INDEPENDENCE OF ℓ AND TRACES ON COHOMOLOGY

MARTIN OLSSON

Abstract. Let k be an algebraically closed field and let $f : X \to X$ be an endomorphism of a separated scheme of finite type over k. We show that for any ℓ invertible in k the alternating sum of traces $\sum_i (-1)^i \mathrm{tr}(f^*|H^i(X, \mathbb{Q}_\ell))$ of pullback on étale cohomology is a rational number independent of ℓ. This is deduced from a more general result for motivic sheaves.

1. Statement of main results

1.1. In this note we explain how to deduce various independence of ℓ results from the formalism of six operations for motives developed by Cisinski and Deglise in [2] and our work in [7].

Thoughout the paper we work over an algebraically closed field k and consider only finite type separated k-schemes. For such a scheme X let $\mathscr{M}(X)$ denote the triangulated category of constructible Beilinson motives over X as defined in [2, §14]. A summary of the basic six operations formalism for this category can be found in [7, §2 and §6].

1.2. Let

$$c = (c_1, c_2) : C \to X \times X$$

be a correspondence over k and let $F \in \mathscr{M}(X)$ be an object equipped with a morphism

$$u : c_1^* F \to c_2^! F$$

in $\mathscr{M}(C)$. For any ℓ invertible in k we have a realization functor (see [3, 7.2.24])

$$R_\ell : \mathscr{M}(X) \to D_c^b(X, \mathbb{Q}_\ell)$$

to the bounded derived category of complexes of $\mathbb{Q}_\ell$-sheaves on the étale site of X with constructible cohomology. The map u induces a map

$$u_\ell : c_1^* R_\ell(F) \to c_2^! R_\ell(F)$$

in $D_c^b(C, \mathbb{Q}_\ell)$.

If c_2 is proper then the map u_ℓ induces an endomorphism

$$u_\ell^* : R\Gamma(X, R_\ell(F)) \to R\Gamma(X, R_\ell(F))$$

defined as the composite

$$R\Gamma(X, R_\ell(F)) \xrightarrow{\mathrm{id} \to c_{1*}c_1^*} R\Gamma(C, c_1^* R_\ell(F)) \xrightarrow{u_\ell} R\Gamma(C, c_2^! R_\ell(F)) \simeq R\Gamma(X, c_{2*}c_2^! R_\ell(F)) \xrightarrow{c_{2*}c_2^! \to \mathrm{id}} R\Gamma(X, R_\ell(F)),$$

where the map $c_{2*}c_2^! \to \mathrm{id}$ is defined using the properness of c_2.

The author was partially supported by NSF grant DMS-1303173 and a grant from the Simons Foundation. Part of this work was done during a visit to the I.H.E.S partially funded by the National Science Foundation under Grant No. 1002477.

The main result of the paper is the following:

Theorem 1.3. *Let $c : C \to X \times X$ be a correspondence with c_2 proper and let $F \in \mathscr{M}(X)$ be an object with a map $u : c_1^*F \to c_2^!F$ in $\mathscr{M}(C)$. Then for any ℓ invertible in k the alternating sum of traces*

$$\sum_i (-1)^i \mathrm{tr}(u_\ell^* | H^i(X, R_\ell(F))) \tag{1.3.1}$$

is in $\mathbb{Q}$ and independent of ℓ.

1.4. Let $f : X \to X$ be an endomorphism of a finite type separated k-scheme X and consider the correspondence

$$\Gamma_f := (f, \mathrm{id}) : X \to X \times X.$$

Taking $F = 1_X$ and $u : f^*1_X \to 1_X$ we have $R_\ell(F) = \mathbb{Q}_\ell$ and in this case the sum (1.3.1) reduces to the trace on cohomology

$$\sum_i (-1)^i \mathrm{tr}(f^* | H^*(X, \mathbb{Q}_\ell)).$$

Theorem 1.3 therefore implies the following:

Corollary 1.5. *If $f : X \to X$ is an endomorphism of a finite type separated k-scheme then for any ℓ invertible in k*

$$\sum_i (-1)^i \mathrm{tr}(f^* | H^i(X, \mathbb{Q}_\ell)) \tag{1.5.1}$$

is in $\mathbb{Q}$ and independent of ℓ.

Remark 1.6. Corollary 1.5 answers a question posed in [6, 3.5 (c)].

Remark 1.7. Corollary 1.5 has also been obtained by Bondarko [1, Discussion following 8.4.1].

1.8. If $f : X \to X$ is proper one can also consider the correspondence

$$\Gamma_f^t := (\mathrm{id}, f) : X \to X \times X,$$

the motivic dualizing complex $\Omega_X^{\mathscr{M}}$ discussed in [7, 2.9 (Duality)], and the canonical isomorphism

$$u : \Gamma_f^{t*}\Omega_X^{\mathscr{M}} = \Omega_X^{\mathscr{M}} \to f^!\Omega_X^{\mathscr{M}}.$$

For any ℓ invertible in k the realization $R_\ell(\Omega_X^{\mathscr{M}})$ is the ℓ-adic dualizing complex Ω_X and $R\Gamma(X, R_\ell(\Omega_X^{\mathscr{M}}))$ is dual to the compactly supported étale cohomology $R\Gamma_c(X, \mathbb{Q}_\ell)$. The endomorphism of $R\Gamma_c(X, \mathbb{Q}_\ell)$ obtained by this identification is the pullback map (defined since f is proper)

$$f^* : R\Gamma_c(X, \mathbb{Q}_\ell) \to R\Gamma_c(X, \mathbb{Q}_\ell).$$

From 1.3 we therefore obtain:

Corollary 1.9. *Let $f : X \to X$ be a proper endomorphism of a separated finite type k-scheme X. Then the alternating sum of traces*

$$\sum_i (-1)^i \mathrm{tr}(f^* | H_c^i(X, \mathbb{Q}_\ell)) \tag{1.9.1}$$

is in $\mathbb{Q}$ and independent of ℓ.

Remark 1.10. Corollary 1.9 can also be obtained by first reducing to the case when k is a finite field and then using Fujiwara's theorem [4, 5.4.5] as discussed in [6, 3.5 (c)].

Remark 1.11. The traces (1.5.1) and (1.9.1) in fact lie in $\mathbb{Z}[1/p]$, where p is the characteristic of k, since they a rational numbers which lie in $\mathbb{Z}_\ell$ for all $\ell \neq p$.

2. Motivic characteristic classes

2.1. Let X and Y denote finite type separated k-schemes and let $p : X \times Y \to X$ (resp. $q : X \times Y \to Y$) be the projection. Recall [7, 7.15] that there is a natural isomorphism

$$\epsilon_{X\times Y} : \Omega_X^{\mathscr{M}} \boxtimes \Omega_Y^{\mathscr{M}} \to \Omega_{X\times Y}^{\mathscr{M}}.$$

This isomorphism induces for any $A \in \mathscr{M}(X)$ and $B \in \mathscr{M}(Y)$ a map

$$\alpha_{A,B} : D_X(A) \boxtimes B \to \mathscr{R}Hom_{X\times Y}(p^*A, q^!B).$$

Indeed giving such a map is by adjunction equivalent to giving a map

$$(D_X(A) \otimes A) \boxtimes B \to q^!B \simeq \mathscr{R}Hom_{X\times Y}(q^*D_Y(B), \Omega_X^{\mathscr{M}} \boxtimes \Omega_Y^{\mathscr{M}}),$$

and using adjunction again giving such a map is equivalent to giving a map

$$(D_X(A) \otimes A) \boxtimes (B \otimes D_Y(B)) \to \Omega_X^{\mathscr{M}} \boxtimes \Omega_Y^{\mathscr{M}}.$$

To define $\alpha_{A,B}$ we take the tensor product of the evaluation maps

$$D_X(A) \otimes A \to \Omega_X^{\mathscr{M}}, \quad B \otimes D_Y(B) \to \Omega_Y^{\mathscr{M}}.$$

2.2. Let $\pi : X' \to X$ be a proper morphism, let $\tilde{\pi} : X' \times Y \to X \times Y$ be the base change of π, and let

$$p' : X' \times Y \to X', \quad q' : X' \times Y \to Y$$

be the projections. Then it follows from the construction that for $A' \in \mathscr{M}(X')$ the diagram

$$\begin{array}{ccc}
D_X(\pi_*A') \boxtimes B & \xrightarrow{\alpha_{\pi_*A',B}} & \mathscr{R}Hom_{X\times Y}(p^*\pi_*A', q^!B) \\
\downarrow{\scriptstyle D_X\pi_*\simeq\pi_*D_{X'}} & & \downarrow{\scriptstyle \tilde{\pi}_*p'^*\simeq p^*\pi_*} \\
\pi_*D_{X'}(A') \boxtimes B & & \mathscr{R}Hom_{X\times Y}(\tilde{\pi}_*p'^*A', q^!B) \\
\downarrow & & \downarrow{\scriptstyle \text{adjunction}} \\
\tilde{\pi}_*(D_{X'}(A') \boxtimes B) & \xrightarrow{\tilde{\pi}_*\alpha_{A',B}} & \tilde{\pi}_*\mathscr{R}Hom_{X'\times Y}(p'^*A', q'^!B)
\end{array}$$

commutes, where all the vertical maps are isomorphisms. In particular, if $\alpha_{A',B}$ is an isomorphism then $\alpha_{\pi_*A',B}$ is also an isomorphism.

2.3. Similarly if $\gamma : Y' \to Y$ is a proper morphism with resulting morphisms

$$\tilde{\gamma} : X \times Y' \to X \times Y, \quad p' : X \times Y' \to X, \quad q' : X \times Y' \to Y',$$

and if $B' \in \mathscr{M}(Y')$ then the resulting diagram

$$\begin{array}{ccc} D_X(A) \boxtimes \gamma_* B' & \xrightarrow{\alpha_{A,\gamma_* B}} & \mathscr{R}Hom_{X \times Y}(p^* A, q^! \gamma_* B') \\ \Big\downarrow & & \Big\downarrow {\scriptstyle q^! \gamma_* \simeq \tilde{\gamma}_* q'^!} \\ & & \mathscr{R}Hom_{X \times Y}(p^* A, \tilde{\gamma}_* q'^! B') \\ & & \Big\downarrow {\scriptstyle \text{adjunction}} \\ \tilde{\gamma}_*(D_X(A) \boxtimes B') & \xrightarrow{\tilde{\gamma}_* \alpha_{A,B'}} & \tilde{\gamma}_* \mathscr{R}Hom_{X \times Y'}(p'^* A, q'^! B') \end{array}$$

commutes, where again the vertical morphisms are isomorphisms. It follows that if $\alpha_{A,B'}$ is an isomorphism then so is $\alpha_{A,\gamma_* B'}$.

Proposition 2.4. *For any $A \in \mathscr{M}(X)$ and $B \in \mathscr{M}(Y)$ the map $\alpha_{A,B}$ is an isomorphism.*

Proof. The key ingredient in the proof is the following fact (see [3, 6.2.6]): If Z is a finite type separated k-scheme and $\mathscr{D} \subset \mathscr{M}(Z)$ is a thick triangulated subcategory of $\mathscr{M}(Z)$ containing all objects of the form $f_*(1_{Z'}(n))$ for $f : Z' \to Z$ a projective morphism and $n \in \mathbb{Z}$ then $\mathscr{D} = \mathscr{M}(Z)$.

We will use this to reduce the proof of 2.4 to the case when $A = 1_X$ and $B = 1_Y$.

Lemma 2.5. *Let Z be a finite type separated k-scheme. Then for any $A \in \mathscr{M}(Z)$ and $n \in \mathbb{Z}$ the natural map*

$$D(A)(n) \to D(A(-n)) \tag{2.5.1}$$

adjoint to the evaluation map

$$D(A)(n) \otimes A(-n) \to \Omega_Z^{\mathscr{M}}$$

is an isomorphism.

Proof. The category $\mathscr{D} \subset \mathscr{M}(Z)$ for which the lemma holds is a thick triangulated subcategory so it suffices to show that it contains objects of the form $f_* 1_{Z'}(n)$ for a projective morphism $f : Z' \to Z$ and $n \in \mathbb{Z}$. Now observe that under the isomorphisms (using that f is proper)

$$D_Z(f_* 1_{Z'})(n) \simeq (f_* D_{Z'}(1_{Z'}))(n), \quad D_Z(f_* 1_{Z'}(n)) \simeq f_* D_{Z'}(1_{Z'}(n))$$

the map (2.5.1) is identified with the pushforward of the corresponding map

$$D_{Z'}(1_{Z'})(n) \to D_{Z'}(1_{Z'}(-n))$$

which is the natural isomorphism

$$\mathscr{R}Hom_{Z'}(1_{Z'}, \Omega_{Z'}^{\mathscr{M}})(n) \to \mathscr{R}Hom_{Z'}(1_{Z'}(-n), \Omega_{Z'}^{\mathscr{M}}).$$

□

This implies in particular that for any morphism $g : W \to Z$, $A \in \mathscr{M}(Z)$, and $n \in \mathbb{Z}$ the natural map

$$(g^! A)(n) \to g^!(A(n))$$

is an isomorphism.

With notation as in 2.1 we then get for any integer $n \in \mathbb{Z}$ a square

$$\begin{array}{ccc} D(A) \boxtimes B(n) & \xrightarrow{\alpha_{A,B(n)}} & \mathscr{R}Hom(p^*A, q^!(B(n))) \\ \downarrow \simeq & & \downarrow \simeq \\ D(A(-n)) \boxtimes B & \xrightarrow{\alpha_{A(-n),B}} & \mathscr{R}Hom(p^*(A(-n)), q^!B), \end{array}$$

where the vertical isomorphisms are obtained from the preceding identifications. Chasing through the definitions one finds that this diagram commutes. In particular $\alpha_{A,B(n)}$ is an isomorphism if and only if $\alpha_{A(-n),B}$ is an isomorphism.

In the case when $A = 1_X$ and $B = 1_Y$ the map

$$\alpha_{1_X,1_Y} : \Omega_X^{\mathscr{M}} \boxtimes 1_Y \to q^! 1_Y \simeq \mathscr{R}Hom_{X\times Y}(q^*\Omega_Y^{\mathscr{M}}, \Omega_X^{\mathscr{M}} \boxtimes \Omega_Y^{\mathscr{M}})$$

is the map denoted $\rho_{X\times Y}$ in [7, 5.4] (with X and Y interchanged) and in particular $\alpha_{1_X,1_Y}$ is an isomorphism by [7, 5.7]. Since the identification $q^!(1_Y(n)) \simeq (q^!1_Y)(n)$ identifies the map $\alpha_{1_X,1_Y(n)}$ with the map obtained from $\alpha_{1_X,1_Y}$ by tensoring with $1_{X\times Y}(n)$, it also follows that $\alpha_{1_X,1_Y(n)}$ is an isomorphism for any integer n.

From this we deduce that $\alpha_{A,1_Y}$ is an isomorphism for any $A \in \mathscr{M}(X)$. Indeed the collection of $A \in \mathscr{M}(X)$ for which $\alpha_{A,1_Y}$ is an isomorphism is a thick triangulated subcategory of $\mathscr{M}(X)$ and by the discussion in 2.2 the map $\alpha_{\pi_*1_{X'}(n),1_Y}$ is an isomorphism for all proper morphisms $\pi : X' \to X$ and all n. Using [3, 6.2.6], it follows that $\alpha_{A,1_Y}$ is an isomorphism for all $A \in \mathscr{M}(X)$, and also that $\alpha_{A,1_Y(n)}$ is an isomorphism for all $A \in \mathscr{M}(X)$ and $n \in \mathbb{Z}$ since the maps $\alpha_{A(-n),1_Y}$ are isomorphisms.

Now consider the collection of $B \in \mathscr{M}(Y)$ for which the map $\alpha_{A,B}$ is an isomorphism for all $A \in \mathscr{M}(X)$. Again this is a thick triangulated subcategory of $\mathscr{M}(Y)$ and by the discussion in 2.3, and the already known case of the $\alpha_{A,1_Y(n)}$'s, it contains all objects of the form $\gamma_*1_{Y'}(n)$ for $\gamma : Y' \to Y$ proper. Using [3, 6.2.6] once again it follows that $\alpha_{A,B}$ is an isomorphism for all A and B as desired. □

2.6. Let

$$c = (c_1, c_2) : C \to X \times X$$

be a correspondence, and let F denote $C \times_{c,X\times X,\Delta} X$ so we have a cartesian square

$$\begin{array}{ccc} F & \xrightarrow{\delta} & C \\ \downarrow c' & & \downarrow c \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

By [3, A.1.10 (5)] we have for $F \in \mathscr{M}(X)$

$$c^!\mathscr{R}Hom_{X\times X}(\mathrm{pr}_1^*F, \mathrm{pr}_2^!F) \simeq \mathscr{R}Hom_C(c_1^*F, c_2^!F).$$

Combining this with 2.4 we get a map

$$\mathrm{Hom}_{\mathscr{M}(C)}(c_1^*F, c_2^!F) \to c^!(D_X(F) \boxtimes F).$$

Composing with the map

$$c^!(D_X(F) \boxtimes F) \xrightarrow{\mathrm{id}\to\Delta_*\Delta^*} c^!\Delta_*(D_X(F) \otimes F) \xrightarrow{\text{evaluation}} c^!\Delta_*\Omega_X^{\mathscr{M}} \xrightarrow{\text{base change}} \delta_*\Omega_F^{\mathscr{M}}$$

we get a morphism

$$\mathrm{Tr} : \mathrm{Hom}_{\mathscr{M}(C)}(c_1^*F, c_2^!F) \to \mathrm{Hom}_{\mathscr{M}(F)}(1_F, \Omega_F^{\mathscr{M}}).$$

The image of a map $u : c_1^*F \to c_2^!F$ under this map is called the *characteristic class* of u.

Remark 2.7. If F is quasi-projective then it is shown in [7, 6.2] that there is a canonical isomorphism

$$A_0(F)_{\mathbb{Q}} \simeq \mathrm{Hom}_{\mathscr{M}(F)}(1_F, \Omega_F^{\mathscr{M}}),$$

where $A_0(F)_{\mathbb{Q}}$ denotes the Chow group of 0-cycles on F tensor $\mathbb{Q}$.

2.8. As in [7, 5.9] the formation of characteristic classes is compatible with morphisms of motivic categories. In particular for ℓ invertible in k we have the étale realization functor [3, 7.2.24]

$$R_\ell : \mathscr{M}(X) \to D_c^b(X, \mathbb{Q}_\ell).$$

If $F \in \mathscr{M}(X)$ is an object and $u : c_1^*F \to c_2^!F$ is a morphism in $\mathscr{M}(C)$ with realization

$$u_\ell : c_1^*R_\ell(F) \to c_2^!R_\ell(F)$$

in $D_c^b(C, \mathbb{Q}_\ell)$, then the corresponding characteristic class $\mathrm{Tr}(u_\ell) \in H^0(F, \Omega_F)$ defined as in [5, III, 4.1] is the image of $\mathrm{Tr}(u)$ under the realization map

$$\mathrm{Hom}_{\mathscr{M}(F)}(1_F, \Omega_F^{\mathscr{M}}) \to H^0(F, \Omega_F).$$

3. Proof of 1.3

3.1. By Nagata's theorem we can find a commutative diagram

$$\begin{array}{ccc} C & \overset{\tilde{j}}{\hookrightarrow} & \overline{C} \\ {\scriptstyle c_2}\downarrow & & \downarrow{\scriptstyle \bar{c}_2} \\ X & \overset{j}{\hookrightarrow} & \overline{X}, \end{array}$$

where $\tilde{j}$ and j are dense open imbeddings and $\overline{C}$ and $\overline{X}$ are proper over k. Since c_2 is proper by assumption this square is cartesian. Modifying $\overline{C}$ along $\overline{C} - C$, which does not change the property that the square is cartesian, we can further arrange that the map $c_1 : C \to X$ extends to a morphism $\bar{c}_1 : \overline{C} \to \overline{X}$.

Let $\overline{F} \in \mathscr{M}(\overline{X})$ denote j_*F. Then using the canonical base change isomorphism $\bar{c}_2^!j_* \simeq \tilde{j}_*c_2^!$ we have

$$\bar{c}_2^!\overline{F} \simeq \tilde{j}_*c_2^!F$$

so giving a morphism

$$\bar{c}_1^*\overline{F} \to \bar{c}_2^!\overline{F}$$

is equivalent to giving a morphism $c_1^*F \to c_2^!F$ and u extends uniquely to a morphism $\bar{u} : \bar{c}_1^*\overline{F} \to \bar{c}_2^!\overline{F}$.

Since the realization functors R_ℓ commute with the six operations we have

$$R_\ell(\overline{F}) = j_*R_\ell(F).$$

We therefore get an isomorphism

$$R\Gamma(\overline{X}, R_\ell(\overline{F})) \simeq R\Gamma(X, R_\ell(F))$$

and

$$\sum_i (-1)^i \mathrm{tr}(\bar{u}_\ell^* | H^i(\overline{X}, R_\ell(\overline{F}))) = \sum_i (-1)^i \mathrm{tr}(u_\ell^* | H^i(X, R_\ell(F))).$$

From this it follows that it suffices to prove 1.3 in the case when X and C are proper over k.

3.2. In this case the Grothendieck-Lefschetz trace formula [5, III, 4.7] gives

$$\sum_i (-1)^i \mathrm{tr}(\bar{u}_\ell^* | H^i(X, R_\ell(F))) = \sum_{Z \in \pi_0(F)} \int_Z \mathrm{Tr}(u_\ell),$$

where on the right the sum is over the connected components of the fixed point locus $F := C \times_{X \times X, \Delta_X} X$, $\mathrm{Tr}(u_\ell)$ is the characteristic class of u_ℓ defined in [5, III, 4.1] and

$$\int_Z : H^0(Z, \Omega_Z) \to \overline{\mathbb{Q}}_\ell$$

is the pushforward map induced by adjunction. By 2.8 there is a class $\mathrm{Tr}(u) \in \mathrm{Hom}_{\mathscr{M}(F)}(1_F, \Omega_F^{\mathscr{M}})$ such that for all ℓ we have $\mathrm{Tr}(u_\ell) = R_\ell(\mathrm{Tr}(u))$. In particular we find that $\int_Z(\mathrm{Tr}(u_\ell))$ is equal to the corresponding pushforward of $\mathrm{Tr}(u)$ to $\mathrm{Ext}^0_{\mathscr{M}(k)}(1_k, 1_k) = \mathbb{Q}$. This proves 1.3 □

References

[1] M. Bondarko, *Differential graded motives: weight complex, weight filtrations and spectral sequences for realizations; Voevodsky vs. Hanamura*, preprint available on arXiv:math/0601713 (2008).

[2] D.-C. Cisinski and F. Déglise, *Triangulated categories of mixed motives*, preprint (2012).

[3] D.-C. Cisinski and F. Déglise, *Étale Motives*, to appear in Comp. Math.

[4] K. Fujiwara, *Rigid geometry, Lefschetz-Verdier trace formula, and Deligne's conjecture*, Inv. Math. **127**, 489–533 (1997).

[5] A. Grothendieck, *Cohomologie l-adic et Fonctions L*, Lectures Notes in Math **589** (1977).

[6] L. Illusie, *Miscellany on traces in ℓ-adic cohomology: a survey*, Jpn. J. Math **1** (2006), 107–136.

[7] M. Olsson, *Motivic cohomology, localized chern classes, and local terms*, Manuscripta Math., DOI:10.1007/s00229-015-0765-3.