Quiz #12 Solutions

Math 55 with Professor Stankova
Discussion Section #108 with GSI James Moody

Wednesday, the 16th of November 2016
Write your name at the top!

Question 1 [12 points] A vending machine accepts \$1 coins, \$1 bills, and \$3 tokens. Let A_n be the set of ways of putting $\$n$ into the machine, where the order you put in the units of currency in matters. Let $a_n = |A_n|$. Find a recurrence relation for a_n, and give the initial conditions.

For $n \geq 1$, we can partition A_n into three disjoint subsets:
$C_n :=$ ways of putting $\$n$ in which end with a \$1 coin
$B_n :=$ of ways of putting $\$n$ in which end with a \$1 bill
$T_n :=$ ways of putting $\$n$ in which end with a \$1 token

We can see that $|C_n| = a_{n-1}$ (since we must have put in exactly $\$(n-1)$ before putting in the final \$1 coin at the end).

We can see that $|B_n| = a_{n-1}$ (since we must have put in exactly $\$(n-1)$ before putting in the final \$1 bill at the end).

We can see that $|T_n| = a_{n-3}$ (since we must have put in exactly $\$(n-3)$ before putting in the final \$3 token at the end).

Since C_n, B_n, and D_n form a partition of A_n (i.e., are disjoint and their union is A_n, we have by the inclusion-exclusion principle that:

$$a_n = |A_n| = |C_n| + |B_n| + |T_n| = a_{n-1} + a_{n-1} + a_{n-3} = 2a_{n-1} + a_{n-3}$$

Our initial conditions are $a_{-2} = 0$, $a_{-1} = 0$, and $a_0 = 1$. Notice that we chose our initial conditions to be precisely the 3 terms (3 = the degree of our relation) immediately preceding the first n for which our recurrence relation holds. Our recurrence relation works for $n \geq 1$, so we gave initial conditions for the 3 terms before a_1.

Alternatively, we could use the initial conditions $a_0 = 1$, $a_1 = 2$, $a_2 = 4$ (which we can find by counting). Either answer is acceptable.

Question 2 [±1 **point**] What is the degree of the characteristic equation for the recurrence relation $a_{n+3} = a_{n-1} - 2a_{n-2}$?

1 —— 2 —— 3 —— 4 —— **5** —— 6

We first replace a_k with x^k everywhere, getting $x^{n+3} = x^{n-1} - 2x^{n-2}$, then we divide out by a power of x to make the lowest degree term a constant, yielding $x^5 = x - 2$. The degree is thus 5.

Question 3 [±1 **point**] Every solution a_n of a homogeneous degree 2 recurrence relation is of the form $Ar_1^n + Br_2^n$, where r_1 and r_2 are roots of the characteristic equation, and A, B are constants.

True —or— **False**?

This is only true if r_1 and r_2 are **distinct** roots!

Question 4 [±1 **point**] If a_n and b_n are solutions to a non-homogeneous recurrence relation, then so is $c_n := a_n + b_n$

True —or— **False**?

This is only true for **homogeneous** recurrence relations. The corresponding theorem for **non-homogeneous equations** is the following:

If p_n is a particular solution to the original equation, and b_n is any solution to the corresponding homogeneous equation, then $p_n + b_n$ is a solution to the original equation.

If you've taken Math 1B or Math 54, this is exactly the same situation that happens with homogeneous versus non-homogeneous linear differential equations.