Estimation: Newton's method and areas

Math 1A, section 103

April 10, 2014

0. (Warmup.) Approximate $\sqrt{3}$ using Newton's method.

1. Try to compute ln(2) without a calculator. In other words, use Newton's method to approximate the solutions of the equation $e^x - 2 = 0$.

   Solution Outline: Start with $x_0 = 1$ as a first approximation. We can approximate e as 2.72 by using the formula $e = 1/0! + 1/1! + 1/2! + \cdots$. The next approximation gives

   $$x_1 = 1 - \frac{e-2}{e} = 2/e \approx \frac{2}{2.72} \approx 0.735.$$

   Now, the next approximation is

   $$x_2 = 0.735 - \frac{e^{2/e} - 2}{e^{2/e}} \approx 0.735 - 1 + 2/e^{0.735}.$$

   To estimate this, let's approximate $0.735 \approx 3/4$, and calculate $e^{3/4}$ by Newton's method. It's the solution to $x^4 - e^3 = 0$, and Newton's method shows the solution is approximately 2.125. Substituting this in for $e^{0.735}$ in x_2, we find $x_2 \approx 0.677$.

   This is pretty close to $\ln(2) = 0.69$!

2. Approximate π.

   Solution Outline: Draw a half-circle of radius 1 centered at $(0, 0)$. This is described by the function $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. The area under this circle should be $\pi/2$, and we can estimate it using rectangles! Using $-1, -0.9, -0.8, \ldots, 1$ as our endpoints of the rectangles, the lower estimate is about 1.45 and the upper is about 1.65. The average is 1.55, and doubling this we find $\pi \approx 3.1$. Pretty close!

3. (Problem 24 in Stewart.)

    (a) Find an expression for the area under the curve $y = x^3$ from 0 to 1 as a limit.

    (b) Evaluate the limit in part (a). You may use the formula

$$1^3 + 2^3 + 3^2 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

4. Using the definition of the definite integral in terms of limits of Riemann sums, prove that

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}.$$

5. Use the fundamental theorem of calculus to find

$$\int_{-1}^{2} x^3 - 2x\,dx.$$