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1 Notation

- G is a reductive algebraic group over $\mathbf{C}$ or a compact Lie group over $\mathbf{R}$
- $T \subset G$ is a maximal torus
- X is the lattice of characters of T and $X^+ \subset X$ is the subset of dominant weights. Let R^+ be the positive roots and let ρ be half the sum of the positive roots.
- W is the Weyl group. For $w \in W$ set $\ell(w) = \#(R^+ \cap w^{-1}R^+)$.
- Let V_λ be the representation corresponding to $\lambda \in X^+$, and let χ_λ be its character. Recall the Weyl character formula

$$\begin{aligned}\chi_\lambda &= \sum_{w \in W} w\left(\frac{x^\lambda}{\prod_{\alpha \in R^+}(1 - x^{-\alpha})}\right) \\ &= \frac{\sum_{w \in W}(-1)^{\ell(w)} w(x^{\lambda+\rho})}{x^\rho \prod_{\alpha \in R^+}(1 - x^{-\alpha})}.\end{aligned}$$

- Let $K_{\lambda,\mu} = \dim(V_\lambda)_\mu$. Then

$$K_{\lambda,\mu} = \sum_{w \in W} (-1)^{\ell(w)} p(w(\lambda + \rho) - \mu - \rho)$$

where $p(\nu) = \#\{(m_\alpha) \in \mathbf{N}^{R^+} \mid \nu = \sum_{\alpha \in R^+} m_\alpha \alpha\}$ is the Kostant partition function.

2 Hall–Littlewood functions

Define

$$P_\lambda(x;t) = \frac{1}{W_\lambda(t)} \sum_{w \in W} w\left(x^\lambda \prod_{\alpha \in R^+} \frac{1 - tx^{-\alpha}}{1 - x^{-\alpha}}\right)$$

where

$$W_\lambda(t) = \sum_{w \in \mathrm{stab}_W(\lambda)} t^{\ell(w)}.$$

Remark 2.1.
- $P_\lambda(x;0) = \chi_\lambda$
- $P_\lambda(x;1)$ is the orbit sum $m_\lambda = x^\lambda +$ similar terms.
- Macdonald's identity: $P_0(x;t) = 1$.
- $P_\lambda(x;t) = \chi_\lambda + \sum_{\mu < \lambda} k_{\lambda,\mu}(t)\chi_\mu$ where $k_{\lambda,\mu}(t) \in \mathbf{Z}[t]$. In particular, we can write $\chi_\lambda = \sum_{\mu \leq \lambda} K_{\lambda,\mu}(t) P_\mu(x;t)$ where $K_{\lambda,\mu}(t) \in \mathbf{Z}[t]$. Then $K_{\lambda,\lambda} = 1$, $K_{\lambda,\mu}(0) = \delta_{\lambda,\mu}$ and $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$. □

We can define a dual basis

$$Q_\mu(x;t) = \sum_{\lambda \geq \mu} K_{\lambda,\mu}(t)\chi_\lambda.$$

Theorem 2.2.
$$Q_\mu(x;t) = \sum_{w\in W} w\left(\frac{x^\lambda}{\prod_{\alpha\in R^+}(1-x^{-\alpha})(1-tx^\alpha)}\right)$$

Define χ_λ for all $\lambda \in X$ by the Weyl character formula. Then

$$\chi_\lambda = \begin{cases} 0 & \text{if } \mathrm{stab}_W(\lambda) \neq 1 \\ (-1)^{\ell(v)}\chi_\theta & \text{if } \nu(\lambda+\rho) = \theta+\rho \text{ for } \theta \in X^+ \end{cases}.$$

Then we get the formula

$$K_{\lambda,\mu}(t) = \sum_{w\in W} (-1)^{\ell(w)} p(w(\lambda+\rho)-(\mu+\rho);t)$$

where

$$p(\nu;t) = \sum_{\nu=\sum m_\alpha \alpha} t^{\sum m_\alpha}.$$

Theorem 2.3. $K_{\lambda,\mu}(t) \in \mathbf{N}[t]$.

2.1 Interpretation 1 of $K_{\lambda,\mu}(t)$

Consider the flag variety G/B and the torus-fixed points $(G/B)^T = W\cdot e$ where e is the trivial coset. The T-weights of $\mathrm{T}^*_e(G/B) = (\mathfrak{g}/\mathfrak{b})^*$ is R^+. Given $\lambda \in X$, let $\mathbf{C}_\lambda$ be the 1-dimensional representation of B. This gives a line bundle $\mathcal{L}_\lambda = G\times_B \mathbf{C}_\lambda \to G/B$. The T-weight of $\mathcal{L}_\lambda(e)$ is λ. By localization,

$$\sum_i (-1)^i \mathrm{char}_T \mathrm{H}^i(G/B;\mathcal{L}_{w_0(\lambda)}) = \sum_{w\in W} w\left(\frac{x^\lambda}{\prod_{\alpha\in R^+}(1-x^{-\alpha})}\right) = \chi_\lambda.$$

For $\lambda \in X^+$, $\mathrm{H}^{>0}(G/B;\mathcal{L}_{w_0(\lambda)}) = 0$ and $\mathrm{H}^0(G/B;\mathcal{L}_{w_0(\lambda)}) = V_\lambda$.

Let $Z = \mathrm{T}^*(G/B)$ be the cotangent bundle. There is now an action of $G\times\mathbf{C}^*$ ($\mathbf{C}^*$ acts on the fibers) and we let t be the generating character of this extra $\mathbf{C}^*$. Let $\pi\colon \mathrm{T}^*(G/B)\to G/B$ be the structure map. Again by localization,

$$\sum_i (-1)^i \mathrm{char}_{T\times\mathbf{C}^*} \mathrm{H}^i(Z;\pi^*\mathcal{L}_{w_0(\lambda)}) = \sum_{w\in W} w\left(\frac{x^\lambda}{\prod_{\alpha\in R^+}(1-x^{-\alpha})(1-tx^\alpha)}\right) = \sum_\lambda K_{\lambda,\mu}(t)\chi_\lambda.$$

Again we have vanishing $\mathrm{H}^{>0}(Z;\pi^*\mathcal{L}_{w_0(\mu)}) = 0$ for $\mu \in X^+$. Hence $K_{\lambda,\mu}(t)$ is the multiplicity of χ_λ in $\mathrm{H}^0(Z;\pi^*\mathcal{L}_{w_0(\mu)})$ under the action of $\mathfrak{g}$.

2.2 Interpretation 2 of $K_{\lambda,\mu}(t)$

Let $W_a = W \ltimes X$. This gives a Hecke algebra over $\mathbf{Q}(t)$ and Kazhdan–Lusztig polynomials $P_{v,w}(t)$. Let $v_\lambda \in W_a$ be the longest element in the double coset $W\cdot\lambda\cdot W$.

Then we have the following theorem proven by Lusztig (in type A) and Kato (in general).

Theorem 2.4. $K_{\lambda,\mu}(t) = P_{v_\mu,v_\lambda}(t)$

The affine Grassmannian is $Y = G^L(\mathbf{C}[q^{\pm1}])/G(\mathbf{C}[q])$ (here G^L is the Langlands dual group). The Schubert cells in Y are in bijection with the T-fixed points and also with X. The Kazhdan–Lusztig polynomials can be interpreted in terms of intersection homology on the Schubert varieties in Y.