bmatrix}122&-431&-1557&-1047&-138\\-431&-1618&-677&383&-173\\-1557&-677&71&143&417\\-1047&383&143&-194&-280\\-138&-173&417&-280&633\end{bmatrix}\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$(6,2):\begin{bmatrix}107&-2070&3772&3183&-877\\-2070&8089&-4303&-7235&-618\\3772&-4303&4198&3116&-1287\\3183&-7235&3116&14047&-7863\\-877&-618&-1287&-7863&2329\end{bmatrix}\begin{bmatrix}-4277&-2710&3327&4344&-1832\\-2710&8849&-5303&-8703&2139\\3327&-5303&3071&-101&1030\\4344&-8703&-101&7592&-3534\\-1832&2139&1030&-3534&1857\end{bmatrix}\begin{bmatrix}-1447&-2171&2877&3036&421\\-2171&5216&-5624&-9508&3094\\2877&-5624&3276&1240&412\\3036&-9508&1240&9665&-3864\\421&3094&412&-3864&1092\end{bmatrix}\begin{bmatrix}-1259&3134&-1777&-2214&242\\3134&-8458&4713&7395&-1530\\-1777&4713&-3821&-1799&-375\\-2214&7395&-1799&-13222&5135\\242&-1530&-375&5135&-2426\end{bmatrix}$$

$$(6,4):\begin{bmatrix}3597&1738&854&-1886&1556\\1738&402&-948&324&1760\\854&-948&-2145&3135&-742\\-1886&324&3135&-4210&-821\\1556&1760&-742&-821&3812\end{bmatrix}\begin{bmatrix}2495&1730&-1151&-1015&-1530\\1730&369&-1423&-1470&3383\\-1151&-1423&-1920&422&880\\-1015&-1470&422&2284&2023\\-1530&3383&880&2023&1802\end{bmatrix}\begin{bmatrix}-3840&-96&-2344&1220&-900\\-96&-503&1121&-519&871\\-2344&1121&-3153&269&-1225\\1220&-519&269&-4757&1171\\-900&871&-1225&1171&3984\end{bmatrix}\begin{bmatrix}-4000&-1740&-156&744&-1348\\-1740&-1494&270&1296&-1110\\-156&270&-161&-414&202\\744&1296&-414&-1556&208\\-1348&-1110&202&208&-4814\end{bmatrix}$$

$$(6,6): \begin{bmatrix} -549 & -1017 & 858 & 486 & 215 \\ -1017 & 3223 & -2873 & -786 & -3829 \\ 858 & -2873 & 1289 & 2072 & 158 \\ 486 & -786 & 2072 & -831 & 2820 \\ 215 & -3829 & 158 & 2820 & 2130 \end{bmatrix} \begin{bmatrix} -1935 & -4601 & 292 & -425 & -1777 \\ -4601 & -3985 & -386 & -450 & -4875 \\ 292 & -386 & -422 & -1221 & 736 \\ -425 & -450 & -1221 & -2488 & -170 \\ -1777 & -4875 & 736 & -170 & 3128 \end{bmatrix} \begin{bmatrix} -757 & -2828 & 1833 & 1509 & 580 \\ -2828 & 802 & -630 & -1442 & -1937 \\ 1833 & -630 & 224 & 356 & 619 \\ 1509 & -1442 & 356 & 162 & 336 \\ 580 & -1937 & 619 & 336 & 4057 \end{bmatrix} \begin{bmatrix} -542 & 1158 & -478 & -733 & 98 \\ 1158 & -4276 & 2006 & 2124 & 2614 \\ -478 & 2006 & -2123 & -824 & -1417 \\ -733 & 2124 & -824 & -1178 & -746 \\ 98 & 2614 & -1417 & -746 & -4443 \end{bmatrix}$$

$$(8,0): \begin{bmatrix} 1097 & 1705 & 642 & -1217 & -850 \\ 1705 & 551 & -293 & 601 & 666 \\ 642 & -293 & 29 & -317 & 896 \\ -1217 & 601 & -317 & -121 & -1328 \\ -850 & 666 & 896 & -1328 & 429 \end{bmatrix} \begin{bmatrix} 1333 & 665 & -682 & -28 & 1177 \\ 665 & -372 & 2 & -347 & -1291 \\ -682 & 2 & 47 & 181 & 695 \\ -28 & -347 & 181 & 100 & 232 \\ 1177 & -1291 & 695 & 232 & -541 \end{bmatrix} \begin{bmatrix} -1280 & 323 & -945 & 748 & -350 \\ 323 & 553 & 161 & 1121 & -849 \\ -945 & 161 & -1109 & -748 & 1643 \\ 748 & 1121 & -748 & -934 & 1774 \\ -350 & -849 & 1643 & 1774 & -2608 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(8,2): \begin{bmatrix} -472 & 3370 & -2386 & -2378 & -2682 \\ 3370 & 3520 & -1998 & 3841 & -2020 \\ -2386 & -1998 & 972 & -913 & -3227 \\ -2378 & 3841 & -913 & -9088 & 2395 \\ -2682 & -2020 & -3227 & 2395 & -5612 \end{bmatrix} \begin{bmatrix} -649 & -12 & 446 & -1229 & -493 \\ -12 & -1400 & -901 & 2861 & -3359 \\ 446 & -901 & 520 & 83 & -830 \\ -1229 & 2861 & 83 & -7865 & 4994 \\ -493 & -3359 & -830 & 4994 & -3740 \end{bmatrix} \begin{bmatrix} -8625 & 3318 & 1654 & -2983 & -4347 \\ 3318 & -990 & 40 & 4904 & -1351 \\ 1654 & 40 & -6552 & -95 & 1573 \\ -2983 & 4904 & -95 & -11363 & 3023 \\ -4347 & -1351 & 1573 & 3023 & -5016 \end{bmatrix} \begin{bmatrix} 409 & -530 & -62 & 861 & -825 \\ -530 & -644 & 424 & -3033 & 2811 \\ -62 & 424 & 148 & -1131 & 381 \\ 861 & -3033 & -1131 & 8889 & -3870 \\ -825 & 2811 & 381 & -3870 & 1575 \end{bmatrix}$$

$$(8,4): \begin{bmatrix} 1333 & 1964 & -2304 & 587 & -3145 \\ 1964 & 697 & 183 & 500 & -1822 \\ -2304 & 183 & -1159 & 1120 & 1126 \\ 587 & 500 & 1120 & 4565 & 1205 \\ -3145 & -1822 & 1126 & 1205 & 3273 \end{bmatrix} \begin{bmatrix} 2001 & 212 & -1246 & -1825 & -2831 \\ 212 & -581 & 141 & 1958 & -1968 \\ -1246 & 141 & -804 & -2569 & 2508 \\ -1825 & 1958 & -2569 & -239 & 3502 \\ -2831 & -1968 & 2508 & 3502 & 4238 \end{bmatrix} \begin{bmatrix} -2749 & 479 & -2744 & -1803 & -2498 \\ 479 & 184 & -2059 & 878 & 334 \\ -2744 & -2059 & -3994 & -4097 & -364 \\ -1803 & 878 & -4097 & 2181 & -1963 \\ -2498 & 334 & -364 & -1963 & -4388 \end{bmatrix} \begin{bmatrix} -3901 & -1036 & 722 & 1015 & 3637 \\ -1036 & -1341 & 853 & -1356 & 1794 \\ 722 & 853 & -650 & 249 & -1524 \\ 1015 & -1356 & 249 & -5822 & -1499 \\ 3637 & 1794 & -1524 & -1499 & -5105 \end{bmatrix}$$

$$(8,6): \begin{bmatrix} 1112 & -4233 & 7 & 1091 & 736 \\ -4233 & -3813 & -1522 & 2847 & 576 \\ 7 & -1522 & 2395 & -2275 & -109 \\ 1091 & 2847 & -2275 & 6570 & -188 \\ 736 & 576 & -109 & -188 & -176 \end{bmatrix} \begin{bmatrix} -5926 & -4140 & 896 & -2563 & -101 \\ -4140 & -1714 & -3920 & 409 & -48 \\ 896 & -3920 & 1496 & -716 & -178 \\ -2563 & 409 & -716 & 6679 & -711 \\ -101 & -48 & -178 & -711 & 168 \end{bmatrix} \begin{bmatrix} -2182 & -3668 & 2678 & 3867 & -2695 \\ -3668 & -285 & -1689 & -1653 & -2816 \\ 2678 & -1689 & 651 & -4505 & 730 \\ 3867 & -1653 & -4505 & 2991 & 2423 \\ -2695 & -2816 & 730 & 2423 & -2799 \end{bmatrix} \begin{bmatrix} -1522 & 2446 & 40 & -573 & -545 \\ 2446 & -4472 & 1340 & 48 & 516 \\ 40 & 1340 & -3652 & 2076 & 974 \\ -573 & 48 & 2076 & -8469 & 72 \\ -545 & 516 & 974 & 72 & -542 \end{bmatrix}$$

$$(10,0): \begin{bmatrix} 913 & -908 & 519 & -223 & 30 \\ -908 & 811 & -106 & 426 & -937 \\ 519 & -106 & -495 & -291 & -2011 \\ -223 & 426 & -291 & 421 & -684 \\ 30 & -937 & -2011 & -684 & -1149 \end{bmatrix} \begin{bmatrix} 926 & 542 & -257 & -1320 & -32 \\ 542 & 568 & 624 & -1797 & -420 \\ -257 & 624 & 668 & 639 & -1142 \\ -1320 & -1797 & 639 & 712 & -602 \\ -32 & -420 & -1142 & -602 & -187 \end{bmatrix} \begin{bmatrix} 1940 & -1206 & 671 & 142 & -669 \\ -1206 & 235 & -1443 & 419 & 48 \\ 671 & -1443 & -1159 & -976 & -170 \\ 142 & 419 & -976 & 71 & 141 \\ -669 & 48 & -170 & 141 & -1083 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(10,2): \begin{bmatrix} -2817 & 554 & -346 & 611 & -2446 \\ 554 & 1468 & 49 & 444 & -1422 \\ -346 & 49 & 226 & -774 & -2352 \\ 611 & 444 & -774 & -537 & 1155 \\ -2446 & -1422 & -2352 & 1155 & -6855 \end{bmatrix} \begin{bmatrix} -1168 & -2331 & -610 & -3375 & -2196 \\ -2331 & -3602 & 1365 & -1269 & -1941 \\ -610 & 1365 & 2773 & 2195 & 1640 \\ -3375 & -1269 & 2195 & 5047 & 2251 \\ -2196 & -1941 & 1640 & 2251 & -1238 \end{bmatrix} \begin{bmatrix} -408 & -256 & -1751 & -83 & 283 \\ -256 & 908 & -698 & 1058 & -1856 \\ -1751 & -698 & -3348 & -1866 & -1251 \\ -83 & 1058 & -1866 & -2054 & 966 \\ 283 & -1856 & -1251 & 966 & -4067 \end{bmatrix} \begin{bmatrix} -32 & -174 & 379 & 1 & -179 \\ -174 & -1817 & -690 & -374 & 1102 \\ 379 & -690 & -1876 & 930 & 117 \\ 1 & -374 & 930 & 240 & -711 \\ -179 & 1102 & 117 & -711 & 534 \end{bmatrix}$$

$$(10,4): \begin{bmatrix} -450 & 5000 & -1562 & 615 & -1099 \\ 5000 & 402 & -332 & -3720 & 25 \\ -1562 & -332 & 158 & 3002 & -216 \\ 615 & -3720 & 3002 & -5377 & -2159 \\ -1099 & 25 & -216 & -2159 & -1635 \end{bmatrix} \begin{bmatrix} 2068 & 2525 & -887 & -2196 & -946 \\ 2525 & 1653 & -724 & -2701 & -2068 \\ -887 & -724 & 807 & 1500 & 186 \\ -2196 & -2701 & 1500 & -2542 & -2831 \\ -946 & -2068 & 186 & -2831 & -2885 \end{bmatrix} \begin{bmatrix} 2196 & 2357 & -819 & -2284 & -417 \\ 2357 & 2203 & -1076 & -3197 & -754 \\ -819 & -1076 & 858 & 1413 & -377 \\ -2284 & -3197 & 1413 & -3254 & -2172 \\ -417 & -754 & -377 & -2172 & -2025 \end{bmatrix} \begin{bmatrix} -2545 & -2768 & 836 & 1938 & 448 \\ -2768 & -2984 & 751 & 2447 & 787 \\ 836 & 751 & -1404 & -1646 & 66 \\ 1938 & 2447 & -1646 & 2364 & 2512 \\ 448 & 787 & 66 & 2512 & 1376 \end{bmatrix}$$

$$(10,6): \begin{bmatrix} -458 & 922 & -1502 & -181 & 2166 \\ 922 & 979 & -1790 & -1664 & -3573 \\ -1502 & -1790 & -1301 & 375 & 418 \\ -181 & -1664 & 375 & -2773 & -4658 \\ 2166 & -3573 & 418 & -4658 & -10774 \end{bmatrix} \begin{bmatrix} -3382 & -100 & -2355 & 115 & 65 \\ -100 & -3223 & 1151 & 3310 & -2419 \\ -2355 & 1151 & 87 & -603 & 1758 \\ 115 & 3310 & -603 & -7784 & -2311 \\ 65 & -2419 & 1758 & -2311 & -5881 \end{bmatrix} \begin{bmatrix} -1609 & 2794 & 1165 & 170 & 340 \\ 2794 & -8412 & -4206 & -2121 & -2424 \\ 1165 & -4206 & -2310 & 2017 & 2928 \\ 170 & -2121 & 2017 & -1949 & -4341 \\ 340 & -2424 & 2928 & -4341 & -4840 \end{bmatrix} \begin{bmatrix} -304 & -858 & 1527 & -175 & -1370 \\ -858 & -1405 & 837 & 1026 & 2339 \\ 1527 & 837 & -1556 & -2200 & -1657 \\ -175 & 1026 & -2200 & 1395 & 3836 \\ -1370 & 2339 & -1657 & 3836 & 3827 \end{bmatrix}$$

$$(10,8): \begin{bmatrix} 9803 & -1770 & 1174 & -537 & 1779 \\ -1770 & 610 & 679 & -635 & -257 \\ 1174 & 679 & -584 & -3023 & 195 \\ -537 & -635 & -3023 & -4013 & -1799 \\ 1779 & -257 & 195 & -1799 & -1052 \end{bmatrix} \begin{bmatrix} 7501 & -958 & 705 & 1920 & 3862 \\ -958 & 291 & 15 & -1316 & -203 \\ 705 & 15 & -3096 & -2102 & 3357 \\ 1920 & -1316 & -2102 & -3705 & -3588 \\ 3862 & -203 & 3357 & -3588 & -4040 \end{bmatrix} \begin{bmatrix} 8097 & -2404 & 285 & -1646 & 3890 \\ -2404 & -5574 & -1249 & -130 & -954 \\ 285 & -1249 & 624 & -1466 & 2179 \\ -1646 & -130 & -1466 & 168 & 1158 \\ 3890 & -954 & 2179 & 1158 & -3413 \end{bmatrix} \begin{bmatrix} -9830 & 1791 & -1207 & 675 & -1929 \\ 1791 & -725 & -404 & 439 & 173 \\ -1207 & -404 & -1505 & 1012 & -1014 \\ 675 & 439 & 1012 & -845 & 449 \\ -1929 & 173 & -1014 & 449 & -1110 \end{bmatrix}$$

$$(12,2): \begin{bmatrix} -3258 & -792 & 5442 & 344 & 1437 \\ -792 & -1888 & 509 & -1967 & -4488 \\ 5442 & 509 & -4912 & 1536 & 4675 \\ 344 & -1967 & 1536 & 2854 & 3178 \\ 1437 & -4488 & 4675 & 3178 & 1652 \end{bmatrix} \begin{bmatrix} 1504 & -660 & -996 & -696 & -430 \\ -660 & 423 & -369 & -565 & -704 \\ -996 & -369 & 954 & -163 & 725 \\ -696 & -565 & -163 & -1188 & -236 \\ -430 & -704 & 725 & -236 & -1370 \end{bmatrix} \begin{bmatrix} -1341 & -539 & -841 & -2201 & 154 \\ -539 & -970 & -920 & -2023 & 1313 \\ -841 & -920 & 592 & -695 & 598 \\ -2201 & -2023 & -695 & -2218 & 714 \\ 154 & 1313 & 598 & 714 & -1476 \end{bmatrix} \begin{bmatrix} -548 & -362 & 666 & 1240 & -504 \\ -362 & -324 & 649 & 617 & -62 \\ 666 & 649 & -929 & 228 & -436 \\ 1240 & 617 & 228 & 584 & 593 \\ -504 & -62 & -436 & 593 & -961 \end{bmatrix}$$

$$(12,4): \begin{bmatrix} 3189 & 680 & 1939 & 3187 & 1533 \\ 680 & 405 & -1300 & 355 & 1162 \\ 1939 & -1300 & 2378 & 273 & 612 \\ 3187 & 355 & 273 & 3165 & 1409 \\ 1533 & 1162 & 612 & 1409 & 1904 \end{bmatrix} \begin{bmatrix} -3340 & -3670 & -2096 & -3556 & 1547 \\ -3670 & -4485 & -4075 & -4227 & 3674 \\ -2096 & -4075 & -1986 & -2355 & 1810 \\ -3556 & -4227 & -2355 & -5912 & 1550 \\ 1547 & 3674 & 1810 & 1550 & -1779 \end{bmatrix} \begin{bmatrix} 613 & 940 & -1593 & 463 & -1255 \\ 940 & 659 & -1638 & -110 & 1046 \\ -1593 & -1638 & 1763 & -2060 & 602 \\ 463 & -110 & -2060 & -2107 & 346 \\ -1255 & 1046 & 602 & 346 & 1961 \end{bmatrix} \begin{bmatrix} -113 & -342 & 285 & -165 & -335 \\ -342 & -596 & 1225 & 357 & -1344 \\ 285 & 1225 & -484 & 1165 & 666 \\ -165 & 357 & 1165 & 1402 & -1194 \\ -335 & -1344 & 666 & -1194 & -881 \end{bmatrix}$$

$$(12,6): \begin{bmatrix} 2471 & -718 & -3844 & -1197 & 503 \\ -718 & 1150 & -16 & 151 & -146 \\ -3844 & -16 & 10001 & 1736 & 1448 \\ -1197 & 151 & 1736 & 693 & -264 \\ 503 & -146 & 1448 & -264 & 924 \end{bmatrix} \begin{bmatrix} -6181 & 1013 & -2606 & -3291 & 3085 \\ 1013 & 15 & 765 & 1145 & -350 \\ -2606 & 765 & 4787 & 465 & 2367 \\ -3291 & 1145 & 465 & -2321 & 2574 \\ 3085 & -350 & 2367 & 2574 & -2806 \end{bmatrix} \begin{bmatrix} -1617 & -3623 & -985 & 2354 & -3851 \\ -3623 & 80 & 433 & 3054 & -1533 \\ -985 & 433 & 5927 & -1292 & 1527 \\ 2354 & 3054 & -1292 & -3572 & 4062 \\ -3851 & -1533 & 1527 & 4062 & -240 \end{bmatrix} \begin{bmatrix} -1396 & 1118 & 1740 & 470 & 92 \\ 1118 & -986 & -848 & -437 & 236 \\ 1740 & -848 & -5452 & -218 & -1992 \\ 470 & -437 & -218 & -237 & 342 \\ 92 & 236 & -1992 & 342 & -1784 \end{bmatrix}$$

$$(12,8): \begin{bmatrix} -1656 & 30 & 4395 & -3192 & 420 \\ 30 & 2958 & -3288 & 1582 & -1954 \\ 4395 & -3288 & 64 & -1226 & -802 \\ -3192 & 1582 & -1226 & 2227 & -439 \\ 420 & -1954 & -802 & -439 & -75 \end{bmatrix} \begin{bmatrix} -7336 & -2605 & -3052 & -2963 & 6220 \\ -2605 & 3529 & -994 & 2557 & -1609 \\ -3052 & -994 & 1212 & -3149 & 2608 \\ -2963 & 2557 & -3149 & -1075 & 374 \\ 6220 & -1609 & 2608 & 374 & -2392 \end{bmatrix} \begin{bmatrix} -944 & -2009 & 4281 & -1237 & 301 \\ -2009 & 3109 & 328 & -552 & -1994 \\ 4281 & 328 & -2289 & 1298 & 271 \\ -1237 & -552 & 1298 & -716 & -372 \\ 301 & -1994 & 271 & -372 & 710 \end{bmatrix} \begin{bmatrix} -1854 & 1902 & -1809 & 2097 & -903 \\ 1902 & -4251 & 1066 & -995 & 1850 \\ -1809 & 1066 & -4770 & 1871 & -43 \\ 2097 & -995 & 1871 & -3073 & 666 \\ -903 & 1850 & -43 & 666 & -910 \end{bmatrix}$$

$$(12,10): \begin{bmatrix} -109 & 1725 & 2680 & -67 & 505 \\ 1725 & -3845 & -1799 & 3564 & -2223 \\ 2680 & -1799 & 639 & 5406 & -2811 \\ -67 & 3564 & 5406 & 1049 & -631 \\ 505 & -2223 & -2811 & -631 & 545 \end{bmatrix} \begin{bmatrix} -2389 & -1019 & -1617 & 3433 & -1020 \\ -1019 & 1050 & 1761 & 1352 & -1788 \\ -1617 & 1761 & 3856 & 5511 & -3130 \\ 3433 & 1352 & 5511 & 254 & 501 \\ -1020 & -1788 & -3130 & 501 & -2049 \end{bmatrix} \begin{bmatrix} 1046 & -937 & -1071 & 1055 & 210 \\ -937 & 523 & 1782 & 640 & -155 \\ -1071 & 1782 & 1597 & 4252 & -1772 \\ 1055 & 640 & 4252 & 2052 & -1740 \\ 210 & -155 & -1772 & -1740 & 520 \end{bmatrix} \begin{bmatrix} -1465 & 800 & -136 & -1001 & -44 \\ 800 & -1364 & -2042 & -704 & 810 \\ -136 & -2042 & -5346 & -3567 & 2242 \\ -1001 & -704 & -3567 & -3134 & 1587 \\ -44 & 810 & 2242 & 1587 & -1074 \end{bmatrix}$$

$$(14,2): \begin{bmatrix} 1592 & 398 & 482 & 3394 & -1228 \\ 398 & 876 & 64 & 706 & -1474 \\ 482 & 64 & 1301 & -1897 & -1240 \\ 3394 & 706 & -1897 & 4609 & -257 \\ -1228 & -1474 & -1240 & -257 & 3017 \end{bmatrix} \begin{bmatrix} -5893 & -452 & 3785 & 4955 & -94 \\ -452 & 392 & 584 & 1828 & -342 \\ 3785 & 584 & -1281 & -4791 & 224 \\ 4955 & 1828 & -4791 & 895 & -1729 \\ -94 & -342 & 224 & -1729 & 369 \end{bmatrix} \begin{bmatrix} -962 & 362 & -1287 & 2701 & 2407 \\ 362 & -549 & -347 & 1527 & -649 \\ -1287 & -347 & -729 & -497 & 3687 \\ 2701 & 1527 & -497 & 1651 & -4922 \\ 2407 & -649 & 3687 & -4922 & -7684 \end{bmatrix} \begin{bmatrix} -236 & -544 & 172 & -1466 & 604 \\ -544 & -356 & 572 & -1566 & 398 \\ 172 & 572 & 4 & 1912 & -662 \\ -1466 & -1566 & 1912 & -2928 & 1603 \\ 604 & 398 & -662 & 1603 & -437 \end{bmatrix}$$

$$(14,4):\begin{bmatrix}1992&-2340&-1393&-1450&1737\\-2340&7044&1161&4922&-1291\\-1393&1161&-3335&-445&-3428\\-1450&4922&-445&2352&-2343\\1737&-1291&-3428&-2343&-1772\end{bmatrix}\begin{bmatrix}-204&152&-1727&-1046&2493\\152&-3061&-1596&4863&-2535\\-1727&-1596&-2076&354&-60\\-1046&4863&354&-295&-382\\2493&-2535&-60&-382&455\end{bmatrix}\begin{bmatrix}1480&-475&-1352&917&1145\\-475&1547&-67&-629&-345\\-1352&-67&-1450&-3&-848\\917&-629&-3&-8830&198\\1145&-345&-848&198&458\end{bmatrix}\begin{bmatrix}-1925&1116&1104&1084&-1535\\1116&-2677&640&-1955&1028\\1104&640&721&240&225\\1084&-1955&240&-1477&948\\-1535&1028&225&948&-1084\end{bmatrix}$$

$$(14,6):\begin{bmatrix}8918&10292&1889&-3645&-5451\\10292&4924&-3508&-1534&-5116\\1889&-3508&-488&-486&876\\-3645&-1534&-486&4331&3200\\-5451&-5116&876&3200&4617\end{bmatrix}\begin{bmatrix}8688&9985&-822&-2584&-6333\\9985&6862&1749&-4849&-4636\\-822&1749&-2673&2318&-566\\-2584&-4849&2318&-563&2407\\-6333&-4636&-566&2407&1353\end{bmatrix}\begin{bmatrix}9834&10212&711&-3608&-6134\\10212&5803&-1783&-2588&-7068\\711&-1783&1760&-1447&2557\\-3608&-2588&-1447&1388&2556\\-6134&-7068&2557&2556&3813\end{bmatrix}\begin{bmatrix}-10922&-8424&-479&4357&5643\\-8424&-9324&1130&2328&5794\\-479&1130&-929&612&-642\\4357&2328&612&-2258&-1865\\5643&5794&-642&-1865&-3789\end{bmatrix}$$

$$(14,8):\begin{bmatrix}1202&-543&-692&4530&-2070\\-543&-2044&-3004&401&4791\\-692&-3004&1839&1570&-990\\4530&401&1570&5718&-7276\\-2070&4791&-990&-7276&3019\end{bmatrix}\begin{bmatrix}1154&1182&1691&3904&-3648\\1182&665&-3021&-1516&2224\\1691&-3021&111&409&-1652\\3904&-1516&409&5314&-6142\\-3648&2224&-1652&-6142&5122\end{bmatrix}\begin{bmatrix}-967&673&-1094&8242&-4124\\673&-2200&1030&-1240&1160\\-1094&1030&1271&1267&-1993\\8242&-1240&1267&729&-4212\\-4124&1160&-1993&-4212&756\end{bmatrix}\begin{bmatrix}-2483&-70&47&-3673&2897\\-70&-1834&1238&1594&-1503\\47&1238&-2837&-343&2758\\-3673&1594&-343&-7275&5115\\2897&-1503&2758&5115&-6269\end{bmatrix}$$

$$(14,10):\begin{bmatrix}3781&4974&-225&1640&963\\4974&2012&-154&4722&2197\\-225&-154&-62&695&-1422\\1640&4722&695&-1237&-2266\\963&2197&-1422&-2266&437\end{bmatrix}\begin{bmatrix}3037&6769&114&1377&-1263\\6769&4623&-411&1630&-267\\114&-411&-1375&1037&1819\\1377&1630&1037&2136&-2231\\-1263&-267&1819&-2231&-1863\end{bmatrix}\begin{bmatrix}3860&4380&-2053&2943&-953\\4380&660&-3861&6622&-1965\\-2053&-3861&-2940&3301&-2183\\2943&6622&3301&-2185&773\\-953&-1965&-2183&773&-82\end{bmatrix}\begin{bmatrix}-4546&-5416&578&-1836&101\\-5416&-7001&1147&-1361&-333\\578&1147&-549&-633&783\\-1836&-1361&-633&-2321&1517\\101&-333&783&1517&-2258\end{bmatrix}$$

$$(14,12):\begin{bmatrix}-3166&469&1384&-1046&1578\\469&2570&-1108&459&322\\1384&-1108&294&203&-3171\\-1046&459&203&-1249&3950\\1578&322&-3171&3950&-639\end{bmatrix}\begin{bmatrix}-3659&2080&-1637&2049&-2356\\2080&554&401&-1499&3168\\-1637&401&633&167&-1911\\2049&-1499&167&-1181&1176\\-2356&3168&-1911&1176&-163\end{bmatrix}\begin{bmatrix}-2527&1599&-24&-681&-2229\\1599&-633&-348&-1013&1637\\-24&-348&1318&-1833&-118\\-681&-1013&-1833&498&-97\\-2229&1637&-118&-97&-3371\end{bmatrix}\begin{bmatrix}-218&543&350&-538&-148\\543&-3165&436&660&-1357\\350&436&-1844&1804&1562\\-538&660&1804&-2180&-1118\\-148&-1357&1562&-1118&-1762\end{bmatrix}$$

$$(16,2):\begin{bmatrix}-1328&-2806&2680&-3044&-4036\\-2806&-607&2526&-1007&-245\\2680&2526&3584&-3104&915\\-3044&-1007&-3104&2656&3763\\-4036&-245&915&3763&1744\end{bmatrix}\begin{bmatrix}-6621&191&3484&-5719&-5252\\191&-5181&2021&747&-828\\3484&2021&4410&-17&-881\\-5719&747&-17&86&3863\\-5252&-828&-881&3863&1275\end{bmatrix}\begin{bmatrix}-2447&-2723&4194&-4339&-2004\\-2723&-837&764&171&404\\4194&764&7668&-130&32\\-4339&171&-130&-1241&2098\\-2004&404&32&2098&4506\end{bmatrix}\begin{bmatrix}2407&2028&-3036&5362&3201\\2028&523&-1172&745&-309\\-3036&-1172&-5356&-58&1454\\5362&745&-58&-756&-3034\\3201&-309&1454&-3034&-3881\end{bmatrix}$$

$$(16,4):\begin{bmatrix}-228&-222&448&-367&-37\\-222&-3252&-1081&2531&1286\\448&-1081&-2676&1843&3684\\-367&2531&1843&-2427&-530\\-37&1286&3684&-530&-4654\end{bmatrix}\begin{bmatrix}-301&714&-1325&-1631&-839\\714&-2195&707&4493&397\\-1325&707&565&1832&25\\-1631&4493&1832&-389&1330\\-839&397&25&1330&157\end{bmatrix}\begin{bmatrix}-402&-270&892&-669&-49\\-270&425&-79&-94&790\\892&-79&-2457&3159&1164\\-669&-94&3159&-1628&-1074\\-49&790&1164&-1074&-364\end{bmatrix}\begin{bmatrix}-273&-279&-106&-225&-86\\-279&-1226&-66&11&-490\\-106&-66&-461&-571&35\\-225&11&-571&-782&89\\-86&-490&35&89&-205\end{bmatrix}$$

$$(16,6):\begin{bmatrix}3761&-715&254&-1303&1680\\-715&-4824&-159&2449&3620\\254&-159&0&-419&1665\\-1303&2449&-419&-1362&-1888\\1680&3620&1665&-1888&428\end{bmatrix}\begin{bmatrix}-371&142&-1302&-1161&-187\\142&862&1624&-759&-509\\-1302&1624&-1128&-1174&2960\\-1161&-759&-1174&336&-69\\-187&-509&2960&-69&-3739\end{bmatrix}\begin{bmatrix}4881&539&855&-1716&526\\539&-1323&-107&-1733&3972\\855&-107&427&-1201&2139\\-1716&-1733&-1201&709&-668\\526&3972&2139&-668&-959\end{bmatrix}\begin{bmatrix}-4995&-930&-1053&1599&-637\\-930&-1526&-1274&947&-1235\\-1053&-1274&-1097&824&-930\\1599&947&824&-881&917\\-637&-1235&-930&917&-1549\end{bmatrix}$$

$$(16,8):\begin{bmatrix}1684&-336&482&1655&930\\-336&-874&-401&690&-105\\482&-401&3136&-2010&1016\\1655&690&-2010&2646&726\\930&-105&1016&726&144\end{bmatrix}\begin{bmatrix}719&79&-5&536&1235\\79&-569&-163&-1839&-660\\-5&-163&3268&-3277&301\\536&-1839&-3277&-3895&-2464\\1235&-660&301&-2464&-4608\end{bmatrix}\begin{bmatrix}-1640&880&825&1882&806\\880&-446&1262&-173&-529\\825&1262&-2064&-1635&4633\\1882&-173&-1635&2860&446\\806&-529&4633&446&-2472\end{bmatrix}\begin{bmatrix}-2514&-120&-953&-1541&-945\\-120&-12&-32&-104&-84\\-953&-32&-3850&2126&-654\\-1541&-104&2126&-3114&-478\\-945&-84&-654&-478&-638\end{bmatrix}$$

$$(16,10):\begin{bmatrix}4961&6046&93&1835&-1011\\6046&5480&-412&-319&-2334\\93&-412&-69&-471&-1920\\1835&-319&-471&-1692&815\\-1011&-2334&-1920&815&-1571\end{bmatrix}\begin{bmatrix}3743&5589&-803&833&-2908\\5589&5820&-1161&1203&-1859\\-803&-1161&-1709&-3301&2060\\833&1203&-3301&527&1830\\-2908&-1859&2060&1830&-1320\end{bmatrix}\begin{bmatrix}5451&5737&-198&340&-2268\\5737&5263&-714&1214&-1984\\-198&-714&51&-361&1317\\340&1214&-361&112&-898\\-2268&-1984&1317&-898&-831\end{bmatrix}\begin{bmatrix}-5669&-5668&355&-893&2055\\-5668&-6289&912&-1164&2404\\355&912&-1253&1387&159\\-893&-1164&1387&-1877&-417\\2055&2404&159&-417&-1434\end{bmatrix}$$

$$(16,12):\begin{bmatrix}1535&-1461&2919&947&-1403\\-1461&-1441&-980&841&-366\\2919&-980&107&-2221&630\\947&841&-2221&3080&961\\-1403&-366&630&961&1001\end{bmatrix}\begin{bmatrix}839&2451&-1666&2797&-2587\\2451&-3157&734&-2787&2044\\-1666&734&-4476&1446&848\\2797&-2787&1446&80&2976\\-2587&2044&848&2976&-436\end{bmatrix}\begin{bmatrix}-192&-397&2993&-3649&-772\\-397&-2170&624&-521&215\\2993&624&-542&1202&565\\-3649&-521&1202&857&2041\\-772&215&565&2041&-32\end{bmatrix}\begin{bmatrix}-4193&972&-1309&913&1133\\972&-809&1046&-1291&63\\-1309&1046&-1574&916&-419\\913&-1291&916&-4691&-850\\1133&63&-419&-850&-1070\end{bmatrix}$$

$$(18,4):\begin{bmatrix}974&30&837&1183&1069\\30&-1689&-412&1980&-651\\837&-412&-3553&-2484&563\\1183&1980&-2484&-3904&-2429\\1069&-651&563&-2429&351\end{bmatrix}\begin{bmatrix}-1113&1604&548&-61&-100\\1604&-4127&-1509&-657&1606\\548&-1509&-1864&-431&2058\\-61&-657&-431&-227&485\\-100&1606&2058&485&-3501\end{bmatrix}\begin{bmatrix}-1294&700&506&-977&-198\\700&-6249&-3108&-1003&-1024\\506&-3108&-2312&-1091&-1174\\-977&-1003&-1091&2561&-2679\\-198&-1024&-1174&-2679&-880\end{bmatrix}\begin{bmatrix}897&-1868&-740&-251&634\\-1868&2668&1353&-769&-241\\-740&1353&912&199&-316\\-251&-769&199&-1185&987\\634&-241&-316&987&-500\end{bmatrix}$$

$$(18,6):\begin{bmatrix}1739&-721&827&-4017&-5980\\-721&-4242&89&92&-3624\\827&89&-2897&-3354&-490\\-4017&92&-3354&-5617&2933\\-5980&-3624&-490&2933&1222\end{bmatrix}\begin{bmatrix}4487&1033&2954&-566&-6042\\1033&-1929&1527&-929&-566\\2954&1527&555&2096&-986\\-566&-929&2096&1065&321\\-6042&-566&-986&321&3766\end{bmatrix}\begin{bmatrix}4147&364&17&-1702&-5673\\364&-2328&2584&412&1051\\17&2584&-8005&469&-1579\\-1702&412&469&341&1894\\-5673&1051&-1579&1894&-437\end{bmatrix}\begin{bmatrix}-4648&152&-1748&1922&4869\\152&436&-854&140&537\\-1748&-854&1223&283&369\\1922&140&283&-885&-1902\\4869&537&369&-1902&-4239\end{bmatrix}$$

$$(18,8):\begin{bmatrix}1045&-3992&-159&-1512&404\\-3992&1238&-2564&-741&-2295\\-159&-2564&1161&253&-10\\-1512&-741&253&643&-68\\404&-2295&-10&-68&1599\end{bmatrix}\begin{bmatrix}636&-3757&2129&-1380&1695\\-3757&582&1224&-1017&-2234\\2129&1224&-1449&2467&1105\\-1380&-1017&2467&-1086&-1056\\1695&-2234&1105&-1056&1854\end{bmatrix}\begin{bmatrix}1527&-2094&944&-330&1565\\-2094&2494&-1968&-736&-1335\\944&-1968&-256&588&3628\\-330&-736&588&-360&-34\\1565&-1335&3628&-34&-3181\end{bmatrix}\begin{bmatrix}-2602&2420&-606&858&-1373\\2420&-2892&1620&-40&1510\\-606&1620&-2084&-972&-62\\858&-40&-972&-1208&-66\\-1373&1510&-62&-66&-2841\end{bmatrix}$$

$$(18,10):\begin{bmatrix}-5212&657&-2057&1387&-1775\\657&-803&-998&-200&1584\\-2057&-998&9326&-3813&-1853\\1387&-200&-3813&4665&829\\-1775&1584&-1853&829&618\end{bmatrix}\begin{bmatrix}-5911&-1163&-6704&4325&-1608\\-1163&449&-984&587&624\\-6704&-984&6841&-1416&-2663\\4325&587&-1416&2229&1258\\-1608&624&-2663&1258&863\end{bmatrix}\begin{bmatrix}873&245&-2063&2810&-1082\\245&-3288&-760&-2081&2578\\-2063&-760&9944&-3787&-2034\\2810&-2081&-3787&3427&1578\\-1082&2578&-2034&1578&-20\end{bmatrix}\begin{bmatrix}-2474&930&2020&-1702&696\\930&-603&619&-19&-746\\2020&619&-9955&3743&2110\\-1702&-19&3743&-4755&-714\\696&-746&2110&-714&-1124\end{bmatrix}$$

$$(18,12):\begin{bmatrix}-331&650&-2187&2320&-13\\650&1892&-1178&2169&-663\\-2187&-1178&-707&1936&-370\\2320&2169&1936&-1272&4316\\-13&-663&-370&4316&1324\end{bmatrix}\begin{bmatrix}-3588&2375&793&1399&5425\\2375&2200&-1261&1267&-1003\\793&-1261&750&-610&354\\1399&1267&-610&-2090&1653\\5425&-1003&354&1653&187\end{bmatrix}\begin{bmatrix}-130&-870&1109&1931&1964\\-870&-2593&-1417&5921&1096\\1109&-1417&-697&1811&859\\1931&5921&1811&-3819&1541\\1964&1096&859&1541&3012\end{bmatrix}\begin{bmatrix}-1378&-700&259&-1187&-1420\\-700&-2857&982&-821&-331\\259&982&-1505&-410&-1374\\-1187&-821&-410&-1470&-2103\\-1420&-331&-1374&-2103&-3473\end{bmatrix}$$

$$(20,4): \begin{bmatrix} 3588 & -704 & -1550 & -2306 & 2434 \\ -704 & 4508 & -1500 & 1075 & 2874 \\ -1550 & -1500 & 3378 & -898 & -2780 \\ -2306 & 1075 & -898 & 5982 & -1807 \\ 2434 & 2874 & -2780 & -1807 & 4531 \end{bmatrix} \begin{bmatrix} 5652 & 1947 & -1099 & 1513 & -360 \\ 1947 & 5248 & -2598 & 1852 & 1918 \\ -1099 & -2598 & 2229 & -3431 & -1100 \\ 1513 & 1852 & -3431 & 3406 & -291 \\ -360 & 1918 & -1100 & -291 & 3269 \end{bmatrix} \begin{bmatrix} 1607 & 737 & -1144 & 104 & -58 \\ 737 & 1970 & -645 & 1933 & 5637 \\ -1144 & -645 & 1961 & -2956 & -1981 \\ 104 & 1933 & -2956 & 1073 & -2666 \\ -58 & 5637 & -1981 & -2666 & -415 \end{bmatrix} \begin{bmatrix} -1928 & 360 & -26 & 30 & -1010 \\ 360 & -4179 & 1947 & -721 & -3177 \\ -26 & 1947 & -2645 & 2578 & 1297 \\ 30 & -721 & 2578 & -3029 & -208 \\ -1010 & -3177 & 1297 & -208 & -3329 \end{bmatrix}$$

$$(20,6): \begin{bmatrix} 1673 & 358 & 4720 & 679 & 4893 \\ 358 & -765 & 945 & -1589 & 769 \\ 4720 & 945 & -517 & 679 & -698 \\ 679 & -1589 & 679 & 6714 & -991 \\ 4893 & 769 & -698 & -991 & 3859 \end{bmatrix} \begin{bmatrix} -2135 & 681 & 2756 & -2322 & 4112 \\ 681 & 1413 & -2410 & -1717 & 822 \\ 2756 & -2410 & -1875 & 1575 & -5700 \\ -2322 & -1717 & 1575 & 4582 & 508 \\ 4112 & 822 & -5700 & 508 & -4597 \end{bmatrix} \begin{bmatrix} 1137 & -1078 & -1361 & -294 & 1211 \\ -1078 & 5826 & 1538 & -4424 & 1231 \\ -1361 & 1538 & 6852 & -767 & -1782 \\ -294 & -4424 & -767 & 6975 & -2448 \\ 1211 & 1231 & -1782 & -2448 & 2336 \end{bmatrix} \begin{bmatrix} -531 & -456 & 417 & 1461 & -999 \\ -456 & -1966 & 1488 & 2162 & -1431 \\ 417 & 1488 & -2747 & -287 & 1945 \\ 1461 & 2162 & -287 & -5963 & 2375 \\ -999 & -1431 & 1945 & 2375 & -2437 \end{bmatrix}$$

$$(20,8): \begin{bmatrix} 513 & -3040 & 1827 & 1056 & 3131 \\ -3040 & -1968 & 718 & -1480 & 2564 \\ 1827 & 718 & 69 & 2687 & 1553 \\ 1056 & -1480 & 2687 & 4643 & 2837 \\ 3131 & 2564 & 1553 & 2837 & -883 \end{bmatrix} \begin{bmatrix} 2662 & 527 & 1241 & 103 & 2217 \\ 527 & 2080 & 470 & -1723 & 1619 \\ 1241 & 470 & -1575 & 3330 & 2282 \\ 103 & -1723 & 3330 & 3345 & 1617 \\ 2217 & 1619 & 2282 & 1617 & 4343 \end{bmatrix} \begin{bmatrix} 2765 & -844 & 1003 & 1028 & 4635 \\ -844 & -6033 & 157 & -1908 & 3630 \\ 1003 & 157 & 108 & 1006 & 804 \\ 1028 & -1908 & 1006 & 2760 & 1809 \\ 4635 & 3630 & 804 & 1809 & 1809 \end{bmatrix} \begin{bmatrix} -3443 & -860 & -882 & -1067 & -3288 \\ -860 & -3338 & 801 & 1375 & -2277 \\ -882 & 801 & -1230 & -2358 & -1533 \\ -1067 & 1375 & -2358 & -4769 & -2949 \\ -3288 & -2277 & -1533 & -2949 & -5850 \end{bmatrix}$$

$$(20,8): \begin{bmatrix} -1059 & 1076 & 1209 & -3580 & 1688 \\ 1076 & 1880 & -4328 & 5476 & 5316 \\ 1209 & -4328 & 8904 & 1919 & 2137 \\ -3580 & 5476 & 1919 & -6532 & -1136 \\ 1688 & 5316 & 2137 & -1136 & 1024 \end{bmatrix} \begin{bmatrix} -2640 & 499 & 1146 & -1190 & 457 \\ 499 & 1671 & -2171 & 2031 & 4035 \\ 1146 & -2171 & 8807 & 700 & 1114 \\ -1190 & 2031 & 700 & 263 & 226 \\ 457 & 4035 & 1114 & 226 & 2731 \end{bmatrix} \begin{bmatrix} 927 & -501 & -1059 & -1143 & 2206 \\ -501 & 4711 & -4637 & 617 & 895 \\ -1059 & -4637 & 6398 & -426 & -568 \\ -1143 & 617 & -426 & -2141 & 980 \\ 2206 & 895 & -568 & 980 & 1260 \end{bmatrix} \begin{bmatrix} -1865 & -338 & -285 & -150 & -2566 \\ -338 & -5921 & 3227 & -1792 & -2370 \\ -285 & 3227 & -9474 & -344 & -922 \\ -150 & -1792 & -344 & -769 & -1055 \\ -2566 & -2370 & -922 & -1055 & -4485 \end{bmatrix}$$

$$(20,12): \begin{bmatrix} 524 & 724 & -3 & -1043 & 488 \\ 724 & -3078 & -3271 & 835 & 2109 \\ -3 & -3271 & 534 & 315 & -945 \\ -1043 & 835 & 315 & 542 & 309 \\ 488 & 2109 & -945 & 309 & 169 \end{bmatrix} \begin{bmatrix} -4128 & -2499 & -2951 & 1538 & -1531 \\ -2499 & -2070 & -2545 & 1093 & -1217 \\ -2951 & -2545 & -437 & 2308 & -3543 \\ 1538 & 1093 & 2308 & -2477 & 3035 \\ -1531 & -1217 & -3543 & 3035 & -1928 \end{bmatrix} \begin{bmatrix} -3046 & 2429 & -755 & -1445 & 1441 \\ 2429 & -4923 & -679 & -2606 & -364 \\ -755 & -679 & 1839 & -524 & -1432 \\ -1445 & -2606 & -524 & -50 & -381 \\ 1441 & -364 & -1432 & -381 & 706 \end{bmatrix} \begin{bmatrix} -541 & -630 & 62 & 968 & -460 \\ -630 & -488 & 924 & 401 & -743 \\ 62 & 924 & -2084 & 538 & 1810 \\ 968 & 401 & 538 & -1579 & -123 \\ -460 & -743 & 1810 & -123 & -1435 \end{bmatrix}$$

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