The 3x + 1 conjecture (Collatz conjecture)

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- Example: 9
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- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \rightarrow 28$
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- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \to 28 \to 14$
The $3x + 1$ conjecture (Collatz conjecture)

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- Define $C : \mathbb{N} \to \mathbb{N}$ by
  \[
  C(x) = \begin{cases} 
  x/2 & \text{if } x \text{ is even} \\
  3x + 1 & \text{if } x \text{ is odd}
  \end{cases}
  .
  
- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \to 28 \to 14 \to 7$
The $3x + 1$ conjecture (Collatz conjecture)

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- Define $C : \mathbb{N} \to \mathbb{N}$ by $C(x) = \begin{cases} x/2 & \text{x is even} \\ 3x + 1 & \text{x is odd} \end{cases}$.

- What is the long-term behaviour of $C$ as a discrete dynamical system?

- Example: $9 \to 28 \to 14 \to 7 \to 22$
The 3x + 1 conjecture (Collatz conjecture)

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- Define $C : \mathbb{N} \rightarrow \mathbb{N}$ by
  $$C(x) = \begin{cases} 
  x/2 & \text{if } x \text{ is even} \\
  3x + 1 & \text{if } x \text{ is odd}
  \end{cases}.$$

- What is the long-term behaviour of $C$ as a discrete dynamical system?

- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11$
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- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34$
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- Famous open problem stated in 1929 by Collatz.
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- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17$
The $3x + 1$ conjecture (Collatz conjecture)

- Famous open problem stated in 1929 by Collatz.
- Define $C : \mathbb{N} \to \mathbb{N}$ by $C(x) = \begin{cases} x/2 & \text{x is even} \\ 3x + 1 & \text{x is odd} \end{cases}$.
- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52$
The $3x + 1$ conjecture (Collatz conjecture)

- Famous open problem stated in 1929 by Collatz.
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 x/2 & \text{if } x \text{ is even} \\
 3x + 1 & \text{if } x \text{ is odd}
 \end{cases}$.
- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26$
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- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5$
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- What is the long-term behaviour of $C$ as a discrete dynamical system?
- Example: $9 \rightarrow 28 \rightarrow 14 \rightarrow 7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16$
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- Example: $9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8$
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- **Collatz Conjecture:** The $C$-orbit $x, C(x), C(C(x)), \ldots$ of every positive integer $x$ eventually enters the cycle containing 1.
The $3x + 1$ conjecture (Collatz conjecture)

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- Can also use $T(x) = \begin{cases} x/2 & \text{x is even} \\ \frac{3x+1}{2} & \text{x is odd} \end{cases}$. 
The $3x + 1$ conjecture (Collatz conjecture)

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The Collatz graph $G$
Two smaller conjectures

- **The Nontrivial Cycles conjecture:** There are no $T$-cycles of positive integers other than the cycle 1, 2.
- **The Divergent Orbits conjecture:** The $T$-orbit of every positive integer is bounded and hence eventually cyclic.
- Together, these suffice to prove the Collatz conjecture.
Two smaller conjectures

- **The Nontrivial Cycles conjecture:** There are no $T$-cycles of positive integers other than the cycle $1, 2$.
- **The Divergent Orbits conjecture:** The $T$-orbit of every positive integer is bounded and hence eventually cyclic.
- Together, these suffice to prove the Collatz conjecture.
- Both still unsolved.
Starting point: sufficiency of arithmetic progressions

- Two positive integers *merge* if their orbits eventually meet.
Starting point: sufficiency of arithmetic progressions

- Two positive integers _merge_ if their orbits eventually meet.
- A set of $S$ positive integers is _sufficient_ if every positive integer merges with an element of $S$. 

Theorem. (K. M. Monks, 2006.) Every arithmetic sequence is sufficient.

In fact, Monks shows that every positive integer relatively prime to 3 can be _back-traced_ to an element of a given arithmetic sequence.

Every integer congruent to 0 mod 3 _forward-traces_ to an integer relatively prime to 3, at which point the orbit contains no more multiples of 3.
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The Collatz graph $G$

\[ x/2 \]
\[ (3x + 1)/2 \]
The pruned Collatz graph $\tilde{G}$
Natural questions arising from the sufficiency of arithmetic progressions

1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?
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1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?

2. For a given \( x \in \mathbb{N} \setminus 3\mathbb{N} \), how “close” is the nearest element of \( \{a + bN\}_{N \geq 0} \) that we can back-trace to?
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3. Starting from \( x = 1 \), can we chain these short back-tracing paths together to find which integers are in an infinite back-tracing path from 1?
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3. Starting from $x = 1$, can we chain these short back-tracing paths together to find which integers are in an infinite back-tracing path from 1?

4. In which infinite back-tracing paths does a given arithmetic sequence $\{a + bN\}$ occur?
Attempting the first question
A family of sparse sufficient sets

Proposition (Monks, Monks, Monks, M.)

For any function $f : \mathbb{N} \rightarrow \mathbb{N}$ and any positive integers $a$ and $b$,

$$\{2^{f(n)}(a + bn) \mid n \in \mathbb{N}\}$$

is a sufficient set.

Proof.

Any positive integer $x$ merges with some number of the form $a + bN$. Then $2^{f(N)}(a + bN)$, which maps to $a + bN$ after $f(N)$ iterations of $T$, also merges with $x$. 

\[ \square \]

Corollary

For any fixed $a$ and $b$, the sequence $(a + bn) \cdot 2^n$ is a sufficient set with asymptotic density zero in the positive integers.
Natural questions arising from the sufficiency of arithmetic progressions

1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?

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1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?
   Yes!

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Attempting the second question
Efficient back-tracing

- Define the *length* of a finite back-tracing path to be the number of red arrows in the path.
Efficient back-tracing

- Define the *length* of a finite back-tracing path to be the number of red arrows in the path.
- Want to find the shortest back-tracing path to an element of the arithmetic sequence $a \mod b$ for various $a$ and $b$. 

Consider three cases: when $b$ is a power of 2, a power of 3, or relatively prime to 2 and 3.
Efficient back-tracing

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Efficient back-tracing

Proposition

Let $b \in \mathbb{N}$ with $\gcd(b, 6) = 1$, and let $a < b$ be a nonnegative integer. Let $e$ be the order of $\frac{3}{2}$ modulo $b$. Then any $x \in \mathbb{N} \setminus 3\mathbb{N}$ can be back-traced to an integer congruent to $a$ mod $b$ via a path of length at most $(b - 1)e$. 
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Proposition
Let $n \geq 1$ and $a < 2^n$ be nonnegative integers. Then any $x \in \mathbb{N} \setminus 3\mathbb{N}$ can be back-traced to an integer congruent to $a$ mod $2^n$ using a path of length at most $\lceil \log_2 a + 1 \rceil$. 
Efficient back-tracing

Proposition

Let \( m \geq 1 \) and \( a < 3^m \) be nonnegative integers. Then any \( x \in \mathbb{N} \setminus 3\mathbb{N} \) can be back-traced to infinitely many odd elements of \( a + 3^m\mathbb{N} \) via an admissible sequence of length 1.
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Let $m \geq 1$ and $a < 3^m$ be nonnegative integers. Then any $x \in \mathbb{N} \setminus 3\mathbb{N}$ can be back-traced to infinitely many odd elements of $a + 3^m\mathbb{N}$ via an admissible sequence of length 1.

Working mod $3^m$ is particularly nice because 2 is a primitive root mod $3^m$. What about when 2 is a primitive root mod $b$?
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Working mod \( 3^m \) is particularly nice because 2 is a primitive root mod \( 3^m \). What about when 2 is a primitive root mod \( b \)?

Proposition

Let \( b \in \mathbb{N} \) with \( \gcd(b, 6) = 1 \) such that 2 is a primitive root mod \( b \). Let \( a \) be such that \( 0 \leq a \leq b \) and \( \gcd(a, b) = 1 \). From any \( x \in \mathbb{N} \setminus 3\mathbb{N} \), there exists a back-tracing path of length at most 1 to an integer \( y \in \mathbb{N} \setminus 3\mathbb{N} \) with \( y \equiv a \pmod{b} \).
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   Yes!

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   Pretty close, depending on \( b \).

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Attempting the third question
Infinite back-tracing

An infinite back-tracing sequence is a sequence of the form $x_0, x_1, x_2, \ldots$ for which $T(x_i) = x_{i-1}$ for all $i \geq 1$. 

An infinite back-tracing parity vector is the binary sequence formed by taking an infinite back-tracing sequence mod 2.

We think of an infinite back-tracing parity vector as an element of $\mathbb{Z}_2$, the ring of 2-adic integers.

Some are simple to describe: those that end in 0. These are the positive integers $\mathbb{N} \subset \mathbb{Z}_2$.

When there are infinitely many 1's, they are much harder to describe.
Infinite back-tracing

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- Some are simple to describe: those that end in \( \bar{0} \). These are the positive integers \( \mathbb{N} \subset \mathbb{Z}_2 \).
Infinite back-tracing

- An *infinite back-tracing sequence* is a sequence of the form

  \[ x_0, x_1, x_2, \ldots \]

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- Some are simple to describe: those that end in \( 0 \). These are the positive integers \( \mathbb{N} \subset \mathbb{Z}_2 \).

- When there are infinitely many 1’s, they are much harder to describe.
Uniqueness of infinite back-tracing vectors

Proposition

Let $x \in \mathbb{N} \setminus 3\mathbb{N}$, and suppose $v$ is a back-tracing parity vector for $x$ containing infinitely many 1’s. If $v$ is also a back-tracing parity vector for $y$, then $x = y$. 

(Barberstein, 1994.) This gives a map $\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ that sends $v$ to the unique 2-adic whose $T$-orbit, taken mod 2, is $v$. Similarly, we can define a map $\Psi : \mathbb{Z}_2 \setminus 3\mathbb{N} \to \mathbb{Z}_3$ that sends $v$ to the unique 3-adic having $v$ as an infinite back-tracing parity vector.
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What are the back-tracing parity vectors starting from positive integers?

Proposition

Every back-tracing parity vector of a positive integer $x$, considered as a 2-adic integer, is either:

(a) a positive integer (ends in 0),
(b) immediately periodic (its binary expansion has the form $v_0...v_k$ where each $v_i \in \{0, 1\}$), or
(c) irrational.

Can we write down an irrational one? The best we can do is a recursive construction, such as the greedy back-tracing vector that follows red whenever possible. Even this is hard to describe explicitly.
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Another look at $\tilde{G}$

\begin{align*}
\frac{x}{2} & \quad \frac{3x+1}{2} \\
1 & \quad 2 & \quad 4 & \quad 8 & \quad 16 & \quad 32 & \quad 64 & \quad 128 & \quad 256 & \quad 512
\end{align*}
Natural questions arising from the sufficiency of arithmetic progressions

1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?
   Yes!

2. For a given \( x \in \mathbb{N} \setminus 3\mathbb{N} \), how “close” is the nearest element of \( \{a + bN\}_{N \geq 0} \) that we can back-trace to?
   Pretty close, depending on \( b \).

3. Starting from \( x = 1 \), can we chain these short back-tracing paths together to find which integers are in an infinite back-tracing path from 1?

4. In which infinite back-tracing paths does a given arithmetic sequence \( \{a + bN\} \) occur?
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   Pretty close, depending on \( b \).

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   This turns out to be very hard to find explicitly.

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Attempting the fourth question
Strong sufficiency in the reverse direction

**Theorem**

Let \( x \in \mathbb{N} \setminus 3\mathbb{N} \). Then every infinite back-tracing sequence from \( x \) contains an element congruent to 2 mod 9.
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Let $x \in \mathbb{N} \setminus 3\mathbb{N}$. Then every infinite back-tracing sequence from $x$ contains an element congruent to $2 \mod 9$.

We say that the set of positive integers congruent to $2 \mod 9$ is strongly sufficient in the reverse direction.
Proof by picture: the pruned Collatz graph mod 9.
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Strong sufficiency in the forward direction

- A similar argument shows that 2 mod 9 is strongly sufficient in the forward direction: the $T$-orbit of every positive integer contains an element congruent to 2 mod 9!
A similar argument shows that 2 mod 9 is *strongly sufficient in the forward direction*: the $T$-orbit of every positive integer contains an element congruent to 2 mod 9!

A set $S$ is *strongly sufficient in the forward direction* if every divergent orbit and nontrivial cycle of positive integers intersects $S$. How this helps: Suppose we can show that, for instance, the set of integers congruent to 1 mod 2 is strongly sufficient for every $n$. Then the nontrivial cycles conjecture is true!
Strong sufficiency in the forward direction

- A similar argument shows that 2 mod 9 is strongly sufficient in the forward direction: the $T$-orbit of every positive integer contains an element congruent to 2 mod 9!

- A set $S$ is strongly sufficient in the forward direction if every divergent orbit and nontrivial cycle of positive integers intersects $S$.

- A set $S$ is strongly sufficient in the reverse direction if every infinite back-tracing sequence containing infinitely many odd elements, other than 1, 2, intersects $S$. 

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- $S$ is strongly sufficient if it is strongly sufficient in both directions.
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- $S$ is strongly sufficient if it is strongly sufficient in both directions.

- **How this helps:** Suppose we can show that, for instance, the set of integers congruent to 1 mod $2^n$ is strongly sufficient for every $n$. Then the nontrivial cycles conjecture is true!
The graphs $\Gamma_k$

Definition
For $k \in \mathbb{N}$, define $\Gamma_k$ to be the two-colored directed graph on $\mathbb{Z}/k\mathbb{Z}$ having a black arrow from $r$ to $s$ if and only if $\exists x, y \in \mathbb{N}$ with

$$x \equiv r \text{ and } y \equiv s \pmod{k}$$

with $x/2 = y$, and a red arrow from $r$ to $s$ if there are such an $x$ and $y$ with $(3x + 1)/2 = y$. 
Example: $\Gamma_9$
Example: $\Gamma_7$
A criterion for strong sufficiency

Theorem

Let $n \in \mathbb{N}$, and let $a_1, \ldots, a_k$ be $k$ distinct residues mod $n$. 

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Let $n \in \mathbb{N}$, and let $a_1, \ldots, a_k$ be $k$ distinct residues mod $n$.

- Let $\Gamma_n'$ be the vertex minor of $\Gamma_n$ formed by deleting the nodes labeled $a_1, \ldots, a_k$ and all arrows connected to them.

If $\Gamma_n''$ is a disjoint union of cycles and isolated vertices, and each of the cycles have length less than $630, 138, 897$, then the set $a_1, \ldots, a_k$ mod $n$ is strongly sufficient.
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**Theorem**

Let $n \in \mathbb{N}$, and let $a_1, \ldots, a_k$ be $k$ distinct residues mod $n$.

- Let $\Gamma'_n$ be the vertex minor of $\Gamma_n$ formed by deleting the nodes labeled $a_1, \ldots, a_k$ and all arrows connected to them.

- Let $\Gamma''_n$ be the graph formed from $\Gamma'_n$ by deleting any edge which is not contained in any cycle of $\Gamma'_n$.

If $\Gamma''_n$ is a disjoint union of cycles and isolated vertices, and each of the cycles have length less than $630$, then the set $a_1, \ldots, a_k$ mod $n$ is strongly sufficient.
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A list of strongly sufficient sets

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Natural questions arising from the sufficiency of arithmetic progressions

1. Can we find a sufficient set with asymptotic density 0 in \( \mathbb{N} \)?
   
   Yes!

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   Pretty close, depending on \( b \).

3. Starting from \( x = 1 \), can we chain these short back-tracing paths together to find which integers are in an infinite back-tracing path from 1?
   
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4. In which infinite back-tracing paths does a given arithmetic sequence \( \{a + bN\} \) occur?
   We’re still working on a general answer, but we know that many (such as \( 2 \mod 9 \)) occur in all of them!
Question 5.

*Which deeper structure theorems about T-orbits can be used to improve on these results?*
Background on percentage of 1’s in a $T$-orbit

- **Theorem.** (Eliahou, 1993.) If a $T$-cycle of positive integers of length $n$ contains $r$ odd positive integers (and $n - r$ even positive integers), and has minimal element $m$ and maximal element $M$, then

$$\frac{\ln(2)}{\ln \left(3 + \frac{1}{m}\right)} \leq \frac{r}{n} \leq \frac{\ln(2)}{\ln \left(3 + \frac{1}{M}\right)}$$

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Looking mod 27

\[ \frac{3x + 1}{2} \mod 27 \]
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\[
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The autoconjugacy $\Omega$
Working with $\Omega$

- $\Omega$ is *solenoidal*, that is, it induces a permutation on $\mathbb{Z}/2^n\mathbb{Z}$ for all $n$.

Example:

$$\Omega(110 \cdots) = \Phi \circ V \circ \Phi^{-1}(110 \cdots) = \Phi(001 \cdots) = 001 \cdots$$

- We say that, mod 8, $\Omega(3) = 4$. 

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Self-duality in $\Gamma_{2n}$

- Define the *color dual* of a graph $\Gamma_k$ to be the graph formed by replacing every red arrow with a black arrow and vice versa.

Theorem

For any $n \geq 1$, the graph $\Gamma_{2n}$ is self-color-dual.

Idea of proof: if we replace each label $a$ with $\Omega(a) \mod 2^n$, we get the color dual of $\Gamma_{2n}$. 
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Example: $\Gamma_8$

$x/2 \mod 8$

$(3x + 1)/2 \mod 8$
Hedlund’s other endomorphisms

- **Discrete derivative map:** $D : \mathbb{Z}_2 \to \mathbb{Z}_2$ by
  
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  is an endomorphism of $T$. 

(M., 2009.) $R$ is a two-to-one map, and $R(\Omega(x)) = R(x)$ for all $x$. 

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The endomorphism $R$
Folding $\Gamma_{2^{n+1}}$ onto $\Gamma_{2^n}$

<table>
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<th>$\Gamma_{2^n}$</th>
<th>$\Gamma_{2^{n+1}}$</th>
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<tbody>
<tr>
<td>$R(x)$</td>
<td>$x$</td>
</tr>
<tr>
<td>$R(y)$</td>
<td>$y$</td>
</tr>
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$\iff$

$x$ or $y$

$\iff$

$x$ or $y$
Folding $\Gamma_8$ onto $\Gamma_4$

$x/2 \mod 8$

$(3x + 1)/2 \mod 8$
Folding $\Gamma_8$ onto $\Gamma_4$

\[
x / 2 \mod 4
\]
\[
(3x + 1) / 2 \mod 4
\]
Folding $\Gamma_8$ onto $\Gamma_4$

\begin{align*}
\frac{x}{2} \mod 4 \\
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\end{align*}
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- The structure of $T$ as a 2-adic dynamical system can be used to obtain properties of the graphs $\Gamma_{2^n}$.
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- Can we find an irrational infinite back-tracing parity vector explicitly, say using algebraic properties?
Acknowledgements

The authors would like to thank Gina Monks for her support throughout this research project.