Midterm 1 Review

Math 53, section 213

October 6, 2014

1. Find the length of the curve given by $x = 3t^2$, $y = 2t^3$, $0 \leq t \leq 2$.

   Solution: The length is $\int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$. We can evaluate this using u substitution:

$$\begin{aligned}\int_0^2 \sqrt{(6t)^2 + (6t^2)^2}dt &= \int_0^2 6t\sqrt{1+t^2}dt \\ &= 3 \cdot \frac{2}{3}(1+t^2)^{3/2} \mid_0^2 \\ &= 2 \cdot (1+4)^{3/2} - 2 \cdot (1+0)^{3/2} \\ &= 10\sqrt{5} - 2.\end{aligned}$$

2. Compute the cross product of $\langle 1, 1, -1\rangle$ and $\langle 2, 4, 6\rangle$. What is the area of the parallelogram spanned by these vectors?

   Solution: The area of the parallelogram is the length of the cross product. The cross product of the two vectors is

$$\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{pmatrix} = 10\mathbf{i} - 8\mathbf{j} + 2\mathbf{k}.$$

   So, the area is the length of this vector: $\sqrt{100 + 64 + 4} = \sqrt{168}$.

3. Find an equation for the plane passing through $(1, 2, -2)$ that contains the line $x = 2t$, $y = 3 - t$, $z = 1 + 3t$.

   Solution: Choose two points on the line, say at $t = 0$ and $t = 1$. These are the points $A = (0, 3, 1)$ and $B = (2, 2, 4)$. Let $C = (1, 2, -2)$ be the other point on the plane. Then the normal vector to the plane is the cross product of two of the vectors that lie in the plane, say $B - A$ and $C - A$. We have $B - A = (2, -1, 3)$ and $C - A = (1, -1, -3)$. Their cross product is

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & -1 & -3 \end{pmatrix} = 6\mathbf{i} + 3\mathbf{j} - \mathbf{k}.$$

Thus the equation of the plane is $6x + 3y - z = c$ for some c. Since $(1, 2, -2)$ lies in the plane, it follows that $c = 6 + 6 - 2 = 10$, so the equation is $6x + 3y - z = 10$.

4. Evaluate the following limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}.$$

What about the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{x^2 - y^2}?$$

Solution: The first limit is 0, which can be shown by setting $x = r\cos(\theta)$ and $y = r\sin(\theta)$ and taking the limit as $r \to 0$. The second limit does not exist since if we approach $(0, 0)$ along the y-axis or x-axis respectively we get limits of 1 and -1.

5. Find all second partial derivatives of $f(x, y, z) = x^k y^l z^m$.

Solution: There are six distinct second partial derivatives: f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yz}, f_{xz}. We will compute only f_{xx} and f_{xy}; the others are similar. We have $f_{xx} = k(k-1)x^{k-2}y^l z^m$ (notice that this is correct even when k is 0 or 1) and $f_{xy} = klx^{k-1}y^{l-1}z^m$.

6. Find equations of the tangent plane and the normal line to the surface $xy + yz + zx = 3$ at the point $(1, 1, 1)$.

Solution: The gradient vector at this point is $(2, 2, 2)$, so the tangent plane is $2x + 2y + 2z = 6$ (or to simplify, $x + y + z = 3$) and the normal line is $x = y = z = 1 + 2t$.

7. Find the maximum rate of change of $f(x, y) = x^2 y + \sqrt{y}$ at the point $(2, 1)$. In which direction does it occur?

Solution: The maximum rate of change occurs in the direction of the gradient vector, which at this point is $(4, 9/2)$. The length of this vector is $\frac{1}{2}\sqrt{145}$, and so this is the maximum rate of change. The unit vector in this direction is $\frac{1}{\sqrt{145}}(8, 9)$.

8. Find the critical points (local maxima, minima, and saddle points) of the function $f(x, y) = x^3 - 6xy + 8y^3$.

   Solution: The critical points occur when both partial derivatives f_x and f_y are zero. The partial derivatives are $3x^2 - 6y$ and $-6x + 24y^2$ respectively. Setting these equal to zero we get $x^2 = 2y$ and $x = 4y^2$. Thus $16y^4 = 2y$ and so y is either 0 or $1/2$. If y is 0 then x is 0; otherwise x is 1. So the two critical points are $(0, 0)$ and $(1, 1/2)$.

   We can then use the second derivative tests to see if these are maxima, minima, or saddle points. The second derivatives are $f_{xx} = 6x$, $f_{yy} = 48y$, and $f_{xy} = -6$. At the point $(0, 0)$ we have that f_{xx} and f_{yy} are both 0 and $f_{xy} = -6$; this means $f_{xx} \cdot f_{yy} - f_{xy}^2 < 0$ and so it is a saddle point. At $(1, 1/2)$ we have that $f_{xx} \cdot f_{yy} - f_{xy}^2 = 6 \cdot 24 - 6^2 > 0$, and since $f_{xx} > 0$ this means it is a local minimum.