# Number theoretic properties of generating functions related to Dyson's rank for partitions into distinct parts.

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# Definitions

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- $p(n)$ is the number of partitions of $n$.
- $Q(n)$ is the number of partitions of $n$ into distinct parts.

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- Are there combinatorial explanations for these elegant identities?

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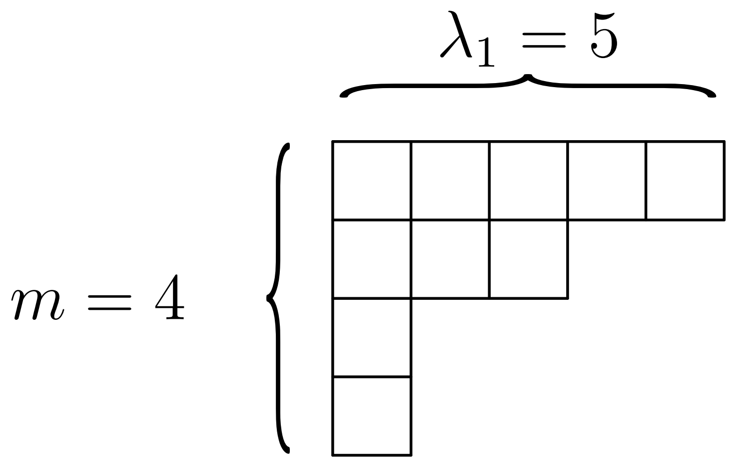

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- Taken modulo $7$, the rank also sorts the partitions of $7n+5$ into $7$ equal-sized groups.
- Failed to explain $p(11n+6) \equiv 0 \pmod{11}$. Garvan discovered the *crank*, which explained this identity along with many other congruences.

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- Gordon and Ono: For any positive integer $j$, the set of integers $n$ for which $Q(n)$ is divisible by $2^j$ is dense in the positive integers.
- Can a rank or similar combinatorial invariant be used to explain congruences for $Q(n)$?
- The rank provides a combinatorial interpretation for $j = 1$ and $j = 2$!

**Theorem** (M.). Define $T(m, k; n)$ to be the number of partitions of $n$ into distinct parts having rank congruent to $m \pmod{k}$. Then

$$T(0,4;n) = T(1,4;n) = T(2,4;n) = T(3,4;n)$$

if and only if $24n + 1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$ such that the largest power of $p$ dividing $24n + 1$ is $p^e$ where $e$ is odd.

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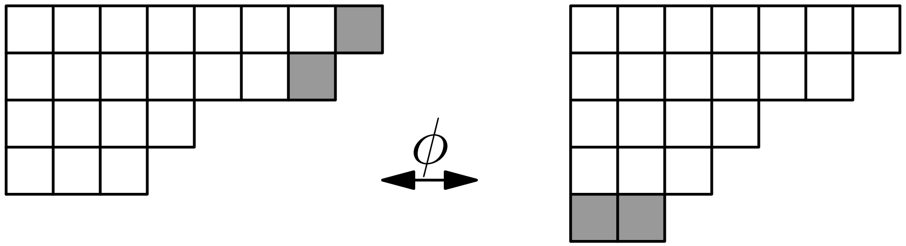

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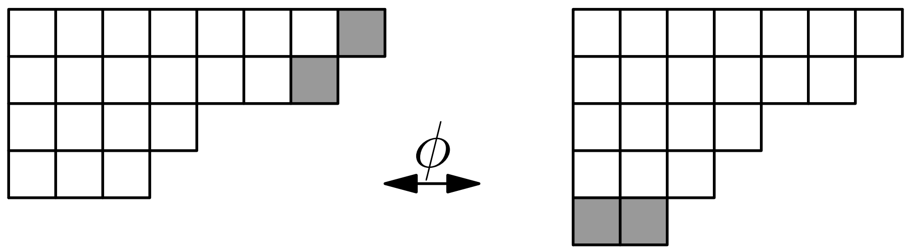


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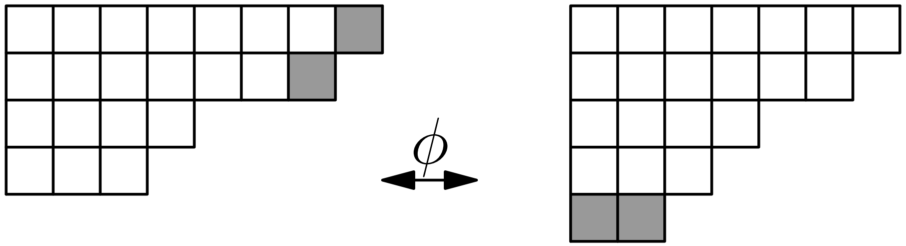


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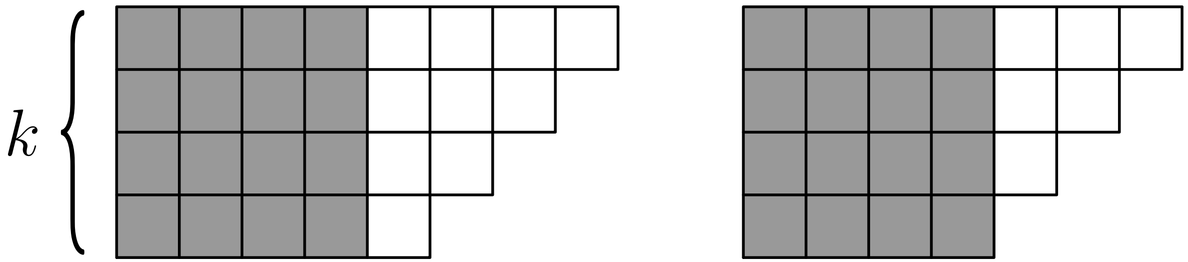

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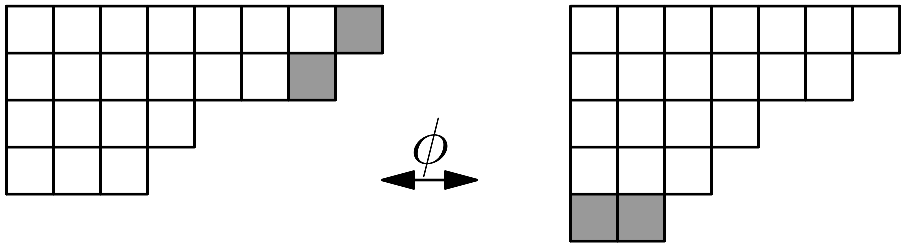


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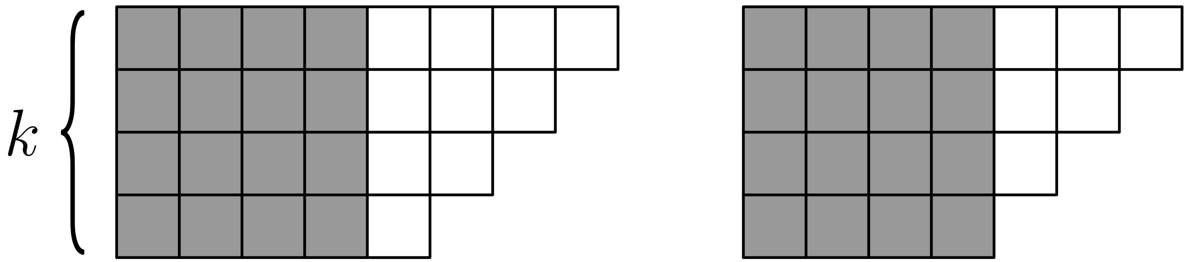

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- Thus $T(0,4;n) = T(1,4;n) = T(2,4;n) = T(3,4;n)$ for such $n$, and the set of such $n$ is dense in the integers. Thus $Q(n)$ is nearly always divisible by $4$.

# Generating functions

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- One can show that

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1-zq)(1-zq^2)\cdots(1-zq^s)}$$

for $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$.

# $G(z,q)$ at fourth roots of unity $z$

**Theorem** (M.). Let $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$. Then

$$\begin{aligned} G(i,q) &= \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2} \\ G(-i,q) &= \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2} \end{aligned}$$

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# A new false theta function (or two)

- It follows that

$$qG(i,q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}$$

and

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- Not true theta functions, but they resemble theta functions in the sense that their coefficients are roots of unity and are $0$ whenever the exponent of $q$ is not a perfect square. Such functions are known as *false theta functions*.

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- $R(-1, q)$ is one of Ramanujan's famous “mock theta functions”.

# The relation between $G$ and $R$

**Theorem** (M.)**.** We have

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or alternatively,

$$\begin{aligned} qR(i,q^{-24}) &= \sum_{n=0}^{\infty}(-1)^n\left(q^{(12n+1)^2} + q^{(12n+5)^2} + q^{(12n+7)^2} + q^{(12n+11)^2}\right) \\ &= q + q^{25} + q^{49} + q^{121} - q^{169} \\ &\quad -q^{289} - q^{361} - q^{529} + q^{625} + \cdots . \end{aligned}$$

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- The analytic behavior of the false theta functions $G(\pm i,q)$ gives the behavior of $R(\pm i,q)$ for $q$ outside the unit disk!

# $L$-values at negative integers

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| $n$ | $1$ | $5$ | $7$ | $11$ | $13$ | $17$ | $19$ | $23$ |
|---|---|---|---|---|---|---|---|---|
| $\chi_0(n)$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_1(n)$ | $1$ | $1$ | $-1$ | $-1$ | $1$ | $1$ | $-1$ | $-1$ |
| $\chi_2(n)$ | $1$ | $-1$ | $1$ | $-1$ | $-1$ | $1$ | $-1$ | $1$ |
| $\chi_3(n)$ | $1$ | $-1$ | $-1$ | $1$ | $-1$ | $1$ | $1$ | $-1$ |
| $\chi_4(n)$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ | $1$ | $-1$ |
| $\chi_5(n)$ | $1$ | $-1$ | $-1$ | $1$ | $1$ | $-1$ | $-1$ | $1$ |
| $\chi_6(n)$ | $1$ | $1$ | $1$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $\chi_7(n)$ | $1$ | $1$ | $-1$ | $-1$ | $-1$ | $-1$ | $1$ | $1$ |

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**Theorem** (M.). We have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_6, -2n) t^n = e^{-t} + e^{-t} \sum_{n=1}^{\infty} \frac{e^{-24nt}}{\prod_{r=1}^{n}(1+e^{-48rt})}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} L(\chi_7, -2n) t^n = i \sum_{n=1}^{\infty} \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1-ie^{-24rt})} - \frac{e^{-(12n^2+12n+1)t}}{\prod_{r=1}^{n}(1+ie^{-24rt})}$$

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| $n$ | 1 | 5 | 7 | 11 | 13 | 17 | 19 | 23 |
|---|---|---|---|---|---|---|---|---|
| $\chi_6(n)$ | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $\chi_7(n)$ | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | 1 | 1 |

# Observations and Future Work

- The generating functions $R(z,q)$ and $G(z,q)$ are related at $z=\pm i$. What happens if we examine these functions at other roots of unity $z$?

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- The rank fails to explain the divisibility of $Q(n)$ by higher powers of $2$. Is there a generalization of the rank that can be used to divide the partitions of $Q(n)$ into $m$ equal-sized groups whenever $Q(n)$ is divisible by $m$ for any positive integer $m$?

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- Thanks to Joe Gallian, Nathan Kaplan, and Ricky Liu for their mentorship and support throughout this research project, and to Ken Ono for his helpful insights and direction. Finally, thanks to my father, Ken Monks, for his continual support and encouragement.