Final Review Worksheet

Math 1A, section 103

May 4, 2014

0. (Warmup.) What is $\lim_{x\to 2} x^2 + 1$?

   Solution: The function is continuous, so we can just plug in $x = 2$ to get an answer of 5.

1. Find $\lim_{x\to\infty} \frac{4x^2+x+1}{x^2+3x+5}$.

   Solution: Dividing top and bottom by x^2, we see that all terms go to 0 except the leading terms, and so the limit is $4/1 = 4$.

2. Find the derivative of $f(x) = \ln(1 - \sin(x))$. What is the domain of f?

   Solution: The derivative is, by the chain rule, $\frac{1}{1-\sin(x)} \cdot (-cos(x)) = \frac{\cos(x)}{\sin(x)-1}$. Since ln is defined on the positive real numbers, the domain consists of all points x for which $1 - \sin(x) > 0$, or $1 > \sin(x)$. This is true for all real numbers x except for those at which $\sin(x) = 1$, namely the numbers of the form $\frac{\pi}{2} + 2\pi k$ for integers k.

3. Use Newton's method to approximate $\sqrt{2}$.

   Solution: The number $\sqrt{2}$ is a root of the equation $x^2-2 = 0$. Starting with a first approximation $x_0 = 1$ and setting $f(x) = x^2 - 2$, we have $f'(x) = 2x$ and so our next approximation is

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2} = 1.5.$$

   Then, the next approximation is $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{0.25}{3} \approx 1.416666...$. The process can be continued to get closer and closer approximations.

4. If a sphere has volume 36π with a maximum possible error margin of π, what is the maximum possible error margin in the measure of the radius of the sphere?

   Solution: We can use differentials to calculate the error. The volume of a sphere is given in terms of its radius by $V = \frac{4}{3}\pi r^3$, and so $dV = 4\pi r^2 dr$. If the sphere has volume 36π, then $\frac{4}{3}\pi r^3$ is 36π, and so $r = 3$. Thus $\pi = dV = 4\pi \cdot 3^2 dr$, and so $dr = \frac{1}{36}$.

5. Use implicit differentiation to find the slope of the tangent line to the curve $2^y + xy = x^2$ at the point $(2, 1)$.

   Solution: If we implicitly differentiate with respect to x, we get $2yy' + xy' + y = 2x$, and so at the point $(2, 1)$ we have $4y' + y' + 2 = 2$. Thus $y' = 0$, so the tangent line at $(2, 1)$ is horizontal.

6. Find the minimum possible distance from a point on the line $y = 3 - 2x$ to the origin.

   Solution: The distance of a point (x, y) to the origin is $\sqrt{x^2 + y^2}$, and to minimize this it suffices to minimize its square, $x^2 + y^2$. Since $y = 3 - 2x$, we are minimizing

   $$x^2 + (3 - 2x)^2 = x^2 + 9 - 12x + 4x^2 = 5x^2 - 12x + 9.$$

   Setting the derivative equal to zero, we have $10x = 12$ and so $x = 6/5$. Thus the distance to the origin is $\sqrt{(6/5)^2 + (3 - 2 \cdot 6/5)^2 = 6/5\sqrt{10}}$.

7. A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level? Express your answer in meters per second.

   Solution: Let θ be the angle that the rider's seat makes with the horizontal from the center of the ferris wheel. The height of the rider as a function of θ is $h(\theta) = 10 + 10\sin(\theta)$. Thinking of θ as a function of time t, since it makes one revolution every two minutes (120 seconds) we have that $\theta'(t) = 2\pi/120 = \pi/60$ radians per second.

   When he is $16m$ above ground level we have that $10\sin(\theta) = 6$, and by the Pythagorean theorem, $10\cos(\theta) = 8$. So the rate of change of the rider's height is

   $$h' = 10\cos(\theta)\theta'$$

   which at that point is $8\theta' = 8 \cdot \pi/60 = 2\pi/15$ meters per second.

8. What is $\int e^{e^x} \cdot e^x \, dx$?

   Solution: Using the u-substitution $u = e^x$, the integral becomes $\int e^u \, du = e^u$. Thus the most general form of the integral is $e^{e^x} + C$.

9. Find $\int_0^{10} \sqrt{100 - x^2} \, dx$.

   Solution: We can interpret this integral as the area under a semicircle of radius 10 centered at the origin in the first quadrant. This is a quarter of a circle of radius 10, so the area is $1/4 \cdot 100\pi = 25\pi$.

10. Suppose a certain donut can be modeled by revolving the circle of radius r centered at $(r, 0)$ about the y-axis. If a donut hole from the same company is a sphere with radius r, how much larger is the donut than the donut hole, by volume?

    Solution: (Done in class.)