Functions worksheet: Solutions

Math 1A, section 103

February 11, 2014

0. (Warmup.) If $f(x) = 2x$, what is $f(2)$?

   Solution: $f(2) = 2 \cdot 2 = 4$.

1. How is the graph of $y = 2\sin x + 1$ related to the graph of $y = \sin x$? Sketch the graph of the former.

   Solution: Given the graph of $y = \sin x$, we stretch vertically by a factor of two, then shift the graph up one unit to get $y = 2\sin x + 1$.

2. How is the graph of $y = \sqrt{x-1}$ related to the graph of $y = \sqrt{x}$? Sketch the graph of the former and find its domain.

   Solution: The graph of $y = \sqrt{x-1}$ is obtained by shifting the graph of $y = \sqrt{x}$ one unit to the right. Therefore, its domain is $[1, \infty)$.

3. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.

   Solution: By completing the square in h, we find $h(x) = (2x+1)^2 + 6 = g(x)^2 + 6$. So, $f(x) = x^2 + 6$ satisfies $f \circ g = h$.

4. If $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$, find a function g such that $f \circ g = h$.

   Solution: If $f(g(x)) = h(x)$, then we have $3g(x) + 5 = 3x^2 + 3x + 2$. Thus $g(x) = x^2 + x - 1$.

5. Find the inverse of the function $f(x) = 3x + 5$. What is $f^{-1} \circ h$ where $h(x) = 3x^2 + 3x + 2$ as above? How does this relate to the answer to the previous problem?

Solution: It turns out to be $x^2 + x - 1$, which is the same as the function g found in the previous problem. This is a general principle: if $f \circ g = h$, then by taking f^{-1} of both sides we have $g = f^{-1} \circ h$.

6. Find the inverse of the function $f(x) = 2 \cdot 3^{x/4}$. What is the domain of the inverse function?

   Solution: We have $x = 2 \cdot 3^{f^{-1}(x)/4}$, so solving for f^{-1} we have:

$$\begin{aligned} x &= 2 \cdot 3^{f^{-1}(x)/4} \\ x/2 &= 3^{f^{-1}(x)/4} \\ \log_3(x/2) &= f^{-1}(x)/4 \\ 4\log_3(x/2) &= f^{-1}(x) \end{aligned}$$

   The domain of f^{-1} is the range of the function $f(x) = 2 \cdot 3^{x/4}$, which is simply a horizontal and vertical scaling of 3^x, so the range is $(0, \infty)$.

7. Find the inverse of the function $f(x) = 2x^2$, which is defined on the domain $x \leq 0$.

   Solution: Since we have restricted f to the domain $x \leq 0$, it is injective and its range is $[0, \infty)$. Thus the inverse function f^{-1} has domain $[0, \infty)$ and range $(-\infty, 0]$, and is an inverse of the function f. The function $f^{-1}(x) = -\sqrt{x/2}$ satisfies these properties.

8. What is the domain of the function $f(x) = \log_2(\log_3(x))$? What is its range? Is it one-to-one? If so, what is its inverse?

   Solution: The function f is defined when $\log_3(x) > 0$, which occurs when $x > 1$, so its domain is $(1, \infty)$. On this domain, $\log_3(x)$ can be any value in $(0, \infty)$, so $\log_2(\log_3(x))$ can be any value in $(-\infty, \infty)$. So the range of f is $\mathbb{R}$. It is one-to-one with inverse $f^{-1}(x) = 3^{2^x}$.

9. Does the function $f(x) = x^4 + 2x^2 + 1$, defined on $\mathbb{R}$, have an inverse? If so, what is its inverse?

   Solution: No; it does not pass the horizontal line test. In particular, $f(x) = (x^2 + 1)^2$, so both $x = -1$ and $x = 1$ yield the same value of $f(x)$.

10. Find the inverse f^{-1} of the function $f(x) = 3x + 5$, and compute $f^{-1} \circ f$ and $f \circ f^{-1}$. What do you get in either case? Is there a general principle here?

    Solution: The composition is always x, because they are inverse functions.

11. Johnny has a play-doh collection, and each year he decides to use his allowance to buy half as much play-doh as he currently has and add to his collection. If he started with three pounds of play-doh when he was 5 years old, approximately how much play-doh will he have when he is 10 years old, to the nearest pound? (You may use a calculator to find an approximate answer.)

    Solution: Each year he multiplies the size of his play-doh collection by 1.5, so after 5 years, he has $3 \cdot 1.5^5$ pounds of play-doh.

12. In music, why do we have a twelve tone scale?

    (a) It is known that simple ratios of frequencies sound nice together to our ears. For instance, when we move up an octave in pitch from the middle C on a piano to the C above, the sound wave doubles in frequency. To move up to the G above middle C, the frequency is multiplied by 1.5. If the frequency of middle C is 261.6, what is the frequency of the C above that? Of the G just above it?

    (b) On a piano, there are seven half-steps from the C to the G above it. Use a calculator to compute $2^{7/12}$. What is it close to?

    (c) Given this, what do you think happens to the frequency when we move up a half step on a piano?

    Solution: (a) The frequency of the C above middle C is $2 \cdot 261.6 = 523.2$, and the frequency of the G between these is $1.5 \cdot 261.6 = 392.4$. (b) $2^{7/12}$ is approximately 1.498, which is very close to 1.5. So, to get from C to G we approximately multiply by $2^{7/12}$. (c) Moving up a half step on a piano (from one white or black key to the next) multiplies the frequency by $2^{1/12}$. We choose the interval of 1/12 so that we get ratios that are very close to nice fractions: $2^{7/12}$ is close to 1.5, so notes that are seven half-steps apart sound harmonious. Then, $2^{2/12}$ is very close to 9/8, so consecutive whole notes also sound harmonious.

13. A confused cell with a terrifying genetic mutation divides into two cells exactly one minute after being created. Then, after another minute each of the two new cells will divide, and so on. Furthermore, at any moment in time that some cells undergo a division, three extra cells with the same terrifying genetic mutation are brought forth from the Void. So, at time $t = 0$ there is 1 cell, at $t = 1$ minute there are $2 + 3 = 5$ cells, and at time

$t = 2$ minutes there are $2 \cdot 5 + 3 = 13$ cells. Find a formula for the number of cells as a function of time in minutes.

Solution: At each minute, the number multiplies by 2 and adds 3. Let us keep track of this without simplifying, first:

Time t	Number of cells
0	1
1	$2 \cdot 1 + 3$
2	$2^2 \cdot 1 + 2 \cdot 3 + 3$
3	$2^3 \cdot 1 + 2^2 \cdot 3 + 2 \cdot 3 + 3$
4	$2^4 \cdot 1 + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3$

In general at time t, we see that the number of cells is

$$2^t + 3(2^{t-1} + 2^{t-2} + \cdots + 1).$$

By the geometric series formula, this simplifies to

$$2^t + 3\left(\frac{2^t - 1}{2 - 1}\right) = 2^t + 3(2^t - 1) = 4 \cdot 2^t - 3.$$