INDEPENDENCE OF $\ell$ AND TRACES ON COHOMOLOGY

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Abstract. Let $k$ be an algebraically closed field and let $f : X \to X$ be an endomorphism of a separated scheme of finite type over $k$. We show that for any $\ell$ invertible in $k$ the alternating sum of traces $\sum (-1)^i \text{tr}(f^*|H^i(X, \mathbb{Q}_\ell))$ of pullback on étale cohomology is a rational number independent of $\ell$. This is deduced from a more general result for motivic sheaves.

1. Statement of main results

1.1. In this note we explain how to deduce various independence of $\ell$ results from the formalism of six operations for motives developed by Cisinski and Deglise in [2] and our work in [7].

Thoughout the paper we work over an algebraically closed field $k$ and consider only finite type separated $k$-schemes. For such a scheme $X$ let $\mathcal{M}(X)$ denote the triangulated category of constructible Beilinson motives over $X$ as defined in [2, §14]. A summary of the basic six operations formalism for this category can be found in [7, §2 and §6].

1.2. Let $c = (c_1, c_2) : C \to X \times X$ be a correspondence over $k$ and let $F \in \mathcal{M}(X)$ be an object equipped with a morphism $u : c_1^* F \to c_2^* F$ in $\mathcal{M}(C)$. For any $\ell$ invertible in $k$ we have a realization functor (see [3, 7.2.24])

$$R_\ell : \mathcal{M}(X) \to D^b_c(X, \mathbb{Q}_\ell)$$

to the bounded derived category of complexes of $\mathbb{Q}_\ell$-sheaves on the étale site of $X$ with constructible cohomology. The map $u$ induces a map $u_\ell : c_1^* R_\ell(F) \to c_2^* R_\ell(F)$ in $D^b_c(C, \mathbb{Q}_\ell)$.

If $c_2$ is proper then the map $u_\ell$ induces an endomorphism $u_\ell^* : R\Gamma(X, R_\ell(F)) \to R\Gamma(X, R_\ell(F))$ defined as the composite

$$R\Gamma(X, R_\ell(F)) \xrightarrow{\text{id} - c_1^* c_2^*} R\Gamma(C, c_1^* R_\ell(F)) \xrightarrow{u_\ell^*} R\Gamma(C, c_2^* R_\ell(F)) \simeq R\Gamma(X, c_2^* c_2^* R_\ell(F)) \xrightarrow{c_2^* c_2^* + \text{id}} R\Gamma(X, R_\ell(F)), $$

where the map $c_2^* c_2^* \to \text{id}$ is defined using the properness of $c_2$.

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The main result of the paper is the following:

**Theorem 1.3.** Let \( c : C \to X \times X \) be a correspondence with \( c_2 \) proper and let \( F \in \mathcal{M}(X) \) be an object with a map \( u : c_1^*F \to c_2^*F \) in \( \mathcal{M}(C) \). Then for any \( \ell \) invertible in \( k \) the alternating sum of traces

\[
\sum_i (-1)^i \text{tr}(u_\ell^*|H^i(X, R_\ell(F)))
\]

is in \( \mathbb{Q} \) and independent of \( \ell \).

**1.4.** Let \( f : X \to X \) be an endomorphism of a finite type separated \( k \)-scheme and consider the correspondence

\[ \Gamma_f := (f, \text{id}) : X \to X \times X. \]

Taking \( F = 1_X \) and \( u : f^*1_X \to 1_X \) we have \( R_\ell(F) = \mathbb{Q}_\ell \) and in this case the sum (1.3.1) reduces to the trace on cohomology

\[
\sum_i (-1)^i \text{tr}(f^*|H^i(X, \mathbb{Q}_\ell)).
\]

Theorem 1.3 therefore implies the following:

**Corollary 1.5.** If \( f : X \to X \) is an endomorphism of a finite type separated \( k \)-scheme then for any \( \ell \) invertible in \( k \)

\[
\sum_i (-1)^i \text{tr}(f^*|H^i(X, \mathbb{Q}_\ell))
\]

is in \( \mathbb{Q} \) and independent of \( \ell \).

**Remark 1.6.** Corollary 1.5 answers a question posed in \([6, 3.5 (c)]\).

**Remark 1.7.** Corollary 1.5 has also been obtained by Bondarko \([1, \text{Discussion following 8.4.1}]\).

**1.8.** If \( f : X \to X \) is proper one can also consider the correspondence

\[ \Gamma_f^t := (\text{id}, f) : X \to X \times X, \]

the motivic dualizing complex \( \Omega_X^{\ell} \) discussed in \([7, 2.9 (\text{Duality})] \), and the canonical isomorphism

\[ u : \Gamma_f^{t*}\Omega_X^{\ell} = \Omega_X^{\ell} \to f^!\Omega_X^{\ell}. \]

For any \( \ell \) invertible in \( k \) the realization \( R_\ell(\Omega_X^{\ell}) \) is the \( \ell \)-adic dualizing complex \( \Omega_X \) and \( R\Gamma(X, R_\ell(\Omega_X^{\ell})) \) is dual to the compactly supported étale cohomology \( R\Gamma_c(X, \mathbb{Q}_\ell) \). The endomorphism of \( R\Gamma_c(X, \mathbb{Q}_\ell) \) obtained by this identification is the pullback map (defined since \( f \) is proper)

\[ f^* : R\Gamma_c(X, \mathbb{Q}_\ell) \to R\Gamma_c(X, \mathbb{Q}_\ell). \]

From 1.3 we therefore obtain:

**Corollary 1.9.** Let \( f : X \to X \) be a proper endomorphism of a separated finite type \( k \)-scheme \( X \). Then the alternating sum of traces

\[
\sum_i (-1)^i \text{tr}(f^*|H^i_c(X, \mathbb{Q}_\ell))
\]

is in \( \mathbb{Q} \) and independent of \( \ell \).
Remark 1.10. Corollary [1.9] can also be obtained by first reducing to the case when $k$ is a finite field and then using Fujiwara’s theorem [4, 5.4.5] as discussed in [6, 3.5 (c)].

Remark 1.11. The traces (1.5.1) and (1.9.1) in fact lie in $\mathbb{Z}[1/p]$, where $p$ is the characteristic of $k$, since they are rational numbers which lie in $\mathbb{Z}\ell$ for all $\ell \neq p$.

2. Motivic characteristic classes

2.1. Let $X$ and $Y$ denote finite type separated $k$-schemes and let $p : X \times Y \rightarrow X$ (resp. $q : X \times Y \rightarrow Y$) be the projection. Recall [7, 7.15] that there is a natural isomorphism

$$\epsilon_{X \times Y} : \Omega^{\#}_X \boxtimes \Omega^{\#}_Y \rightarrow \Omega^{\#}_{X \times Y}.$$ 

This isomorphism induces for any $A \in \mathcal{M}(X)$ and $B \in \mathcal{M}(Y)$ a map

$$\alpha_{A,B} : D_X(A) \boxtimes B \rightarrow \mathcal{R}\text{Hom}_{X \times Y}(p^*A, q^!B).$$

Indeed giving such a map is by adjunction equivalent to giving a map

$$(D_X(A) \otimes A) \boxtimes B \rightarrow q^!B \simeq \mathcal{R}\text{Hom}_{X \times Y}(q^*D_Y(B), \Omega^{\#}_X \boxtimes \Omega^{\#}_Y),$$

and using adjunction again giving such a map is equivalent to giving a map

$$(D_X(A) \otimes A) \boxtimes (B \otimes D_Y(B)) \rightarrow \Omega^{\#}_X \boxtimes \Omega^{\#}_Y.$$ 

To define $\alpha_{A,B}$ we take the tensor product of the evaluation maps

$$D_X(A) \otimes A \rightarrow \Omega^{\#}_X, \quad B \otimes D_Y(B) \rightarrow \Omega^{\#}_Y.$$ 

2.2. Let $\pi : X' \rightarrow X$ be a proper morphism, let $\tilde{\pi} : X' \times Y \rightarrow X \times Y$ be the base change of $\pi$, and let

$$p' : X' \times Y \rightarrow X', \quad q' : X' \times Y \rightarrow Y$$

be the projections. Then it follows from the construction that for $A' \in \mathcal{M}(X')$ the diagram

$$
\begin{align*}
D_X(\pi_*A') \boxtimes B & \xrightarrow{\alpha_{\pi_*A',B}} \mathcal{R}\text{Hom}_{X \times Y}(p^{\#}\pi_*A', q^!B) \\
\pi_*D_{X'}(A') \boxtimes B & \xrightarrow{\tilde{\pi}^*p^{\#}\pi_*} \mathcal{R}\text{Hom}_{X \times Y}(\tilde{\pi}^*p^{\#}A', q^!B) \\
\tilde{\pi}_*(D_{X'}(A') \boxtimes B) & \xrightarrow{\tilde{\pi}^*\alpha_{A',B}} \tilde{\pi}_*\mathcal{R}\text{Hom}_{X \times Y}(p^{\#}A', q^!B)
\end{align*}
$$

commutes, where all the vertical maps are isomorphisms. In particular, if $\alpha_{A',B}$ is an isomorphism then $\alpha_{\pi_*A',B}$ is also an isomorphism.

2.3. Similarly if $\gamma : Y' \rightarrow Y$ is a proper morphism with resulting morphisms

$$\tilde{\gamma} : X \times Y' \rightarrow X \times Y, \quad p' : X \times Y' \rightarrow X, \quad q' : X \times Y' \rightarrow Y',$$
and if $B' \in \mathcal{M}(Y')$ then the resulting diagram
\[
\begin{array}{c}
D_X(A) \boxtimes \gamma_* B' \xrightarrow{\alpha_A,\gamma_* B'} \mathcal{R}Hom_{X \times Y}(p^* A, q^! \gamma_* B') \\
\downarrow \sim \gamma_* q' \\
\mathcal{R}Hom_{X \times Y}(p^* A, \tilde{\gamma}_* q' B') \xrightarrow{\text{adjunction}} \nabla \end{array}
\]
commutes, where again the vertical morphisms are isomorphisms. It follows that if $\alpha_{A,B'}$ is an isomorphism then so is $\alpha_{A,\gamma_* B'}$.

**Proposition 2.4.** For any $A \in \mathcal{M}(X)$ and $B \in \mathcal{M}(Y)$ the map $\alpha_{A,B}$ is an isomorphism.

**Proof.** The key ingredient in the proof is the following fact (see [3, 6.2.6]): If $Z$ is a finite type separated $k$-scheme and $D \subset \mathcal{M}(Z)$ is a thick triangulated subcategory of $\mathcal{M}(Z)$ containing all objects of the form $f_* (1_{Z'}(n))$ for $f : Z' \to Z$ a projective morphism and $n \in \mathbb{Z}$ then $D = \mathcal{M}(Z)$.

We will use this to reduce the proof of 2.4 to the case when $A = 1_X$ and $B = 1_Y$.

**Lemma 2.5.** Let $Z$ be a finite type separated $k$-scheme. Then for any $A \in \mathcal{M}(Z)$ and $n \in \mathbb{Z}$ the natural map
\[(2.5.1) \quad D(A)(n) \to D(A(-n))\]
adjoint to the evaluation map
\[D(A)(n) \otimes A(-n) \to \Omega^a_Z\]
is an isomorphism.

**Proof.** The category $D \subset \mathcal{M}(Z)$ for which the lemma holds is a thick triangulated subcategory so it suffices to show that it contains objects of the form $f_* (1_{Z'}(n))$ for a projective morphism $f : Z' \to Z$ and $n \in \mathbb{Z}$. Now observe that under the isomorphisms (using that $f$ is proper)
\[D_Z(f_* 1_{Z'})(n) \simeq (f_* D_{Z'}(1_{Z'}))(n), \quad D_Z(f_* 1_{Z'}(n)) \simeq f_* D_{Z'}(1_{Z'}(n))\]
the map (2.5.1) is identified with the pushforward of the corresponding map
\[D_{Z'}(1_{Z'}(n)) \to D_{Z'}(1_{Z'}(-n))\]
which is the natural isomorphism
\[\mathcal{R}Hom_{Z'}(1_{Z'}, \Omega^a_{Z'})(n) \to \mathcal{R}Hom_{Z'}(1_{Z'}(-n), \Omega^a_{Z'}).\]

\[\square\]

This implies in particular that for any morphism $g : W \to Z$, $A \in \mathcal{M}(Z)$, and $n \in \mathbb{Z}$ the natural map
\[(g^! A)(n) \to g^!(A(n))\]
is an isomorphism.
With notation as in \([2.1]\) we then get for any integer \(n \in \mathbb{Z}\) a square

\[
D(A) \boxtimes B(n) \xrightarrow{\alpha_{A,B(n)}} \mathcal{H}om(p^*A, q^!(B(n)))
\]

where the vertical isomorphisms are obtained from the preceding identifications. Chasing through the definitions one finds that this diagram commutes. In particular \(\alpha_{A,B(n)}\) is an isomorphism if and only if \(\alpha_{A(-n),B}\) is an isomorphism.

In the case when \(A = 1_X\) and \(B = 1_Y\) the map

\[
\alpha_{1_X,1_Y} : \Omega_X^\# \boxtimes 1_Y \rightarrow q^!1_Y \simeq \mathcal{H}om_{X \times Y}(q^*\Omega_Y^\#, \Omega_X^\# \boxtimes \Omega_Y^\#)
\]

is the map denoted \(\rho_{X,Y}^\#\) in \([7, 5.4]\) (with \(X\) and \(Y\) interchanged) and in particular \(\alpha_{1_X,1_Y}\) is an isomorphism by \([7, 5.7]\). Since the identification \(q^!(1_Y(n)) \simeq (q^!1_Y)(n)\) identifies the map \(\alpha_{1_X,1_Y(n)}\) with the map obtained from \(\alpha_{1_X,1_Y}\) by tensoring with \(1_X \times Y(n)\), it also follows that \(\alpha_{1_X,1_Y(n)}\) is an isomorphism for any integer \(n\).

From this we deduce that \(\alpha_{A,1_Y}\) is an isomorphism for any \(A \in \mathcal{M}(X)\). Indeed the collection of \(A \in \mathcal{M}(X)\) for which \(\alpha_{A,1_Y}\) is an isomorphism is a thick triangulated subcategory of \(\mathcal{M}(X)\) and by the discussion in \([2.2]\) the map \(\alpha_{\pi_{X,Y}(n),1_Y}\) is an isomorphism for all proper morphisms \(\pi : X' \rightarrow X\) and all \(n\). Using \([3, 6.2.6]\), it follows that \(\alpha_{A,1_Y}\) is an isomorphism for all \(A \in \mathcal{M}(X)\), and also that \(\alpha_{A,1_Y(n)}\) is an isomorphism for all \(A \in \mathcal{M}(X)\) and \(n \in \mathbb{Z}\) since the maps \(\alpha_{A(-n),1_Y}\) are isomorphisms.

Now consider the collection of \(B \in \mathcal{M}(Y)\) for which the map \(\alpha_{A,B}\) is an isomorphism for all \(A \in \mathcal{M}(X)\). Again this is a thick triangulated subcategory of \(\mathcal{M}(Y)\) and by the discussion in \([2.3]\) and the already known case of the \(\alpha_{A,1_Y(n)}\)'s, it contains all objects of the form \(\gamma_{s,1_Y}(n)\) for \(\gamma : Y' \rightarrow Y\) proper. Using \([3, 6.2.6]\) once again it follows that \(\alpha_{A,B}\) is an isomorphism for all \(A\) and \(B\) as desired.

\(\square\)

2.6. Let

\[
c = (c_1, c_2) : C \rightarrow X \times X
\]

be a correspondence, and let \(F\) denote \(C \times_{c,X \times X, \Delta} X\) so we have a cartesian square

\[
\begin{array}{ccc}
F & \xrightarrow{\delta} & C \\
\downarrow{c'} & & \downarrow{c} \\
X & \xrightarrow{\Delta} & X \times X.
\end{array}
\]

By \([3, A.1.10 (5)]\) we have for \(F \in \mathcal{M}(X)\)

\[
c^!\mathcal{H}om_{X \times X}(\text{pr}_1^*F, \text{pr}_2^*F) \simeq \mathcal{H}om_C(c_1^*F, c_2^!F).
\]

Combining this with \([2.4]\) we get a map

\[
\text{Hom}_{\mathcal{M}(C)}(c_1^*F, c_2^!F) \rightarrow c^!(D_X(F) \boxtimes F).
\]

Composing with the map

\[
c^!(D_X(F) \boxtimes F) \xrightarrow{\text{id} \Delta_\ast} c^!\Delta_\ast(D_X(F) \boxtimes F) \xrightarrow{\text{valuation}} c^!\Delta_\ast|_{X} \xrightarrow{\text{base change}} \delta_\ast|_{F}\Omega^\#_F
\]
we get a morphism
\[ \text{Tr} : \text{Hom}_{\mathcal{M}(C)}(c_1^*F, c_2^! F) \to \text{Hom}_{\mathcal{M}(F)}(1_F, \Omega_{F}^{\#}). \]

The image of a map \( u : c_1^* F \to c_2^! F \) under this map is called the \textit{characteristic class} of \( u \).

**Remark 2.7.** If \( F \) is quasi-projective then it is shown in [7, 6.2] that there is a canonical isomorphism
\[ A_0(F)_{\mathbb{Q}} \simeq \text{Hom}_{\mathcal{M}(F)}(1_F, \Omega_{F}^{\#}), \]
where \( A_0(F)_{\mathbb{Q}} \) denotes the Chow group of 0-cycles on \( F \) tensor \( \mathbb{Q} \).

**2.8.** As in [7, 5.9] the formation of characteristic classes is compatible with morphisms of motivic categories. In particular for \( \ell \) invertible in \( k \) we have the \( \acute{\text{e}} \text{tale realization functor} \) [3, 7.2.24]
\[ R_\ell : \mathcal{M}(X) \to D^b_c(X, \mathbb{Q}_\ell). \]
If \( F \in \mathcal{M}(X) \) is an object and \( u : c_1^* F \to c_2^! F \) is a morphism in \( \mathcal{M}(C) \) with realization
\[ u_\ell : c_1^* R_\ell(F) \to c_2^! R_\ell(F) \]
in \( D^b_c(C, \mathbb{Q}_\ell) \), then the corresponding characteristic class \( \text{Tr}(u_\ell) \in H^0(F, \Omega_F) \) defined as in [5, III, 4.1] is the image of \( \text{Tr}(u) \) under the realization map
\[ \text{Hom}_{\mathcal{M}(F)}(1_F, \Omega_{F}^{\#}) \to H^0(F, \Omega_F). \]

### 3. Proof of 1.3

**3.1.** By Nagata’s theorem we can find a commutative diagram
\[ \begin{array}{ccc}
C & \xrightarrow{j} & \mathcal{C} \\
\downarrow{c_2} & & \downarrow{c_2} \\
X & \xleftarrow{j} & \overline{X},
\end{array} \]
where \( j \) and \( j \) are dense open imbeddings and \( \mathcal{C} \) and \( \overline{X} \) are proper over \( k \). Since \( c_2 \) is proper by assumption this square is cartesian. Modifying \( \mathcal{C} \) along \( \mathcal{C} - C \), which does not change the property that the square is cartesian, we can further arrange that the map \( c_1 : C \to X \) extends to a morphism \( \overline{c}_1 : \overline{C} \to \overline{X} \).

Let \( F \in \mathcal{M}(\overline{X}) \) denote \( j_* F \). Then using the canonical base change isomorphism \( \overline{c}_2^! j_* \simeq j_! c_2^! \) we have
\[ \overline{c}_2^! F \simeq \overline{j}_* c_2^! F \]
so giving a morphism
\[ \overline{c}_2^! F \to \overline{c}_2^! F \]
is equivalent to giving a morphism \( c_1^* F \to c_2^! F \) and \( u \) extends uniquely to a morphism \( \overline{u} : \overline{c}_1^* F \to \overline{c}_2^! F \).

Since the realization functors \( R_\ell \) commute with the six operations we have
\[ R_\ell(F) = j_* R_\ell(F). \]

We therefore get an isomorphism
\[ R\Gamma(\overline{X}, R_\ell(F)) \simeq R\Gamma(X, R_\ell(F)). \]
and
\[
\sum_i (-1)^i \text{tr}(\bar{u}_\ell^*|H^i(X, R_\ell(F))) = \sum_i (-1)^i \text{tr}(u_\ell^*|H^i(X, R_\ell(F))).
\]

From this it follows that it suffices to prove 1.3 in the case when \(X\) and \(C\) are proper over \(k\).

3.2. In this case the Grothendieck-Lefschetz trace formula \([5, \text{III, 4.7}]\) gives
\[
\sum_i (-1)^i \text{tr}(\bar{u}_\ell^*|H^i(X, R_\ell(F))) = \sum_{Z \in \pi_0(F)} \int_Z \text{Tr}(u_\ell),
\]
where on the right the sum is over the connected components of the fixed point locus \(F := C \times_{X \times X, \Delta_X} X\), \(\text{Tr}(u_\ell)\) is the characteristic class of \(u_\ell\) defined in \([5, \text{III, 4.1}]\) and
\[
\int_Z : H^0(Z, \Omega_Z) \to \overline{Q}_\ell
\]
is the pushforward map induced by adjunction. By \([2,8]\) there is a class \(\text{Tr}(u) \in \text{Hom}_{\mathcal{M}(F)}(1_F, \Omega_{\mathcal{M}(F)})\) such that for all \(\ell\) we have \(\text{Tr}(u_\ell) = R_\ell(\text{Tr}(u))\). In particular we find that \(\int_Z(\text{Tr}(u_\ell))\) is equal to the corresponding pushforward of \(\text{Tr}(u)\) to \(\text{Ext}^0_{\mathcal{M}(k)}(1_k, 1_k) = \mathbb{Q}\). This proves 1.3 \(\square\)

REFERENCES