INDEPENDENCE OF ℓ AND SURFACES

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Abstract. For a surface X_0 over a finite field $\mathbb{F}_q$ with base extension X to an algebraic closure $\overline{\mathbb{F}}_q$, we show that the characteristic polynomial of Frobenius acting on the various cohomology groups $H^i(X, \mathbb{Q}_\ell)$, $H^i_c(X, \mathbb{Q}_\ell)$, and $IH^i(X, \mathbb{Q}_\ell)$ (cohomology, compactly supported cohomology, and intersection cohomology respectively) have rational coefficients independent of ℓ. An application to Brauer groups of surfaces is also discussed.

1. Introduction

1.1. Let X_0 be a separated finite type $\mathbb{F}_q$-scheme, with $p = \text{char } \mathbb{F}_q$. Fix an algebraic closure k of $\mathbb{F}_q$, and define $X := X_0 \times_{\mathbb{F}_q} k$. Then for each prime $\ell \neq p$, we have ℓ-adic cohomology groups $H^*(X, \mathbb{Q}_\ell)$, compactly supported cohomology groups $H^*_c(X, \mathbb{Q}_\ell)$, and intersection cohomology groups $IH^*(X, \mathbb{Q}_\ell)$. Let $F : X \to X$ be the geometric Frobenius morphism. Then F acts on each of the preceding cohomology groups, and we define

$$P^i_\ell(t) := \det(1 - tF|H^i(X, \mathbb{Q}_\ell)),$$
$$P^i_{\ell,c}(t) := \det(1 - tF|H^i_c(X, \mathbb{Q}_\ell)), \text{ and}$$
$$IP^i_\ell(t) := \det(1 - tF|IH^i(X, \mathbb{Q}_\ell)).$$

These polynomials are a priori in $\mathbb{Q}_\ell[t]$. A well-known open problem is to show that each polynomial has rational coefficients independent of ℓ, i.e, $P^i_\ell(t) = P^i_{\ell'}(t)$ for primes $\ell \neq \ell'$, and similarly for $P^i_{\ell,c}(t)$ and $IP^i_\ell(t)$. In case X is smooth and proper, this follows from Deligne's work on weights [Del80]. A theorem of Gabber [Fuj, Thm 3] shows that when X is proper and equidimensional, $IP^i_\ell(t)$ has rational coefficients independent of ℓ. But the case of general varieties is still open. For a 1-dimensional X, this follows easily from the Grothendieck trace formula. It seems that no proof has appeared in the literature for the case of a 2-dimensional X, although this case may be known to experts. One of our main goals is to supply a proof of this fact:

Theorem 1.2. *Let X be a separated finite type $\mathbb{F}_q$-scheme of dimension ≤ 2, and define the polynomials $P^i_\ell(t)$, $P^i_{\ell,c}(t)$ and $IP^i_\ell(t)$ as above. Then each polynomial has rational coefficients that do not depend on ℓ.*

Remark 1.3. In fact, it follows from Theorem 1.2 that $P^i_\ell(t)$, $P^i_{\ell,c}(t)$ and $IP^i_\ell(t)$ have integer coefficients (independent of ℓ). This follows from Deligne's integrality theorem: Recall that for a separated finite type $\mathbb{F}_q$-scheme X_0 and a complex $K \in D^b_c(X_0, \mathbb{Q}_\ell)$, we say that K is integral if for each closed point $x \in |X_0|$ and each i, the eigenvalues of geometric Frobenius acting on $(\mathscr{H}^i(K))_{\overline{x}}$ are algebraic integers, where $\overline{x}$ is a geometric point lying over x. Then Deligne's integrality theorem, as stated in [Ill, Thm 4.2], is as follows:

Theorem 1.4. *(Deligne) Let $f : X_0 \to Y_0$ be a morphism between separated finite type $\mathbb{F}_q$-schemes, and let $K \in D^b_c(X_0, \mathbb{Q}_\ell)$ be an integral complex. Then $Rf_!K$ and Rf_*K are integral as well.*

Applying this to the structure morphism $p : X_0 \to \mathbb{F}_q$ shows that $P^i_\ell(t)$ and $P^i_{\ell,c}(t)$ have integer coefficients. One deduces the same statement for intersection cohomology since the above theorem combined with [BBD, 2.1.11] shows that the intersection complex IC_{X_0} is integral.

1.5. This paper is organized as follows. Section 2 consists of some elementary lemmas and notation. Section 3 proves the following result, which is presumably well-known to experts:

Proposition 1.6. *Let G be a connected, commutative group scheme of finite type over an algebraically closed field k.*

(i) For any $\ell \neq \mathrm{char}\ k$, the Tate module $T_\ell G := \varprojlim_n G[\ell^n]$ is a free $\mathbb{Z}_\ell$-module, where $G[\ell^n]$ is the ℓ^n-torsion points of $G(k)$.

(ii) For any group scheme homomorphism $\varphi : G \to G$ the reversed characteristic polynomial $P_\ell(\varphi, t) := \det(1 - t\varphi|T_\ell G)$ has integer coefficients independent of ℓ.

In Section 4 we relate $H^1_c(X, \mathbb{Q}_\ell)$ to the Picard scheme of X when X is proper, or to the Picard scheme of a compactification of X if X is not proper. This combined with the above result prove independence of ℓ for $P^1_{\ell,c}(t)$. We also develop some Kummer theory related to cycle class maps which will be used in Sections 5 and 7.

In Section 5 we prove independence of ℓ for smooth 2-dimensional schemes of finite type over $\mathbb{F}_q$. In fact, we prove something slightly more general which we will use in our study of independence of ℓ for intersection cohomology: Let $\overline{X}_0/\mathbb{F}_q$ be a smooth proper 2-dimensional $\mathbb{F}_q$-scheme, and suppose given a divisor with simple normal crossings

$$Z_0 \hookrightarrow \overline{X}_0$$

and a decomposition

$$Z_0 = D_0 \cup E_0,$$

where D_0 and E_0 are also divisors with simple normal crossings. Let X_0 denote $\overline{X}_0 - E_0$. We then get inclusions

$$X_0 - (D_0 \cap X_0) \xhookrightarrow{j_0} X_0 \xhookrightarrow{s_0} \overline{X}_0.$$

Let the absence of the subscript 0 denote base change to $k = \overline{\mathbb{F}}_q$. Then define

$$H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell) := H^i(X, j_!\mathbb{Q}_{\ell, X-(D\cap X)}).$$

We can then define the polynomial

$$P^i_{\ell,D,E}(t) := \det(1 - tF|H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell)).$$

Section 5 shows that this polynomial has rational coefficients independent of ℓ.

In Section 6 we prove independence of ℓ for $P^i_{\ell,c}(t)$ and in Sections 7 and 8 for $P^i_\ell(t)$. The methods here are very analogous to those of [BVS], where they construct 1-motives realizing the cohomology groups $H^1(X, \mathbb{Z})$ and $H^{2d-1}(X, \mathbb{Z})$ for X a separated finite type scheme over an algebraically closed field of characteristic zero. The characteristic zero assumption is mostly unnecessary for our purposes, and their methods of proof go through.

Finally, section 9 completes the proof of Theorem 1.2 by proving independence of ℓ for the intersection cohomology polynomials $IP^i_\ell(t)$.

1.7. In Section 10 we apply Theorem 1.2 to study the Brauer group of a smooth surface, and in particular we prove the following theorem:

Theorem 1.8. *Let k be a separably closed field of characteristic exponent p, and let X/k be a smooth 2-dimensional k-scheme. Let $\widetilde{\mathrm{Br}}(X)$ denote the quotient of the Brauer group of X by its p-torsion, and let*

$$\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \subset \widetilde{\mathrm{Br}}(X)$$

be the subgroup of divisible elements. Then the quotient

$$\widetilde{\mathrm{Br}}(X)/\widetilde{\mathrm{Br}}(X)_{\mathrm{div}}$$

is a finite group, and there exists an integer r such that

$$\widetilde{\mathrm{Br}}(X)_{\mathrm{div}} \simeq F_p^{\oplus r},$$

where F_p denotes the quotient of $\mathbb{Q}/\mathbb{Z}$ by its p-torsion subgroup.

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1.10. **Convention.** Given a morphism $f : X \to Y$, then unless otherwise noted, by f_* we will mean the map in the derived category $Rf_* : D^b_c(X, \mathbb{Q}_\ell) \to D^b_c(Y, \mathbb{Q}_\ell)$. If we wish to refer to the pushforward on sheaves we will write $R^0 f_*$. Similar remarks apply to f^*, $f_!$, $f^!$, etc.

2. Some Preliminary Lemmas and Terminology

Recall the following definition which goes back to [Ser, p. I-11]:

Definition 2.1. Let S be a non-empty set of prime numbers. An *S-system of endomorphisms of $\mathbb{Q}_\ell$-vector spaces* (or simply an *S-system*) is a family $(V_\ell, F_\ell : V_\ell \to V_\ell)_{\ell \in S}$, where for each $\ell \in S$, V_ℓ is a $\mathbb{Q}_\ell$-vector space and F_ℓ is a $\mathbb{Q}_\ell$-linear endomorphism of V_ℓ. For each V_ℓ we define $P_\ell(t) := \det(1 - tF_\ell)$. The S-system (V_ℓ, F_ℓ) is said to be *compatible* if for all $\ell \in S$, $P_\ell(t)$ has rational coefficients and for $\ell \neq \ell'$, $P_\ell(t) = P_{\ell'}(t)$.

Example 2.2. Let X_0 be a separated finite-type $\mathbb{F}_q$-scheme of dimension d, and let $p = \mathrm{char}\, \mathbb{F}_q$. Let S be the set of all primes except p. Then for each i between 0 and $2d$, there are S-systems

$$(H^i(X, \mathbb{Q}_\ell), F|H^i(X, \mathbb{Q}_\ell))_{\ell \in S},$$
$$(H^i_c(X, \mathbb{Q}_\ell), F|H^i_c(X, \mathbb{Q}_\ell))_{\ell \in S}, \text{ and}$$
$$(IH^i(X, \mathbb{Q}_\ell), F|IH^i(X, \mathbb{Q}_\ell))_{\ell \in S}.$$

Here F is the geometric Frobenius, and in all cases the endomorphisms are those induced by F. Theorem 1.2 says that each of these systems is compatible in case $d \leq 2$.

Definition 2.3. Let (V_ℓ, F_ℓ) and (V'_ℓ, F'_ℓ) be S-systems. A *morphism* $g : (V_\ell, F_\ell) \to (V'_\ell, F'_\ell)$ *of S-systems* is a family $g = (g_\ell)_{\ell \in S}$ of linear transformations, where $g_\ell : V_\ell \to V'_\ell$ is a linear transformation such that $F'_\ell \circ g_\ell = g_\ell \circ F_\ell$.

With this definition, the category of all S-systems becomes an abelian category, with kernels and cokernels defined term-by-term.

Note that the category of *compatible* S-systems does not form an abelian subcategory of the category of S-systems: the kernel and cokernel of a morphism $g : (V_\ell, F_\ell) \to (V'_\ell, F'_\ell)$ of compatible S-systems need not be compatible. However, there are some partial results in this direction that we will find useful.

Lemma 2.4. *Let $0 \to A^0 \to A^1 \to ... \to A^n \to 0$ be an exact sequence of vector spaces, and let $F = (F^i : A^i \to A^i)$ be a collection of endomorphisms such that the diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^0} & & \downarrow{\scriptstyle F^1} & & & & \downarrow{\scriptstyle F^n} & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & ... & \longrightarrow & A^n & \longrightarrow & 0 \end{array}$$

commutes. Then $1 = \prod_i P^i(t)^{(-1)^i}$, where $P^i(t)$ denotes $\det(1 - tF^i)$.

Proof. We show this by induction on n, the base case being $n = 1$ which is immediate. For the case $n = 2$, note that in this case there exists a basis for A^1 with respect to which F^1 takes the form

$$F^1 = \begin{bmatrix} F^0 & * \\ 0 & F^2 \end{bmatrix},$$

which implies that $P^1(t) = P^0(t)P^2(t)$ as desired. For the inductive step, let $A^{n-1,\prime}$ denote $\mathrm{Ker}(A^{n-1} \to A^n)$, and let $F^{n-1,\prime} : A^{n-1,\prime} \to A^{n-1,\prime}$ denote the map induced by F^{n-1}. We then have commutative diagrams with exact rows

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1,\prime} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^0} & & \downarrow{\scriptstyle F^1} & & & & \downarrow{\scriptstyle F^2} & & \downarrow{\scriptstyle F^{n-1,\prime}} & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & \cdots & \longrightarrow & A^{n-2} & \longrightarrow & A^{n-1,\prime} & \longrightarrow & 0, \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^{n-1,\prime} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F^{n-1,\prime}} & & \downarrow{\scriptstyle F^{n-1}} & & \downarrow{\scriptstyle F^n} & & \\ 0 & \longrightarrow & A^{n-1,\prime} & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & 0. \end{array}$$

By induction and the case $n = 2$ we therefore have

$$1 = \det(1 - tF^{n-1,\prime})^{(-1)^{n-1}} \cdot \prod_{i=1}^{n-2} P^i(t)^{(-1)^i}, \quad \det(1 - tF^{n-1,\prime}) = P^{n-1}(t)P^n(t)^{-1},$$

which implies that $1 = \prod_{i=1}^n P^i(t)^{(-1)^i}$. ☐

Lemma 2.5. *Let S be a set of primes, and let $0 \to V^0 \to V^1 \to ... \to V^n \to 0$ be an exact sequence of S-systems, where $V^i = (V^i_\ell, F^i_\ell)_{\ell \in S}$. Let $j \in [0, n]$ be an integer, and suppose that for $i \neq j$ the system V^i is compatible. Then V^j is a compatible system as well.*

Proof. Write $P^i_\ell(t) = \det(1 - tF^i_\ell)$. By the previous lemma, we have

$$1 = \prod_i P^i_\ell(t)^{(-1)^i}.$$

Therefore we can express $P^j_\ell(t)$ in terms of the $P^i_\ell(t)$ for $i \neq j$. Since by assumption $P^i_\ell(t)$ has rational coefficients and $P^i_\ell(t) = P^i_{\ell'}(t)$ for $\ell \neq \ell'$, the same is true for $P^j_\ell(t)$. □

The following lemma will be useful for analyzing the terms in the long exact sequence in cohomology resulting from an exact triangle in $D^b_c(X, \mathbb{Q}_\ell)$:

Lemma 2.6. *Let S be a set of primes, and suppose that*

$$0 \to ... \to C^{i-1} \overset{h^{i-1}}{\to} A^i \overset{f^i}{\to} B^i \overset{g^i}{\to} C^i \overset{h^i}{\to} A^{i+1} \to ...$$

is a bounded-below long exact sequence of S-systems (so A^i is actually a system $(A^i_\ell)_{\ell \in S}$, etc.). Suppose that for each i, the systems A^i and B^i are compatible, and suppose in addition that one of S-systems

(2.6.1) $$C^i,$$

(2.6.2) $$\ker(f^i : A^i \to B^i), \text{ or}$$

(2.6.3) $$\operatorname{coker}(f^i : A^i \to B^i)$$

is compatible for each i (i.e., either C^i is compatible for all i, $\ker(f^i)$ is compatible for all i, or $\operatorname{coker}(f^i)$ is compatible for all i). Then all three S-systems (2.6.1), (2.6.2), (2.6.3) are compatible for all i.

Proof. First suppose C^i is compatible for each i. Notice that for all i we have the following exact sequences:

(2.6.4) $$0 \to \text{coker } f^i \to C^i \to \ker f^{i+1} \to 0,$$

(2.6.5) $$0 \to \ker f^i \to A^i \to B^i \to \text{coker } f^i \to 0.$$

Since we are working with bounded below complexes, there exists some k such that for $i \leq k$,

$$C^i = \ker f^i = \text{coker} f^i = 0.$$

Then ker f^{k+1} is compatible by sequence 2.6.4. Since A^{k+1} and B^{k+1} are compatible, sequence 2.6.5 shows that coker f^{k+1} is compatible as well. Then compatibility of C^{k+1} and sequence 2.6.4 show that ker f^{k+2} is compatible, and we proceed by induction.

Next suppose coker f^i is compatible for each i. Then sequence 2.6.5 and Lemma 2.5 show that ker f^i is compatible for each i. Then appying Lemma 2.5 to sequence 2.6.4 shows that C^i is compatible for each i. A similar argument shows that when ker f^i is compatible for each i, so are coker f^i and C^i. □

Next we establish some notation and properties of Tate modules.

Definition 2.7. Let A be an abelian group. We denote by $A[n]$ the kernel of $\cdot n : A \to A$, for any $n > 0$. For a prime ℓ let $T_\ell A$ denote the $\mathbb{Z}_\ell$-module $\varprojlim_n A[\ell^n]$, and set $V_\ell A := T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Notice that the functors $A \mapsto A[n]$, $A \mapsto T_\ell A$, and $A \mapsto V_\ell A$ are all left exact functors, but not exact in general. However, we have the following partial result:

Lemma 2.8. *Let A, B, and C be commutative group schemes of finite type over an algebraically closed field k, and suppose we have an exact sequence*

$$0 \to A(k) \to B(k) \to C(k) \to 0.$$

Let ℓ be a prime distinct from $p = \operatorname{char} k$. Then if A is connected, we have exact sequences

$$0 \to A[\ell^n] \to B[\ell^n] \to C[\ell^n] \to 0 \quad \text{and}$$
$$0 \to T_\ell A \to T_\ell B \to T_\ell C \to 0.$$

Whether A is connected or not, we have an exact sequence

$$0 \to V_\ell A \to V_\ell B \to V_\ell C \to 0.$$

Proof. First off, notice that $A[\ell^n] = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/\ell^n\mathbb{Z}, A)$. Since $\operatorname{Ext}^1(\mathbb{Z}/\ell^n\mathbb{Z}, A) = A/\ell^n A$, we have an exact sequence

$$0 \to A[\ell^n] \to B[\ell^n] \to C[\ell^n] \to A/\ell^n A.$$

Consider the group scheme homomorphism $\cdot \ell^n : A \to A$. This map is étale (since it acts as multiplication by ℓ^n on the Lie algebra), and so is an open mapping. But since it is a (quasi-compact) group scheme homomorphism, its image is also closed [SGA3, Exp.VI$_B$, 1.2], so its image must include the connected component of the identity A^0 of A. Therefore if A is connected, we have an exact sequence

$$0 \to A[\ell^n] \to B[\ell^n] \to C[\ell^n] \to 0.$$

Since $A[\ell^n]$ is finite for each n, the projective system $(A[\ell^n])_n$ satisfies the Mittag-Leffler condition, implying that we get an exact sequence

$$0 \to T_\ell A \to T_\ell B \to T_\ell C \to 0.$$

Now we drop the assumption that A is connected. We still have that the image of $\cdot \ell^n : A \to A$ contains the connected component of the identity A^0 of A, so $A/\ell^n A$ is of order at most equal to the number of connected components of A (and in particular is bounded independently of n). Therefore $\varprojlim_n A/\ell^n A$ is a torsion $\mathbb{Z}_\ell$-module. On taking inverse limits of the exact sequences

$$0 \to A[\ell^n] \to B[\ell^n] \to C[\ell^n] \to A/\ell^n A$$

(and using the fact that $\varprojlim$ is an exact functor when restricted to inverse systems of finite abelian groups), we get an exact sequence

$$0 \to T_\ell A \to T_\ell B \to T_\ell C \to \varprojlim_n A/\ell^n A.$$

Since the group on the right is a torsion $\mathbb{Z}_\ell$-module, on tensoring with $\mathbb{Q}_\ell$ we get an exact sequence

$$0 \to V_\ell A \to V_\ell B \to V_\ell C \to 0$$

as desired. □

3. Tate modules of commutative group schemes

Let G be a locally finite type commutative group scheme over an algebraically closed field k. Then $G(k)$ is in particular an abelian group, and for any prime ℓ we can consider the Tate module $T_\ell G(k)$, usually simply written $T_\ell G$. Let $\varphi : G \to G$ be a group homomorphism. Then φ must send ℓ^n-torsion points to ℓ^n-torsion points, and so there is an induced homomorphism $\varphi_\ell : T_\ell G \to T_\ell G$.

Proposition 3.1. *Let G be a locally finite type group scheme over k as above, and suppose that $(G/G^0)(k)$ is a finitely generated abelian group, where G_0 is the connected component of the identity. Let $\varphi : G \to G$ be as above and let ℓ be a prime distinct from the characteristic p of k.*

(i) The $\mathbb{Z}_\ell$-module $T_\ell G$ is a free of finite rank, and so in particular we can consider the characteristic polynomial $P_\ell(\varphi) := \det(1 - t\varphi_\ell)$, where $\varphi_\ell : T_\ell G \to T_\ell G$ is the action induced by φ.

(ii) The polynomial $P_\ell(\varphi)$ has integer coefficients, and for any other $\ell' \neq p$, $P_\ell(\varphi) = P_{\ell'}(\varphi)$.

Proof. By the following lemma, it suffices to consider the case where G is connected.

Lemma 3.2. *Let G be a commutative group scheme over an algebraically closed field k, and let G^0 be the connected component of the identity. Suppose that G^0 is of finite type over k and the k-points of G/G^0 form a finitely generated abelian group. Then $T_\ell G \cong T_\ell G^0$.*

Proof. Since k is algebraically closed, we have an exact sequence

$$0 \to G^0(k) \to G(k) \to (G/G^0)(k) \to 0,$$

so it suffices to show that $T_\ell(G/G^0) = 0$. This follows since a finitely generated abelian group has no ℓ-divisible points (i.e., if A is a finitely generated abelian group, then $\cap_n \ell^n A = 0$). □

So we may assume G is connected, in which case it is also of finite type [Kle, 9.5.1]. Furthermore, we may assume G is reduced, since the proposition depends only on the k-points of G. Now in the case when G is reduced and connected, there is by Chevalley's theorem [Con02, Thm 1.1] a connected affine algebraic subgroup H of G and an exact sequence

$$1 \to H \to G \to A \to 1,$$

where A is an abelian variety. Note that the endomorphism φ must send H to itself: this is because H (being affine) does not admit a non-constant map into an abelian variety; hence the composition $H \overset{\varphi|_H}{\to} G \to A$ is the zero map. Thus φ acts on the whole exact sequence, so we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 1 \\ & & \varphi\downarrow & & \varphi\downarrow & & \varphi\downarrow & & \\ 1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 1. \end{array}$$

By (2.8), we have an exact sequence $0 \to T_\ell H \to T_\ell G \to T_\ell A \to 0$. Therefore if the proposition is true for H and for A, $T_\ell G$ will be an extension of free $\mathbb{Z}_\ell$-modules and hence free, and the action of φ_ℓ on $T_\ell G$ will have characteristic polynomial equal to the product of those on $T_\ell H$ and $T_\ell A$ (2.4). Thus it suffices to consider the two cases of G an abelian variety, and G an affine connected reduced group scheme.

The case of an abelian variety can be found for example in [Dem, p. 96].

In the second case, G is an extension

$$1 \to D \to G \to U \to 1,$$

where U is a unipotent group and D is a diagonalizable group (in fact G is the product of U and D, by taking semi-simple and unipotent parts of any element of G). Moreover, by [Spr, 2.4.8(ii)], the homomorphism φ fixes D and U. Thus φ acts on the whole exact sequence. We claim that $T_\ell U = 0$. It suffices to show that U has no ℓ-torsion elements. If char $k = 0$ then U has no torsion at all (any torsion element of $GL(n, k)$ in characteristic 0 is diagonalizable). If char $k = p > 0$, then every element $u \in U(k)$ satisfies $u^{p^m} = 1$ for some m. Then u cannot also be ℓ-torsion unless $u = 1$, since ℓ is relatively prime to p^m.

Thus $T_\ell G = T_\ell D$, and we can assume G is diagonalizable. Then, since G is connected, G is isomorphic [Spr, 2.5.8] to a torus $(\mathbb{G}_m)^n$ for some n. The map φ acts as an element $\varphi' \in \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m^n) \cong M_n(\mathbb{Z})$. More precisely, if $x_1, ..., x_n$ are the generators in $k[\mathbb{G}_m^n] \cong k[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}]$, then there exists a matrix $\varphi' = (f_{ij})_{i,j=1}^n \in M_n(\mathbb{Z})$ such that x_i maps to $x_1^{f_{i1}} x_2^{f_{i2}} ... x_n^{f_{in}}$. It is then clear that φ acts on the Tate module $T_\ell(\mathbb{G}_m)^n \cong (\mathbb{Z}_\ell)^n$ via the same matrix φ', for each prime $\ell \neq p$. Thus the characteristic polynomial of φ acting on G has integer coefficients independent of ℓ. □

4. Kummer Theory

4.1. Let k be an algebraically closed field, and let $\overline{X}$ be a proper separated finite type k-scheme. Recall that the Picard group Pic $\overline{X}$ is the k-points of a locally finite type commutative group scheme **Pic** $\overline{X}$ [SGA6, Exp. XII, 1.5], which has a connected component of the identity $\mathbf{Pic}^0$ $\overline{X}$ which is of finite type over k. We let $\mathrm{Pic}^0\ \overline{X}$ denote the k-points of $\mathbf{Pic}^0$ $\overline{X}$. Then we have an extension

$$0 \to \mathrm{Pic}^0\ \overline{X} \to \mathrm{Pic}\ \overline{X} \to NS\ \overline{X} \to 0,$$

where $NS\ \overline{X}$ (the *Neron-Severi* group of $\overline{X}$) is a finitely generated abelian group [SGA6, Exp. XIII, Thm 5.1].

4.2. Now let D be a closed subscheme of $\overline{X}$. Define

$$\begin{aligned}
\mathrm{Pic}(\overline{X}, D) &:= \ker(\mathrm{restriction} : \mathrm{Pic}\ \overline{X} \to \mathrm{Pic}\ D),\\
\mathrm{Pic}^0(\overline{X}, D) &:= \mathrm{Pic}(\overline{X}, D) \cap \mathrm{Pic}^0\ \overline{X}, \quad \text{and}\\
NS(\overline{X}, D) &:= \mathrm{Pic}(\overline{X}, D)/\mathrm{Pic}^0(\overline{X}, D).
\end{aligned}$$

We then have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Pic}^0(\overline{X}, D) & \to & \mathrm{Pic}(\overline{X}, D) & \to & NS(\overline{X}, D) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Pic}^0\ \overline{X} & \to & \mathrm{Pic}\ \overline{X} & \to & NS\ \overline{X} & \to & 0,
\end{array}$$

where the vertical maps are inclusions. In particular $NS(\overline{X}, D)$ is finitely generated. Observe also that since

$$\mathrm{Pic}^0(\overline{X}, D) = \mathrm{Ker}(\mathrm{Pic}^0\ \overline{X} \to \mathrm{Pic}^0\ D),$$

the group $\mathrm{Pic}^0(\overline{X}, D)$ is the k-points of a finite type k-group scheme $\mathbf{Pic}^0(\overline{X}, D)$.

Remark 4.3. This definition of $\mathrm{Pic}(\overline{X}, D)$ is closely related to, but different from, the more standard definition of the relative Picard group given in [MVW, 7.10]. If we denote the relative Picard group given there by $\mathrm{Pic}'(\overline{X}, D)$, we have an exact sequence

$$\mathcal{O}^*(\overline{X}) \to \mathcal{O}^*(D) \to \mathrm{Pic}'(\overline{X}, D) \to \mathrm{Pic}(\overline{X}, D) \to 0.$$

4.4. The group $\mathrm{Pic}(\overline{X}, D)$ is related to étale cohomology as follows. Fix a prime ℓ invertible in k, and consider the Kummer sequences (on $\overline{X}$ and D respectively)

$$0 \longrightarrow \mu_{\ell^n,\overline{X}} \longrightarrow \mathbb{G}_{m,\overline{X}} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,\overline{X}} \longrightarrow 0,$$

and

$$0 \longrightarrow \mu_{\ell^n,D} \longrightarrow \mathbb{G}_{m,D} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m,D} \longrightarrow 0.$$

Taking cohomology we get isomorphisms

$$H^1(\overline{X}, \mu_{\ell^n}) \simeq \mathrm{Pic}(\overline{X})[\ell^n],$$

and

$$H^1(D, \mu_{\ell^n}) \simeq \mathrm{Pic}(D)[\ell^n].$$

Passing to the inverse limit over n we get isomorphisms

$$H^1(\overline{X}, \mathbb{Z}_\ell(1)) \simeq T_\ell(\mathrm{Pic}(\overline{X})), \quad H^1(D, \mathbb{Z}_\ell(1)) \simeq T_\ell(\mathrm{Pic}(D)).$$

Using the exact sequence

$$0 \to \mathrm{Pic}(\overline{X}, D) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(D)$$

and the left exactness of the functor $T_\ell(-)$, we obtain a canonical isomorphism between $T_\ell(\mathrm{Pic}(\overline{X}, D))$ and the kernel of the restriction map

$$H^1(\overline{X}, \mathbb{Z}_\ell(1)) \to H^1(D, \mathbb{Z}_\ell(1)).$$

Summarizing:

Corollary 4.5. *There is a natural isomorphism*

$$T_\ell(\mathrm{Pic}(\overline{X}, D)) \simeq \mathrm{Ker}(H^1(\overline{X}, \mathbb{Z}_\ell(1)) \to H^1(D, \mathbb{Z}_\ell(1))).$$

The following are immediate consequences of these considerations:

Proposition 4.6. *Let $\overline{X}$ be a proper separated finite type scheme over an algebraically closed field k, and let $\varphi : \overline{X} \to \overline{X}$ be an endomorphism. Then for any prime ℓ not equal to* char k, *$H^1(\overline{X}, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-module, and the characteristic polynomial $P^1_\ell(\varphi, t) = \det(1 - t\varphi | H^1(\overline{X}, \mathbb{Z}_\ell))$ has coefficients in $\mathbb{Z}$ that are independent of ℓ (in particular $(H^1(\overline{X}, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system with respect to the geometric Frobenius endomorphism).*

Proof. We have that $H^1(\overline{X}, \mathbb{Z}_\ell(1))$ is isomorphic to the Tate module of $\mathbf{Pic}^0\,\overline{X}$, and the action of φ on $H^1(\overline{X}, \mathbb{Z}_\ell(1))$ is induced by a group homomorphism $\mathbf{Pic}^0\,\overline{X} \to \mathbf{Pic}^0\,\overline{X}$. According to (1.6), the characteristic polynomial of this action has coefficients in $\mathbb{Z}$ that are independent of ℓ. □

Proposition 4.7. *Let X_0 be a separated finite type $\mathbb{F}_q$-scheme with* char $\mathbb{F}_q = p$*, and let $X = X_0 \times_{\mathbb{F}_q} k$. Then the system $(H^1_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system (with respect to the induced actions of Frobenius).*

Proof. Fix a compactification $j : X_0 \hookrightarrow \overline{X_0}$ with $\overline{X_0}$ proper over $\mathbb{F}_q$, and let $i : D_0 \hookrightarrow \overline{X_0}$ be the closed complement. Consider the exact sequence

$$0 \to j_!\mathbb{Q}_{\ell,X} \to \mathbb{Q}_{\ell,\overline{X}} \to i_*\mathbb{Q}_{\ell,D} \to 0.$$

Taking cohomology, we get an exact sequence

$$0 \to C_\ell \to H^1_c(X, \mathbb{Q}_\ell) \to \mathrm{Ker}(H^1(\overline{X}, \mathbb{Q}_\ell) \to H^1(D, \mathbb{Q}_\ell)) \to 0,$$

where C_ℓ denotes

$$\mathrm{Coker}(\mathbb{Q}_\ell^{\pi_0(\overline{X})} \to \mathbb{Q}_\ell^{\pi_0(D)}),$$

where the map is that of pulling back connected components via the inclusion $D \hookrightarrow \overline{X}$. The characteristic polynomial of Frobenius on C_ℓ has integer coefficients independent of ℓ, since the dimension of C_ℓ is clearly independent of ℓ, and the action of F on $\mathbb{Q}_\ell^{\pi_0(D)}$ is just to pull back connected components of D via F. To prove the proposition it therefore suffices to show that the characteristic polynomial of Frobenius acting on

$$\mathrm{Ker}(H^1(\overline{X}, \mathbb{Q}_\ell) \to H^1(D, \mathbb{Q}_\ell))$$

is in $\mathbb{Q}[T]$ and is independent of ℓ. This is clear for by (4.5) this kernel is isomorphic to $V_\ell\mathrm{Pic}^0(\overline{X}, D)(-1)$ which (after twisting) is the Tate module of $\mathbf{Pic}^0(\overline{X}, D)$. □

4.8. Now let X be a smooth, possibly non-proper variety over an algebraically closed field k, and suppose that X is an open dense subscheme of a proper smooth variety $\overline{X}$ with closed complement E. Suppose that D is a closed subscheme of $\overline{X}$ contained in X, so that $D \cap E = \emptyset$. We can then consider a relative Picard group for the non-proper scheme X and its closed subscheme D. Define

$$\mathrm{Pic}(X, D) := \mathrm{Ker}(\mathrm{Pic}(X) \to \mathrm{Pic}(D)),$$

let $\mathrm{Pic}^0(X, D) \subset \mathrm{Pic}(X, D)$ be the image of $\mathrm{Pic}^0(\overline{X}, D)$ under the restriction map

$$\mathrm{Pic}(\overline{X}, D) \to \mathrm{Pic}(X, D),$$

and set

$$NS(X, D) := \mathrm{Pic}(X, D)/\mathrm{Pic}^0(X, D).$$

Notice that since $D \subset X$, the group $\mathrm{Pic}(\overline{X}, D)$ is equal to the preimage of $\mathrm{Pic}(X, D)$ under the surjective (since $\overline{X}$ is smooth) map

$$\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X).$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}, D) & \longrightarrow & \mathrm{Pic}(\overline{X}, D) & \longrightarrow & NS(\overline{X}, D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X, D) & \longrightarrow & \mathrm{Pic}(X, D) & \longrightarrow & NS(X, D) & \longrightarrow & 0, \end{array}$$

where the vertical maps are surjective. In particular $NS(X, D)$ is finitely generated, and $\mathrm{Pic}^0(X, D)$ is an extension of a finite group by a divisible group.

4.9. Let

$$j : \overline{X} - D \hookrightarrow \overline{X}, \quad i : D \hookrightarrow \overline{X}$$

be the inclusions. Then for every $n > 0$ and prime ℓ invertible in k we have a commutative diagram of sheaves on $\overline{X}_{\mathrm{et}}$

(4.9.1)
$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & j_!\mu_{\ell^n} & \longrightarrow & \mu_{\ell^n} & \longrightarrow & i_*\mu_{\ell^n} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathbb{G}_m & \longrightarrow & i_*\mathbb{G}_{m,D} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \\ 0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathbb{G}_m & \longrightarrow & i_*\mathbb{G}_{m,D} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0, & & \end{array}$$

where the rows and columns are exact sequences and $\mathscr{H}$ is defined to be the kernel of the map

$$\mathbb{G}_m \to i_*\mathbb{G}_{m,D}.$$

Taking cohomology of this diagram restricted to X, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & H^1(X, j_!\mu_{\ell^n}) & \longrightarrow & H^1(X, \mu_{\ell^n}) & \longrightarrow & H^1(D, \mu_{\ell^n}) \\ & & \downarrow & & \downarrow & & \downarrow \\ (k^*)^{\pi_0(D)} & \longrightarrow & H^1(X, \mathscr{H}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(D) \\ \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} & & \downarrow{\scriptstyle \cdot\ell^n} \\ (k^*)^{\pi_0(D)} & \longrightarrow & H^1(X, \mathscr{H}) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(D) \\ & & \downarrow & & & & \\ & & H^2(X, j_!\mu_{\ell^n}). & & & & \end{array}$$

Since

$$\cdot\ell^n : (k^*)^{\pi_0(D)} \to (k^*)^{\pi_0(D)}$$

is surjective, we get an isomorphism

$$\text{Coker}(\cdot\ell^n : H^1(X, \mathscr{H}) \to H^1(X, \mathscr{H})) \simeq \text{Pic}(X, D)/\ell^n\text{Pic}(X, D) \tag{4.9.2}$$

and an inclusion

$$\text{Pic}(X, D)/\ell^n\text{Pic}(X, D) \hookrightarrow H^2(X, j_!\mu_{\ell^n}).$$

Passing to the projective limit in n and tensoring with $\mathbb{Q}_\ell$ we get an inclusion

$$(\varprojlim_n \text{Pic}(X, D)/\ell^n\text{Pic}(X, D)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow H^2(X, j_!\mathbb{Q}_\ell(1)). \tag{4.9.3}$$

Proposition 4.10. *The map* (4.9.3) *factors through an inclusion*

$$NS(X, D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \hookrightarrow H^2(X, j_!\mathbb{Q}_\ell(1)). \tag{4.10.1}$$

Proof. Indeed the kernel $\text{Pic}^0(X, D)$ of the surjection

$$\text{Pic}(X, D) \to NS(X, D)$$

is an extension of a finite group by a divisible group, and therefore there exists an integer N such that for every n the kernel of the surjection

$$\text{Pic}(X, D)/\ell^n\text{Pic}(X, D) \to NS(X, D)/\ell^n NS(X, D)$$

is annihilated by N. In particular we get an isomorphism

$$(\varprojlim_n(\text{Pic}(X, D)/\ell^n\text{Pic}(X, D))) \otimes \mathbb{Q} \simeq (\varprojlim_n NS(X, D)/\ell^n NS(X, D)) \otimes \mathbb{Q}.$$

Since $NS(X, D)$ is a finitely generated abelian group, which implies that

$$(\varprojlim_n NS(X, D)/\ell^n NS(X, D)) \otimes \mathbb{Q} \simeq NS(X, D) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

we obtain the result. □

5. Independence for Smooth Surfaces

Let $\overline{X}_0/\mathbb{F}_q$ be a smooth proper 2-dimensional $\mathbb{F}_q$-scheme, and suppose given a divisor with simple normal crossings

$$Z_0 \hookrightarrow \overline{X}_0$$

and a decomposition

$$Z_0 = D_0 \cup E_0,$$

where D_0 and E_0 are also divisors with simple normal crossings. Let X_0 denote $\overline{X}_0 - E_0$. We then get inclusions

$$\begin{array}{ccc} \overline{X}_0 - Z_0 & \xrightarrow{j'_{D0}} & \overline{X}_0 - E_0 \\ \downarrow{\scriptstyle j'_{E0}} & & \downarrow{\scriptstyle j_{E0}} \\ \overline{X}_0 - D_0 & \xrightarrow{j_{D0}} & \overline{X}_0. \end{array}$$

As before, let the absence of the subscript 0 denote base change to $k = \overline{\mathbb{F}}_q$, e.g., $D := D_0 \times_{\mathbb{F}_q} k$, etc. Then define

$$H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell) := H^i(X, j'_{D!}\mathbb{Q}_{\ell, X-(D\cap X)}).$$

The geometric Frobenius F acts naturally on these cohomology groups. For $0 \le i \le 4$ we define $P^i_{\ell,D,E}(t) := \det(1 - tF|H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell))$. The main result of this section is the following:

Theorem 5.1. *The polynomial $P^i_{\ell,D,E}(t)$ has rational coefficients independent of ℓ.*

The proof occupies the remainder of this section.

Special Case 5.2. *Suppose $D \cap E = \emptyset$. Then 5.1 is true for $P^1_{\ell,D,E}(t)$.*

Proof. Let

$$E \xhookrightarrow{i_E} \overline{X} \xhookleftarrow{j_E} X = \overline{X} - E$$

and

$$D \xhookrightarrow{i_D} \overline{X} \xhookleftarrow{j_D} \overline{X} - D$$

be the inclusion maps into $\overline{X}$. In addition, let $j'_D : X - D \hookrightarrow X$ be the inclusion into $X = \overline{X} - E$, so that $H^1_{D,E}(X, \mathbb{Q}_\ell) = H^1(X, j'_{D!}\mathbb{Q}_{\ell,X-D})$. Consider the triangle on $\overline{X}$

$$i_{E*}i^!_E\mathbb{Q}_{\ell,\overline{X}} \longrightarrow \mathbb{Q}_{\ell,\overline{X}} \longrightarrow j_{E*}\mathbb{Q}_{\ell,X} \longrightarrow i_{E*}i^!_E\mathbb{Q}_{\ell,\overline{X}}[1].$$

Restricting this sequence to $\overline{X} - D$ and then applying $j_{D!}$ we get a distinguished triangle (recall that $E \cap D = \emptyset$)

$$i_{E*}i^!_E\mathbb{Q}_{\ell,\overline{X}} \longrightarrow j_{D!}\mathbb{Q}_{\ell,\overline{X}-D} \longrightarrow j'_{D!}\mathbb{Q}_{\ell,X-D} \longrightarrow i_{E*}i^!_E\mathbb{Q}_{\ell,\overline{X}}[1].$$

Taking cohomology we then get an exact sequence

$$0 \longrightarrow H^1_c(\overline{X} - D, \mathbb{Q}_\ell) \longrightarrow H^1(X, j'_{D!}\mathbb{Q}_{\ell,X-D}) \longrightarrow \mathbb{Q}_\ell(-1)^{\pi_0(E)} \xrightarrow{\partial} H^2_c(\overline{X} - D, \mathbb{Q}_\ell).$$

Here we have used the fact that E is a simple normal crossings divisor, and hence $H^2_E(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1)^{\pi_0(E)}$. Therefore it suffices to show that the characteristic polynomial of Frobenius acting on $\mathrm{Ker}(\partial)$ is in $\mathbb{Q}[T]$ and is independent of ℓ.

Let

$$\tau : \mathbb{Q}^{\pi_0(E)} \to NS(\overline{X}, D) \otimes \mathbb{Q}$$

be the natural map defined by the irreducible components of E (each of which is smooth by assumption). Then the map ∂ is obtained by tensoring the map τ with $\mathbb{Q}_\ell$, twisting by $\mathbb{Q}_\ell(-1)$, and then applying the injective map 4.10.1 (we leave to the reader the verification that the two definitions of the cycle class used here agree). We conclude that the characteristic polynomial of Frobenius acting on $\mathrm{Ker}(\partial)$ is equal to the polynomial

$$\det(1 - TF|\mathrm{Ker}(\tau) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell(-1)),$$

which is evidently in $\mathbb{Q}[T]$ and independent of ℓ. □

Special Case 5.3. *Theorem 5.1 is true if $E \cap D = \emptyset$.*

Proof. We deduce this from the preceding special case using Verdier duality as follows. Consider the commutative diagram

$$\begin{array}{ccc} \overline{X} - (D \cup E) & \xhookrightarrow{j'_D} & \overline{X} - E \\ \Big\downarrow {\scriptstyle j'_E} & & \Big\downarrow {\scriptstyle j_E} \\ \overline{X} - D & \xhookrightarrow{j_D} & \overline{X}, \end{array}$$

and let $\mathscr{D}_{\overline{X}}(-)$ be the Verdier duality functor. We then have

$$\mathscr{D}_{\overline{X}} j_{D*} j'_{E!}\mathbb{Q}_\ell \simeq j_{D!} j'_{E*}\mathscr{D}_{\overline{X}-(D\cup E)}\mathbb{Q}_\ell. \tag{5.3.1}$$

Now observe that since $\overline{X}$ is smooth of dimension 2 we have

$$\mathscr{D}_{\overline{X}-(D\cup E)}\mathbb{Q}_\ell = \mathbb{Q}_\ell(2)[4].$$

so the right side of (5.3.1) can be written as

$$j_{D!} j'_{E*}\mathbb{Q}_\ell(2)[4].$$

On the other hand, since D and E are disjoint there is a natural isomorphism

$$j_{D!} j'_{E*}\mathbb{Q}_\ell \simeq j_{E*} j'_{D!}\mathbb{Q}_\ell.$$

Thus we obtain an isomorphism

$$\mathscr{D}_{\overline{X}} j_{D*} j'_{E!}\mathbb{Q}_\ell \simeq j_{E*} j'_{D!}\mathbb{Q}_\ell(2)[4].$$

This shows that we have a perfect duality pairing between

$$H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell(2))$$

and

$$H^{4-i}_{E,D}(\overline{X}, \mathbb{Q}_\ell).$$

Since 5.1 holds for $P^1_{\ell,D,E}(t)$ by 5.2, this implies that 5.1 also holds for $P^3_{\ell,D,E}(t)$. Further, the result for $i = 0$ is immediate, and hence by duality we also have it for $i = 4$. The theorem of Gabber [Fuj, Thm. 2] says that the product

$$\prod_{i=0}^{4} \det(1 - tF | H^i_{D,E}(\overline{X}_k, \mathbb{Q}_\ell))^{(-1)^i}$$

is in $\mathbb{Q}[t]$ and independent of ℓ, and this implies the result also for $i = 2$. □

General Case 5.4. *Proof of 5.1 in the case when $D \cap E$ is not necessarily empty.*

Let

$$b : \overline{Y} \to \overline{X}$$

be the blowup of $E \cap D$, and let $E' \subset \overline{Y}$ (resp. $D' \subset \overline{Y}$) be the strict transform of E (resp. D). Note that

$$E' \cap D' = \emptyset.$$

Let $F' \subset \overline{Y}$ be the exceptional divisor, so we have

$$b^{-1}(D \cup E) = E' \cup D' \cup F'.$$

Let Y denote $\overline{Y} - E'$, and let

$$u : Y \hookrightarrow \overline{Y}$$

be the inclusion. Also define

$$v : Y - D' \hookrightarrow Y$$

to be the inclusion of the complement of D'. Finally, let

$$X - (D \cap X) \xhookrightarrow{j} X \xhookrightarrow{s} \overline{X}$$

be the natural inclusions. There is a natural map in $D^b_c(\overline{X}, \mathbb{Q}_\ell)$

$$b_*(u_*v_!\mathbb{Q}_{\ell,Y-D'}) \to s_*j_!\mathbb{Q}_{\ell,X-(D\cap X)}. \tag{5.4.1}$$

For this note that giving such a map is equivalent by adjunction to specifying a map

$$s^*b_*(u_*v_!\mathbb{Q}_{\ell,Y-D'}) \to j_!\mathbb{Q}_{\ell,X-(D\cap X)}.$$

But there is a canonical such isomorphism because b is an isomorphism away from $D \cap E$. The following lemma shows that the map 5.4.1 is an isomorphism, and therefore in particular we have $H^i_{D,E}(\overline{X}, \mathbb{Q}_\ell) \simeq H^i_{D',E'}(\overline{Y}, \mathbb{Q}_\ell)$. This combined with 5.3 implies 5.1.

Lemma 5.5. *The map 5.4.1 is an isomorphism.*

Proof. Let $x \in D \cap E$ be a point. It suffices to show that the stalk complexes (in the derived category of abelian groups)

$$(b_*(u_*v_!\mathbb{Q}_{\ell,Y-D'}))_x, \quad (s_*j_!\mathbb{Q}_{\ell,X-(D\cap X)})_x$$

are both 0, since the map 5.4.1 is clearly an isomorphism away from $D \cap E$.

To compute $(s_*j_!\mathbb{Q}_{\ell,X-(D\cap X)})_x$, let

$$\bar{i} : D \hookrightarrow \overline{X}, \quad i : D \cap X \hookrightarrow X, \quad q : D \cap X \hookrightarrow D$$

be the inclusions. We then have an exact triangle

$$j_!\mathbb{Q}_{\ell,X-(D\cap X)} \to \mathbb{Q}_{\ell,X} \to i_*\mathbb{Q}_{\ell,D\cap X} \to j_!\mathbb{Q}_{\ell,X-(D\cap X)}[1],$$

which upon applying s_* and taking stalks at x gives a distinguished triangle

$$(s_*j_!\mathbb{Q}_{\ell,X-(D\cap X)})_x \to (s_*\mathbb{Q}_{\ell,X})_x \to (q_*\mathbb{Q}_{\ell,D})_x \to (s_*j_!\mathbb{Q}_{\ell,X-(D\cap X)})_x[1].$$

It therefore suffices to show that

$$(s_*\mathbb{Q}_{\ell,X})_x \to (s_*\mathbb{Q}_{\ell,D})_x$$

is an isomorphism. For this, consider the following diagram of inclusions:

$$\begin{array}{ccccc} D \cap X & \xrightarrow{q} & D & \xleftarrow{\gamma} & D \cap E \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{i}} & & \downarrow{\scriptstyle \delta} \\ X & \xrightarrow{s} & \overline{X} & \xleftarrow{\alpha} & E. \end{array}$$

This diagram induces a morphism of exact triangles

$$\begin{array}{ccccccc} \alpha_*\alpha^!\mathbb{Q}_{\ell,\overline{X}} & \longrightarrow & \mathbb{Q}_{\ell,\overline{X}} & \longrightarrow & s_*\mathbb{Q}_{\ell,X} & \longrightarrow & \alpha_*\alpha^!\mathbb{Q}_{\ell,\overline{X}}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{i}_*\gamma_*\gamma^!\mathbb{Q}_{\ell,D} & \longrightarrow & \bar{i}_*\mathbb{Q}_{\ell,D} & \longrightarrow & \bar{i}_*q_*\mathbb{Q}_{\ell,D\cap X} & \longrightarrow & \bar{i}_*\gamma_*\gamma^!\mathbb{Q}_{\ell,D}[1]. \end{array} \tag{5.5.1}$$

It is clear that for $x \in D \cap E$, the map $(\mathbb{Q}_{\ell,\overline{X}})_x \to (\bar{i}_*\mathbb{Q}_{\ell,D})_x$ is an isomorphism. Moreover, since E is smooth, by cohomological purity we have a commuting diagram

$$\begin{array}{ccc} \alpha_*\alpha^!\mathbb{Q}_{\ell,\overline{X}} & \xrightarrow{\cong} & \alpha_*\mathbb{Q}_{\ell,E}(-1)[-2] \\ \downarrow & & \downarrow \\ \bar{i}_*\gamma_*\gamma^!\mathbb{Q}_{\ell,D} & \xrightarrow{\cong} & \bar{i}_*\gamma_*\mathbb{Q}_{\ell,D\cap E}(-1)[-2], \end{array}$$

and the map on the right is that induced by the inclusion $D \cap E \hookrightarrow E$. Therefore the vertical maps are isomorphisms when localized at $x \in D \cap E$, and so by applying the five-lemma to 5.5.1 localized at $x \in D \cap E$ we get that $(s_*\mathbb{Q}_{\ell,X})_x \to (s_* i_* \mathbb{Q}_{\ell, D\cap X})_x$ is an isomorphism for $x \in D \cap E$, as was to be shown.

Finally we compute $(b_*(u_* v_! \mathbb{Q}_\ell))_x$. Let F_x denote the fiber of b over x, so $F_x \simeq \mathbb{P}^1_k$. By the proper base change theorem we have

$$(b_*(u_* v_! \mathbb{Q}_\ell))_x \simeq R\Gamma(F_x, (u_* v_! \mathbb{Q}_\ell)|_{F_x}).$$

Choose an isomorphism $F_x \simeq \mathbb{P}^1_k$ so that $F_x \cap D' = \{0\}$, and $F_x \cap E' = \{\infty\}$, and let

$$i_0 : \mathrm{Spec}(k) \hookrightarrow \mathbb{P}^1, \quad i_\infty : \mathrm{Spec}(k) \hookrightarrow \mathbb{P}^1$$

be the inclusions. We then have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} \mathbb{P}^1 & \xhookleftarrow{f} & \mathbb{A}^1 & \xhookleftarrow{g} & \mathbb{A}^1 - \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{Y} & \xhookleftarrow{u} & Y & \xhookleftarrow{v} & Y - D'. \end{array}$$

Since D' and E' are disjoint, we have

$$(u_* v_! \mathbb{Q}_\ell)|_{\mathbb{P}^1} \simeq f_* g_! \mathbb{Q}_{\ell, \mathbb{A}^1 - \{0\}}.$$

Therefore it suffices to show that

$$R\Gamma(\mathbb{A}^1, g_! \mathbb{Q}_\ell) = 0.$$

Using the short exact sequence of sheaves

$$0 \to g_! \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_{0*} \mathbb{Q}_\ell \to 0$$

this is equivalent to the statement that the restriction map

$$R\Gamma(\mathbb{A}^1, \mathbb{Q}_\ell) \to R\Gamma(\mathrm{Spec}(k), \mathbb{Q}_\ell) = \mathbb{Q}_\ell$$

is an isomorphism, which is standard. □

This completes the proof of 5.1. □

6. Independence for $H^*_c(X, \mathbb{Q}_\ell)$

Let X_0 be a 2-dimensional separated scheme of finite type over $\mathbb{F}_q$. This section is devoted to the proof of the following theorem:

Theorem 6.1. *Let X_0 be a separated 2-dimensional finite type $\mathbb{F}_q$-scheme, and let $X = X_0 \times_{\mathbb{F}_q} k$. Then for each i, the system $(H^i_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible for the Frobenius action.*

6.2. For $i = 0$, this is trivial: $H^0_c(X, \mathbb{Q}_\ell)$ is canonically isomorphic to the $\mathbb{Q}_\ell$-vector space with basis the proper connected components of X, and Frobenius acts by pulling back connected components. We have already shown (Prop. 4.7) that $(H^1_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system. By the Leschetz trace formula, if we can show compatibility for $H^3_c(X, \mathbb{Q}_\ell)$ and $H^4_c(X, \mathbb{Q}_\ell)$, then we will complete the proof of (6.1). Therefore the rest of the section is devoted to the proof of the following:

Proposition 6.3. *Let X_0 be a 2-dimensional separated finite type $\mathbb{F}_q$-scheme, and let $X = X_0 \times_{\mathbb{F}_q} k$ for k a fixed algebraic closure of $\mathbb{F}_q$. Then the systems $(H^3_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ and $(H^4_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ are both compatible for the Frobenius action.*

The proof is inspired by the construction in [BVS] of the Albanese 1-motive $\mathrm{Alb}^+(X)$ associated to a variety X over a field k of characteristic 0.

6.4. In proving (6.3) we may assume X_0 is reduced since $(X_0)_{red}$ has the same cohomology as X. Moreover, we can assume X_0 is equidimensional of dimension 2: to see this, note first that the locus where 1-dimensional and 2-dimensional irreducible components of X_0 intersect is 0-dimensional, so by removing a finite collection $P_0 = \{p_1, ..., p_n\}$ of points from X_0 we can arrange that there are no intersections between 1-dimensional and 2-dimensional components. Let $Y_0 = X_0 - P_0$, and let

$$j_0 : Y_0 \hookrightarrow X_0, \quad i_0 : P_0 \hookrightarrow X_0$$

be the inclusions. The natural maps

$$H^3_c(Y, \mathbb{Q}_\ell) \to H^3_c(X, \mathbb{Q}_\ell),$$
$$H^4_c(Y, \mathbb{Q}_\ell) \to H^4_c(X, \mathbb{Q}_\ell)$$

induced by the exact triangle

$$j_{0!}\mathbb{Q}_{\ell,Y_0} \to \mathbb{Q}_{\ell,X_0} \to i_{0*}\mathbb{Q}_{\ell,P_0} \to j_{0!}\mathbb{Q}_{\ell,Y_0}[1]$$

are isomorphisms since P_0 is 0-dimensional. The cohomology groups $H^3_c(Y, \mathbb{Q}_\ell)$ and $H^4_c(Y, \mathbb{Q}_\ell)$ depend only on the purely 2-dimensional locus of Y, so by replacing X_0 by Y_0 and considering only the 2-dimensional components we may assume X_0 is purely 2-dimensional.

6.5. So let X_0 be a separated, reduced, purely 2-dimensional scheme of finite type over $\mathbb{F}_q$. Then let U_0 be the (non-empty) smooth locus of X_0 and S_0 the singular locus, and choose a resolution of singularities $\pi_0 : X'_0 \to X_0$, (see [Lip] for a proof of resolution of singularities for surfaces) so that π_0 is an isomorphism over U_0 and $S'_0 := \pi_0^{-1}(S_0)$ is a simple normal crossings divisor. Diagramatically, we have

$$\begin{array}{ccccc} U_0 & \xrightarrow{j'_0} & X'_0 & \xleftarrow{i'_0} & S'_0 \\ \| & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0|_{S'_0}} \\ U_0 & \xrightarrow{j_0} & X_0 & \xleftarrow{i_0} & S_0. \end{array}$$

In addition, fix compactifications $\alpha_0 : X_0 \hookrightarrow \overline{X}_0$ of X_0 and $\alpha'_0 : X'_0 \hookrightarrow \overline{X'_0}$ with $\overline{X'_0}$ smooth and a morphism $\overline{\pi_0} : \overline{X'_0} \to \overline{X_0}$ making the diagram

$$\begin{array}{ccc} X'_0 & \xrightarrow{\alpha'_0} & \overline{X'_0} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \overline{\pi_0}} \\ X_0 & \xrightarrow{\alpha_0} & \overline{X}_0 \end{array}$$

commute.

6.6. Let A_ℓ denote the cone of $\mathbb{Q}_{\ell,X} \to \pi_*\mathbb{Q}_{\ell,X'}$, and consider the following commuting diagram in $D^b_c(X, \mathbb{Q}_\ell)$, where the rows and columns are exact triangles:

$$\begin{array}{ccccccc}
j_!\mathbb{Q}_{\ell,U} & \longrightarrow & \mathbb{Q}_{\ell,X} & \longrightarrow & i_*\mathbb{Q}_{\ell,S} & \longrightarrow & j_!\mathbb{Q}_{\ell,U}[1] \\
\| & & \downarrow & & \downarrow & & \downarrow \\
\pi_*j'_!\mathbb{Q}_{\ell,U} & \longrightarrow & \pi_*\mathbb{Q}_{\ell,X'} & \longrightarrow & i_*\pi|_{S'*}\mathbb{Q}_{\ell,S'} & \longrightarrow & \pi_*j'_!\mathbb{Q}_{\ell,U}[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & A_\ell & \xrightarrow{\simeq} & i_*i^*A_\ell & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
j_!\mathbb{Q}_{\ell,U}[1] & \longrightarrow & \mathbb{Q}_{\ell,X}[1] & \longrightarrow & i_*\mathbb{Q}_{\ell,S}[1] & &
\end{array}$$

Applying $\alpha_!$ to this diagram (where $\alpha : X \hookrightarrow \overline{X}$ is the base extension of α_0) and taking cohomology of the middle two columns, we get an isomorphism

$$H^4_c(X, \mathbb{Q}_\ell) \simeq H^4_c(X', \mathbb{Q}_\ell) \tag{6.6.1}$$

as well as a commuting diagram

(6.6.2)

$$\begin{array}{ccccccccc}
 & & H^2_c(X', \mathbb{Q}_\ell) & \longrightarrow & H^2(\overline{X}, \alpha_!A_\ell) & \longrightarrow & H^3_c(X, \mathbb{Q}_\ell) & \longrightarrow & H^3_c(X', \mathbb{Q}_\ell) \longrightarrow 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
H^2_c(S, \mathbb{Q}_\ell) & \longrightarrow & H^2_c(S', \mathbb{Q}_\ell) & \longrightarrow & H^2(\overline{X}, \alpha_!A_\ell) & \longrightarrow & 0 & &
\end{array}$$

where the rows are exact. Since X' is a smooth variety, the systems $(H^3_c(X', \mathbb{Q}_\ell))_{\ell \neq p}$ and $(H^4_c(X', \mathbb{Q}_\ell))_{\ell \neq p}$ are compatible by Theorem 5.1. Therefore (6.6.1) shows that $(H^4_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system. Moreover, we have an exact sequence

$$0 \to C_\ell \to H^3_c(X, \mathbb{Q}_\ell) \to H^3_c(X', \mathbb{Q}_\ell) \to 0,$$

where

$$C_\ell := \mathrm{coker}(H^2_c(X', \mathbb{Q}_\ell) \to H^2(\overline{X}, \alpha_!A_\ell)).$$

Therefore by Lemma 2.5, to show $(H^3_c(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible it suffices to show that the system $(C_\ell)_{\ell \neq p}$ is compatible.

6.7. Let $C_\ell(1)^\vee$ be the Tate twisted dual of C_ℓ, i.e.,

$$C_\ell(1)^\vee := \mathrm{Hom}(C_\ell(1), \mathbb{Q}_\ell).$$

We proceed to give a 'motivic' description of $C_\ell(1)^\vee$, which will in particular show that $(C_\ell(1)^\vee)_{\ell \neq p}$ is a compatible sytem. By exactness of the bottom row of the diagram (6.6.2), we can identify $H^2(\overline{X}, \alpha_!A_\ell)$ with $\mathrm{coker}(H^2_c(S, \mathbb{Q}_\ell) \to H^2_c(S', \mathbb{Q}_\ell))$. Taking duals, we find a Frobenius-equivariant isomorphism

$$C_\ell(1)^\vee \cong \ker(H^2_c(S', \mathbb{Q}_\ell(1))^\vee \to H^2_c(X', \mathbb{Q}_\ell(1))^\vee \oplus H^2_c(S, \mathbb{Q}_\ell(1))^\vee). \tag{6.7.1}$$

Let $\tilde{S}_0 \to S_0$ and $\tilde{S}'_0 \to S'_0$ be the normalizations of the 1-dimensional schemes S and S', respectively. Since these are isomorphisms on dense open subsets of S_0 and S'_0 respectively, they induce isomorphisms $H^2_c(S, \mathbb{Q}_\ell) \cong H^2_c(S', \mathbb{Q}_\ell)$ and $H^2_c(S', \mathbb{Q}_\ell) \cong H^2_c(\tilde{S}', \mathbb{Q}_\ell)$. Then let

$I(S)$ (resp. $I(S')$) be the set of 1-dimensional irreducible components of S (resp. S'). Since $\tilde{S}$ and $\tilde{S}'$ are smooth, Poincaré duality induces canonical isomorphisms

$$H^2_c(S,\mathbb{Q}_\ell(1))^\vee \cong H^2_c(\tilde{S},\mathbb{Q}_\ell(1))^\vee \cong \mathbb{Z}^{I(S)} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \quad \text{and}$$
$$H^2_c(S',\mathbb{Q}_\ell(1))^\vee \cong H^2_c(\tilde{S}',\mathbb{Q}_\ell(1))^\vee \cong \mathbb{Z}^{I(S')} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell.$$

Under these isomorphisms, the map $H^2_c(S',\mathbb{Q}_\ell(1))^\vee \to H^2_c(S,\mathbb{Q}_\ell(1))^\vee$ corresponds to the proper pushforward of Weil divisors $\mathbb{Z}^{I(S')} \to \mathbb{Z}^{I(S)}$, tensored with $\mathbb{Q}_\ell$.

6.8. Now consider the map $H^2_c(S',\mathbb{Q}_\ell(1))^\vee \to H^2_c(X',\mathbb{Q}_\ell(1))^\vee$ induced by the closed immersion $S' \hookrightarrow X'$. Since X' is smooth, Poincaré duality induces an isomorphism $H^2_c(X',\mathbb{Q}_\ell(1))^\vee \cong H^2(X',\mathbb{Q}_\ell)(1)$, and the corresponding map

$$\mathbb{Z}^{I(S')} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2(X',\mathbb{Q}_\ell)(1)$$

is precisely the cycle class map $\mathbb{Z}^{I(S)} \to H^2(X',\mathbb{Q}_\ell)(1)$ by [SGA4h, Cycle, 2.3.6]. By Proposition 4.10 in the case $D = \emptyset$, we have a factorization

$$\mathbb{Z}^{I(S')} \otimes \mathbb{Q}_\ell \to NS(X) \otimes \mathbb{Q}_\ell \hookrightarrow H^2(X,\mathbb{Q}_\ell(1)).$$

Therefore we have

$$\ker(H^2(S',\mathbb{Q}_\ell(1))^\vee \to H^2(X',\mathbb{Q}_\ell)^\vee) \simeq \ker(\mathbb{Z}^{I(S')} \to NS(X')) \otimes \mathbb{Q}_\ell,$$

where the map $\mathbb{Z}^{I(S')} \to NS(X')$ is the canonical map from Weil divisors supported on S' to $NS(X')$.

6.9. In summary, the vector space $C_\ell(1)^\vee$ of 6.7.1 can be expressed as the kernel of the map

$$\mathbb{Z}^{I(S')} \otimes \mathbb{Q}_\ell \to (NS(X') \otimes \mathbb{Q}_\ell) \oplus (\mathbb{Z}^{I(S)} \otimes \mathbb{Q}_\ell),$$

where the maps are those given by the cycle class map of $\mathbb{Z}^{I(S)}$ into $NS(X')$ and proper pushforward of divisors of S' onto divisors of S. From this description it is clear that $(C_\ell^\vee)_{\ell\neq p}$ is a compatible system, and hence so is $(C_\ell)_{\ell\neq p}$. This completes the proof that $(H^3_c(X,\mathbb{Q}_\ell))_{\ell\neq p}$ is a compatible system and hence also completes the proof of Proposition 6.3 and Theorem 6.1.

7. Independence for $H^1(X,\mathbb{Q}_\ell)$

This section is devoted to proving the following:

Proposition 7.1. *Let X_0 be a separated finite type $\mathbb{F}_q$-scheme, let $\mathbb{F}_q \hookrightarrow k$ be an algebraic closure, and let $X = X_0 \times_{\mathbb{F}_q} k$. Then $(H^1(X,\mathbb{Q}_\ell))_{\ell\neq p}$ is a compatible system.*

This proposition has essentially appeared before in the context of 1-motives, in [BVS] and [Ram], although it is not stated in this form. We give a translation of their proof which does not use the theory of 1-motives. We use simplicial schemes and cohomological descent throughout the argument; a good reference for the facts we will need (and more) is [Con01].

7.2. By [dJng] as explained in [Con01, 4.7], there exists a proper hypercover $X_{0,\bullet} \to X_0$ with $X_{0,\bullet}$ a smooth simplicial variety, and a compactification

$$X_{0,\bullet} \overset{j_\bullet}{\hookrightarrow} \overline{X}_{0,\bullet} \overset{i_\bullet}{\hookleftarrow} D_{0,\bullet}$$

with $\overline{X}_{0,\bullet}$ proper and smooth, and where the complement $D_{0,\bullet} \hookrightarrow \overline{X}_{0,\bullet}$ is such that each $D_{0,n}$ is a divisor with simple normal crossings in $\overline{X}_{0,n}$. By cohomological descent for proper morphisms, we have a Frobenius-equivariant isomorphism $\mathbb{H}^1(X_\bullet, \mathbb{Q}_\ell) \cong H^1(X, \mathbb{Q}_\ell)$, where $X_\bullet := X_{0,\bullet} \times_{\mathbb{F}_q} k$. We claim (in analogy to the standard case of a scheme $\overline{X}$) that there is an exact triangle in $D_c^b(\overline{X}_\bullet, \mathbb{Q}_\ell)$

$$i_{\bullet *} i_\bullet^! \mathbb{Q}_\ell \to \mathbb{Q}_\ell \to j_{\bullet *} j_\bullet^* \mathbb{Q}_\ell \to i_{\bullet *} i_\bullet^! \mathbb{Q}_\ell[1]. \tag{7.2.1}$$

To see this, let $\mathscr{I}^\bullet$ be a complex of injective simplicial ℓ-adic sheaves on $\overline{X}_\bullet$. We claim that we have an exact sequence of complexes of sheaves

$$0 \to i_{\bullet *} i_\bullet^! \mathscr{I}^\bullet \to \mathscr{I}^\bullet \to j_{\bullet *} j_\bullet^* \mathscr{I}^\bullet \to 0.$$

To see this, it suffices to check that for each n, the complex

$$0 \to i_{n*} i_n^! \mathscr{I}^n \to \mathscr{I}^n \to j_{n*} j_n^* \mathscr{I}^n \to 0$$

of sheaves on $\overline{X}_n$ is exact. But since each $\mathscr{I}^n$ is injective [Con01, 6.4], this is standard.

Taking cohomology in 7.2.1, we get an exact sequence

$$\mathbb{H}^1_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^1(X_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell).$$

Note that we have a spectral sequence $H^q_{D_p}(\overline{X}_p, \mathbb{Q}_\ell) \Rightarrow \mathbb{H}^{p+q}_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell)$. By cohomological purity and the fact that the divisors D_i have simple normal crossings, we have

$$H^q_{D_p}(\overline{X}_p, \mathbb{Q}_\ell) = 0 \text{ for } q = 0, 1.$$

Therefore $\mathbb{H}^1_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) = 0$, and so we have an exact sequence

$$0 \to \mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^1(X_\bullet, \mathbb{Q}_\ell) \to \ker(\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell)) \to 0. \tag{7.2.2}$$

Thus we need to check that $\mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell)$ and $K := \mathrm{Ker}(\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell))$ have Frobenius action independent of ℓ. To analyze $\mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell)$, start with the simplicial Kummer sequence

$$0 \to \mu_{\ell^n, \overline{X}_\bullet} \to \mathbb{G}_{m, \overline{X}_\bullet} \xrightarrow{\cdot \ell^n} \mathbb{G}_{m, \overline{X}_\bullet} \to 0.$$

Note that we have an isomorphism [BVS, 4.1]

$$\mathbb{H}^1(\overline{X}_\bullet, \mathbb{G}_m) \cong \mathrm{Pic}\, \overline{X}_\bullet,$$

where the group in the right is the Picard group of isomorphism classes of line bundles on $\overline{X}_\bullet$. Recall that giving such a line bundle is equivalent [BVS, p.40] to giving a pair $(\mathscr{L}, \gamma)$, where $\mathscr{L}$ is a line bundle on $\overline{X}_0$, and $\gamma : p_0^* \mathscr{L} \cong p_1^* \mathscr{L}$ is an isomorphism where $p_0, p_1 : \overline{X}_1 \to \overline{X}_0$ are the projections, such that γ satisfies a cocycle condition on the pullbacks to $\overline{X}_2$. Define the Picard functor $\mathrm{Pic}_{X_\bullet/k} : \mathrm{Sch}/k \to \mathrm{Set}$, via the rule $Y \mapsto \mathrm{Pic}(\overline{X}_\bullet \times Y)$.

Theorem 7.3. *Let $\overline{X}_\bullet$ be a simplicial scheme over a field k with each $\overline{X}_i$ proper, and consider the Picard functor $Y \mapsto \mathrm{Pic}(\overline{X}_\bullet \times Y)$. Then the fpqc-sheafification of this functor is representable by a group scheme* $\mathbf{Pic}\ \overline{X}_\bullet$*, locally of finite type over k.*

Proof. See [BVS, p. 75]. □

Thus when k is algebraically closed, $\mathbb{H}^1(\overline{X}_\bullet, \mathbb{G}_m)$ is naturally the k-points of a group scheme $\mathbf{Pic}\ \overline{X}_\bullet$. Let $\mathbf{Pic}^0\ \overline{X}_\bullet$ be the connected component of the identity. We have the following:

Proposition 7.4. *Let* $NS(\overline{X}_\bullet) = (\mathrm{Pic}\ \overline{X}_\bullet)/(\mathrm{Pic}^0\ \overline{X}_\bullet)$*. Then* $NS(\overline{X}_\bullet)$ *is a finitely generated abelian group.*

Proof. Let $\pi_i : \overline{X}_i \to \mathrm{Spec}\ k$ be the structure morphisms. Then by cohomological descent, we have a spectral sequence

$$E_1^{pq} = R^q\pi_{p*}\mathbb{G}_{m,\overline{X}_p} \Rightarrow R^{p+q}\pi_{\bullet *}\mathbb{G}_{m,\overline{X}_\bullet}.$$

Taking the exact sequence of low degree terms, we get an exact sequence of $fppf$ sheaves

$$0 \to \frac{\ker(\pi_{1*}\mathbb{G}_{m,\overline{X}_1} \to \pi_{2*}\mathbb{G}_{m,\overline{X}_2})}{\mathrm{im}(\pi_{0*}\mathbb{G}_{m,\overline{X}_0} \to \pi_{1*}\mathbb{G}_{m,\overline{X}_1})} \to \mathbf{Pic}\ \overline{X}_\bullet \to \ker(\mathbf{Pic}\ \overline{X}_0 \to \mathbf{Pic}\ \overline{X}_1).$$

The conclusion then follows since the group on the left is a subquotient of a torus, and $NS(\overline{X}_0)$ is finitely generated. □

Combining all of these facts, we see that we have an isomorphism $\mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell(1)) \cong V_\ell(\mathrm{Pic}^0\ \overline{X}_\bullet)$ which has Frobenius characteristic polynomial with rational coefficients independent of ℓ by (1.6).

7.5. To complete the proof of independence of ℓ for $H^1(X, \mathbb{Q}_\ell)$, we must show that the other term

$$K_\ell := \mathrm{Ker}(\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell))$$

of equation 7.2.2 has Frobenius characteristic polynomial independent of ℓ; i.e., $(K_\ell)_{\ell\neq p}$ is a compatible system. Note that the map $\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell)$ is induced by a map of spectral sequences

$$H^q_{D_p}(\overline{X}_p, \mathbb{Q}_\ell) \to H^q(\overline{X}_p, \mathbb{Q}_\ell). \tag{7.5.1}$$

Since the first two rows ($q = 0$ and $q = 1$) of the double complex $H^q_{D_p}(\overline{X}_p, \mathbb{Q}_\ell)$ are 0 by cohomological purity, we have

$$\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \cong \ker(H^2_{D_0}(\overline{X}_0, \mathbb{Q}_\ell) \xrightarrow{p_0^*-p_1^*} H^2_{D_1}(\overline{X}_1, \mathbb{Q}_\ell)), \tag{7.5.2}$$

where $p_0, p_1 : \overline{X}_1 \to \overline{X}_0$ are the projections. Since D_i is a divisor with strict normal crossings, cohomological purity gives a Frobenius-equivariant isomorphism

$$H^2_{D_i}(\overline{X}_i, \mathbb{Q}_\ell) \cong \mathbb{Z}^{I(D_i)} \otimes \mathbb{Q}_\ell(-1),$$

where $I(D_i)$ denotes the set of irreducible components of D_i. Under this identification $\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell)$ can be written

$$\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \cong (\ker(\mathbb{Z}^{I(D_0)} \xrightarrow{p_0^*-p_1^*} \mathbb{Z}^{I(D_1)})) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(-1), \tag{7.5.3}$$

where $p_0^*, p_1^* : \mathbb{Z}^{I(D_0)} \to \mathbb{Z}^{I(D_1)}$ act by pulling back cycles under $p_0, p_1 : \overline{X}_1 \to \overline{X}_0$. This shows that the Frobenius action on $\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell)$ has characteristic polynomial with rational coefficients independent of ℓ.

7.6. Returning to the map $\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell)$, we see by (7.5.2) that we can write this map as a composition

$$\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \cong \ker(H^2_{D_0}(\overline{X}_0, \mathbb{Q}_\ell) \xrightarrow{p_0^*-p_1^*} H^2_{D_1}(\overline{X}_1, \mathbb{Q}_\ell)) \to H^2(\overline{X}_0, \mathbb{Q}_\ell) \hookrightarrow \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell).$$

The rightmost map in this composition is an injection because $H^2(\overline{X}_0, \mathbb{Q}_\ell)$ is in the leftmost column of the spectral sequence $H^q(\overline{X}_p, \mathbb{Q}_\ell) \Rightarrow \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell)$. This implies that $K_\ell := \mathrm{Ker}(\mathbb{H}^2_{D_\bullet}(\overline{X}_\bullet, \mathbb{Q}_\ell) \to \mathbb{H}^2(\overline{X}_\bullet, \mathbb{Q}_\ell))$ is equal to

$$\ker(H^2_{D_0}(\overline{X}_0, \mathbb{Q}_\ell) \xrightarrow{p_0^*-p_1^*} H^2_{D_1}(\overline{X}_1, \mathbb{Q}_\ell)) \to H^2(\overline{X}_0, \mathbb{Q}_\ell).$$

Using (7.5.3) above, we see that we can write

$$K_\ell = \ker((\ker(\mathbb{Z}^{I(D_0)} \xrightarrow{p_0^*-p_1^*} \mathbb{Z}^{I(D_1)})) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(-1) \to H^2(\overline{X}_0, \mathbb{Q}_\ell)).$$

More simply, after Tate twist this is the cycle class map from divisors in $\mathbb{Z}^{I(D_0)}$ to $H^2(\overline{X}_0, \mathbb{Q}_\ell)(1)$, restricted to divisors which are equalized under p_0^* and p_1^*. By 4.10 we have a factorization

$$(\ker(\mathbb{Z}^{I(D_0)} \to \mathbb{Z}^{I(D_1)}) \otimes \mathbb{Q}_\ell(-1)) \to (NS(\overline{X}_0) \otimes \mathbb{Q}_\ell(-1)) \hookrightarrow H^2(\overline{X}_0, \mathbb{Q}_\ell),$$

implying that

$$K_\ell = \ker(\mathbb{Z}^{I(D_0)} \longrightarrow \mathbb{Z}^{I(D_1)} \oplus NS(\overline{X}_0)) \otimes \mathbb{Q}_\ell(-1),$$

where the map $\mathbb{Z}^{I(D_0)} \to \mathbb{Z}^{I(D_1)}$ is the map $p_0^* - p_1^*$ with $p_0, p_1 : \overline{X}_1 \to \overline{X_0}$ the projections, and $\mathbb{Z}^{I(D_0)} \to NS(\overline{X}_0)$ is the map from Weil divisors supported in D_0 to their associated line bundles on $\overline{X}_0$. This shows that $(K_\ell)_{\ell \neq p}$ is a compatible system since it manifestly has $\mathbb{Q}_\ell$-dimension independent of ℓ, with Frobenius acting (after Tate twist) by pulling back cycles supported in D_0.

7.7. This completes the proof that $(H^1(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system, since the terms $\mathbb{H}^1(\overline{X}_\bullet, \mathbb{Q}_\ell)$ and K_ℓ of equation 7.2.2 were shown to have Frobenius characteristic polynomial independent of ℓ.

8. Independence for $H^*(X, \mathbb{Q}_\ell)$

Now we are ready to prove independence of ℓ for $H^i(X, \mathbb{Q}_\ell)$ for all i:

Theorem 8.1. *Let X_0 be a 2-dimensional separated finite type $\mathbb{F}_q$-scheme, let $\mathbb{F}_q \hookrightarrow k$ be an algebraic closure, and let $X = X_0 \times_{\mathbb{F}_q} k$. Then for each i, the system $(H^i(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible.*

The proof occupies the remainder of this section.

8.2. For $i = 0$ we have $H^0(X, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\pi_0(X)}$, the free $\mathbb{Q}_\ell$-vector space on the connected components of X with Frobenius action induced by the action on $\pi_0(X)$. This implies that the system $(H^0(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible. Proposition 7.1 shows that $(H^1(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system. By a theorem of Gabber [Fuj, Thm. 2], if we let $P^i_\ell(t) = \det(1 - tF|H^i(X, \mathbb{Q}_\ell))$, then the alternating product

$$\prod_i P^i_\ell(t)^{(-1)^{i+1}}$$

has rational coefficients independent of ℓ. Therefore we need only show that for 2 values of i out of $i = 2, 3, 4$, the polynomial $P^i_\ell(t)$ has rational coefficients independent of ℓ. We will show this for $i = 3$ and $i = 4$. The method is essentially the same as the proof of 6.1, and we use the same notation as in the proof of that theorem to the extent that this is reasonable.

8.3. Let $j : U_0 \to X_0$ be the inclusion of the smooth locus, let $i : S_0 \to X_0$ be the closed complement, and let $\pi_0 : X_0' \to X_0$ be a resolution of singularities, by which we mean that π_0 is an isomorphism over U_0 and the complement $X_0' - U_0$ is a divisor S_0' with simple normal crossings:

$$\begin{array}{ccccc} U_0 & \xrightarrow{j_0'} & X_0' & \xleftarrow{i_0'} & S_0' \\ \| & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0|_{S_0}} \\ U_0 & \xrightarrow{j_0} & X_0 & \xleftarrow{i_0} & S_0 \end{array}$$

In addition, choose a compactification $\alpha_0 : X_0 \hookrightarrow \overline{X}_0$ of X_0 with complement $\beta_0 : Z_0 \hookrightarrow X_0$, and a resolution $\overline{\pi}_0 : \overline{X}_0' \to \overline{X}_0$ such that $\overline{\pi}_0^{-1}(Z_0) = Z_0'$ is a divisor with simple normal crossings:

$$\begin{array}{ccccc} X_0' & \xrightarrow{\alpha_0'} & \overline{X}_0{}' & \xleftarrow{\beta_0'} & Z_0' \\ {\scriptstyle \pi_0}\downarrow & & {\scriptstyle \overline{\pi}_0}\downarrow & & {\scriptstyle \overline{\pi}_0|_{Z_0'}}\downarrow \\ X_0 & \xrightarrow{\alpha_0} & \overline{X}_0 & \xleftarrow{\beta_0} & Z_0 \end{array}$$

We can arrange so that $\overline{S}_0' \cup Z_0'$ is a simple normal crossings divisor in $\overline{X}_0'$, where $\overline{S}_0'$ is the closure of S_0' in $\overline{X}_0'$.

8.4. As usual, let the absence of the subscript 0 denote base change to $k = \overline{\mathbb{F}}_q$. We have the following commutative diagram in $D_c^b(X, \mathbb{Q}_\ell)$, where the rows and columns are exact triangles:

$$\begin{array}{ccccccc} j_!\mathbb{Q}_{\ell,U} & \longrightarrow & \mathbb{Q}_{\ell,X} & \longrightarrow & i_*\mathbb{Q}_{\ell,S} & \longrightarrow & j_!\mathbb{Q}_{\ell,U}[1] \\ \| & & \downarrow & & \downarrow & & \downarrow \\ \pi_*j'_!\mathbb{Q}_{\ell,U} & \longrightarrow & \pi_*\mathbb{Q}_{\ell,X'} & \longrightarrow & i_*\pi_*\mathbb{Q}_{\ell,S'} & \longrightarrow & \pi_*j'_!\mathbb{Q}_{\ell,U}[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_\ell & \xrightarrow{\sim} & i_*i^*A_\ell & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ j_!\mathbb{Q}_{\ell,U}[1] & \longrightarrow & \mathbb{Q}_{\ell,X}[1] & \longrightarrow & i_*\mathbb{Q}_{\ell,Z}[1] & & \end{array}$$

Here the complex A_ℓ, defined to be the cone of $\mathbb{Q}_{\ell,X} \to \pi_*\mathbb{Q}_{\ell,X'}$, is supported on S since π is an isomorphism over U. Taking cohomology of the middle two columns, we get an isomorphism

$$H^4(X, \mathbb{Q}_\ell) \cong H^4(X', \mathbb{Q}_\ell) \tag{8.4.1}$$

as well as a commutative diagram

(8.4.2)

$$\begin{array}{ccccccccccc} & & H^2(X', \mathbb{Q}_\ell) & \longrightarrow & H^2(X, A_\ell) & \longrightarrow & H^3(X, \mathbb{Q}_\ell) & \longrightarrow & H^3(X', \mathbb{Q}_\ell) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \downarrow & & \\ H^2(S, \mathbb{Q}_\ell) & \longrightarrow & H^2(S', \mathbb{Q}_\ell) & \longrightarrow & H^2(S, A_\ell) & \longrightarrow & 0 & \longrightarrow & 0 & & \end{array}$$

where the rows are exact. Equation 8.4.1 immediately shows that $(H^4(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system since X' is smooth.

8.5. It remains only to show that $(H^3(X,\mathbb{Q}_\ell))_{\ell\neq p}$ is compatible. Note that the upper row of (8.4.2) gives an exact sequence

$$0 \to C_\ell \to H^3(X,\mathbb{Q}_\ell) \to H^3(X',\mathbb{Q}_\ell) \to 0, \tag{8.5.1}$$

where

$$C_\ell := \mathrm{coker}(H^2(X',\mathbb{Q}_\ell) \to H^2(X,A_\ell)).$$

Therefore to prove $(H^3(X,\mathbb{Q}_\ell))_{\ell\neq p}$ is compatible, we need only show that $(C_\ell)_{\ell\neq p}$ is compatble. By (8.4.2), $H^2(X,A_\ell)$ can be identified with $\mathrm{coker}(H^2(S,\mathbb{Q}_\ell) \to H^2(S',\mathbb{Q}_\ell))$. Let $C_\ell(1)^\vee$ be obtained from C_ℓ by twisting by 1 and then taking the vector space dual. We see that

$$C_\ell(1)^\vee \cong \mathrm{Ker}(H^2(S',\mathbb{Q}_\ell(1))^\vee \to H^2(S,\mathbb{Q}_\ell(1))^\vee \oplus H^2(X',\mathbb{Q}_\ell(1))^\vee).$$

To show that $(C_\ell)_{\ell\neq p}$ forms a compatible system, it suffices to show that $(C_\ell(1)^\vee)_{\ell\neq p}$ is a compatible system. To this end we give a 'motivic' description of $C_\ell(1)^\vee$, similar to that in [BVS, 3.1]. First off, let $\tilde{S}_0 \to S_0$ and $\tilde{S}'_0 \to S'_0$ be the normalizations of S_0 and S'_0, and let

$$P(S) \quad (\text{resp. } P(S'))$$

denote the set of proper, 1-dimensional irreducible components of S (resp. S'). Then Poincaré duality applied to $\tilde{S}$ and $\tilde{S}'$ yields isomorphisms

$$H^2(S,\mathbb{Q}_\ell(1))^\vee \cong H^2(\tilde{S},\mathbb{Q}_\ell(1))^\vee \cong \mathbb{Z}^{P(S)} \otimes \mathbb{Q}_\ell \quad \text{and}$$
$$H^2(S',\mathbb{Q}_\ell(1))^\vee \cong H^2(\tilde{S}',\mathbb{Q}_\ell(1))^\vee \cong \mathbb{Z}^{P(S')} \otimes \mathbb{Q}_\ell.$$

Under these isomorphisms, the map $H^2(S',\mathbb{Q}_\ell)^\vee \to H^2(S,\mathbb{Q}_\ell)^\vee$ corresponds to the proper pushforward of Weil divisors $\mathbb{Z}^{P(S')} \to \mathbb{Z}^{P(S)}$.

8.6. Next consider the map $H^2(S',\mathbb{Q}_\ell(1))^\vee \to H^2(X',\mathbb{Q}_\ell(1))^\vee$, which we would like to interpret in terms of cycle class maps. We will do this in several steps. First let $D \in P(S')$, and let $\gamma : D \hookrightarrow X'$ be the inclusion. Then $\alpha' \circ \gamma : D \hookrightarrow \overline{X}'$ is also a closed immersion since D is proper. Notice that the map $H^2(X',\mathbb{Q}_\ell) \to H^2(D,\mathbb{Q}_\ell)$ is induced by the map in $D^b_c(\overline{X}',\mathbb{Q}_\ell)$

$$\alpha'_*\mathbb{Q}_{\ell,X'} \to (\alpha'\circ\gamma)_*\mathbb{Q}_{\ell,D}.$$

If we apply Poincaré duality to this map, we get a map

$$(\alpha'\circ\gamma)_*(\alpha'\circ\gamma)^!\mathbb{Q}_{\ell,\overline{X}'} = \alpha'_!\gamma_*\gamma^!\mathbb{Q}_{\ell,X'} \to \alpha'_!\mathbb{Q}_{\ell,X'}. \tag{8.6.1}$$

Here we have used the fact that D is proper to conclude that $(\alpha'\circ\gamma)_! = (\alpha'\circ\gamma)_*$. Since $\overline{X}'$ is smooth, up to Tate twist applying Poincaré duality is the same as applying the dual of vector spaces and shifting. Therefore the map

$$H^2(D,\mathbb{Q}_\ell(1))^\vee \to H^2(S',\mathbb{Q}_\ell(1))^\vee \to H^2(X',\mathbb{Q}_\ell(1))^\vee$$

induced by the inclusions $D \hookrightarrow S' \hookrightarrow X'$ corresponds under Poincaré duality to the map

$$H^2_D(\overline{X}',\mathbb{Q}_\ell(1)) \to H^2_c(X',\mathbb{Q}_\ell(1)) \tag{8.6.2}$$

induced in degree 2 by the map of sheaves (8.6.1).

8.7. Now for any n prime to p, consider the following commuting diagram in $D^b_c(\overline{X}', \mathbb{Q}_\ell)$ where the rows are exact triangles:

$$\begin{array}{ccccccc}
\alpha'_!\gamma_*\gamma^!\mu_n & \longrightarrow & \alpha'_!\gamma_*\gamma^!\mathbb{G}_m & \xrightarrow{\cdot n} & \alpha'_!\gamma_*\gamma^!\mathbb{G}_m & \longrightarrow & \alpha'_!\gamma_*\gamma^!\mu_n[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\alpha'_!\mu_n & \longrightarrow & \alpha'_!\mathbb{G}_m & \xrightarrow{\cdot n} & \alpha'_!\mathbb{G}_m & \longrightarrow & \alpha'_!\mu_n.
\end{array}$$

Taking cohomology, we get (in particular) a commutative diagram

$$\begin{array}{ccc}
H^1_D(\overline{X}', \mathbb{G}_m)/nH^1_D(\overline{X}', \mathbb{G}_m) & \xhookrightarrow{\cong} & H^2_D(\overline{X}', \mu_n) \\
\downarrow & & \downarrow \\
H^1_c(X', \mathbb{G}_m)/nH^1_c(X', \mathbb{G}_m) & \hookrightarrow & H^2_c(X', \mu_n).
\end{array}$$

The upper-hand map is an isomorphism since $H^2_D(\overline{X}', \mu_n)$ is a one-dimensional free $\mathbb{Z}/n\mathbb{Z}$-module, generated by the fundamental class $cl(D) \in H^1_D(\overline{X}', \mathbb{G}_m)$ [SGA4h, Cycle, p. 10]. This shows that the map $H^2_D(\overline{X}', \mathbb{Q}_\ell(1)) \to H^2_c(X', \mathbb{Q}_\ell(1))$ factors through an injection

$$(\varprojlim_n H^1_c(X', \mathbb{G}_m)/nH^1_c(X', \mathbb{G}_m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow H^2_c(X', \mathbb{Q}_\ell(1)).$$

Next, note that we have the following isomorphism:

$$(8.7.1) \qquad (\varprojlim_n H^1_c(X', \mathbb{G}_m)/nH^1_c(X', \mathbb{G}_m)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong NS(\overline{X}', Z') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

where we recall that $Z' = \overline{X}' - X$. Indeed, by definition we have an exact sequence

$$0 \to \mathscr{O}^*(Z')/\mathscr{O}^*(\overline{X}') \to H^1_c(X', \mathbb{G}_m) \to \mathrm{Pic}(X, Z') \to 0.$$

Since multiplication by n is surjective on $\mathscr{O}^*(Z')/\mathscr{O}^*(\overline{X}')$, we get

$$H^1_c(X', \mathbb{G}_m)/nH^1_c(X', \mathbb{G}_m) \cong \mathrm{Pic}(\overline{X}', Z')/n\mathrm{Pic}(\overline{X}', Z').$$

Therefore, the isomorphism (8.7.1) follows from (4.10.1).

8.8. The above calculation shows that for any proper component $D \in \mathrm{Prop}(S')$, we have a commutative diagram

$$(8.8.1) \qquad \begin{array}{ccc}
\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \xrightarrow{1 \mapsto [D]} & NS(\overline{X}', Z') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \\
\downarrow \cong & & \downarrow \\
H^2_D(\overline{X}', \mathbb{Q}_\ell(1)) & \xrightarrow{8.6.2} & H^2_c(X', \mathbb{Q}_\ell(1))
\end{array}$$

where the upper map sends 1 to the class $[D] \in NS(\overline{X}', Z')$ and the lower map is (8.6.2).

8.9. Now recall that Poincaré duality gives an isomorphism $H^2(S', \mathbb{Q}_\ell(1)) \cong \mathbb{Z}^{P(S)} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$. Therefore taking the direct sum of the maps in the diagram (8.8.1) and using the Poincaré duality isomorphism $H^2_c(X, \mathbb{Q}_\ell(1)) \cong H^2(X', \mathbb{Q}_\ell(1))^\vee$, we get a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^{P(S')} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \longrightarrow & NS(\overline{X}', Z') \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \\ \downarrow \cong & & \downarrow \\ H^2(S', \mathbb{Q}_\ell(1)^\vee & \longrightarrow & H^2(X', \mathbb{Q}_\ell(1))^\vee. \end{array}$$

Combining this with the desciption in 8.5 of the map $H^2(S', \mathbb{Q}_\ell(1)^\vee \to H^2(S, \mathbb{Q}_\ell(1))^\vee$ in terms of pushforward of Weil divisors, we see that

$$C_\ell(1)^\vee := \mathrm{Ker}(H^2(S', \mathbb{Q}_\ell(1))^\vee \to H^2(X', \mathbb{Q}_\ell(1))^\vee \oplus H^2(S, \mathbb{Q}_\ell(1)^\vee)$$

is isomorphic to

$$B_{S,0} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell,$$

where $B_{S,0}$ consists of 1-dimensional cycles in X' supported in S' which

(1) map to 0 in $NS(\overline{X}', Z')$ and
(2) map to 0 in S, under proper pushforward of Weil divisors.

Thus $C_\ell(1)^\vee$ is a $\mathbb{Q}_\ell$-vector space of dimension independent of ℓ, with Frobenius acting by pulling back divisors in B_S via F. Thus $C_\ell(1)^\vee$ and hence also C_ℓ have Frobenius characteristic polynomials with rational coefficients independent of ℓ. Using the exact sequence (8.5.1) we then get that $(H^3(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is a compatible system, thereby completing the proof of (8.1). □

9. Independence for $IH^*(X, \mathbb{Q}_\ell)$

9.1. Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$. Recall the definition of the intersection complex IC_{X_0}: let U_0 be any open dense subset of the the smooth locus of X_0 and Z_0 its complement, so we have maps $U_0 \overset{j_0}{\hookrightarrow} X_0 \overset{i_0}{\hookleftarrow} Z_0$. For any perverse sheaf $B \in Perv(U_0)$, the intermediate extension $j_{0!*}B \in Perv(X_0)$ is characterized [KW, III.5.1(3)] as the unique perverse sheaf on X such that:

(1) $j_0^*(j_{0!*}B) \cong B$.
(2) ${}^p\mathscr{H}^0(i_0^* j_{0!*}B) = {}^p\mathscr{H}^0(i_0^! j_{0!*}B) = 0$.

Suppose that U_0 is a disjoint union $U_0 = \coprod_i U_0^i$, with U_0^i of pure dimension d_i. The intersection complex IC_{X_0} is then defined to be $j_{0!*}Q$, where $Q|_{U_0^i} = \mathbb{Q}_{\ell, U_0^i}[d_i]$. Fix an algebraic closure k of $\mathbb{F}_q$, and define $X = X_0 \times_{\mathbb{F}_q} k$. Let $\pi : X \to X_0$ be the projection. Then the intersection complex IC_X on X satisfies $IC_X = \pi^* IC_{X_0}$. We define

$$IH^i(X, \mathbb{Q}_\ell) = H^i(X, IC_X[-d]),$$

where $d = \dim X$. Since $IC_X = \pi^* IC_{X_0}$, the geometric Frobenius endomorphism acts naturally on $IH^i(X, \mathbb{Q}_\ell)$.

Our goal in this section is to prove the following:

Theorem 9.2. *Let X_0 be a 2-dimensional separated scheme of finite type over $\mathbb{F}_q$, and let $X = X_0 \times_{\mathbb{F}_q} k$. Then for each i the system $(IH^i(X, \mathbb{Q}_\ell))_{\ell \neq p}$ is compatible.*

Remark 9.3. In the case of proper equidimensional surfaces this result is a special case of a theorem of Gabber [Fuj, Thm 1].

The proof occupies the remainder of this section. First note that we can reduce to the case that X_0 is normal using the following:

Proposition 9.4. *Let X be a separated scheme of finite type over a field k, and let $\pi : X' \to X$ be its normalization, so we have a commutative diagram*

$$\begin{array}{ccccc} U & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \| & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z, \end{array}$$

where U is the smooth locus of X, Z' is its complement in X', and Z is its complement in X. Then for any perverse sheaf F on U we have $\pi_* j'_{!*}F \simeq j_{!*}F$. In particular $\pi_* IC_{X'} \simeq IC_X$.

Proof. First we note that $\pi_* j'_{!*}F$ is a perverse sheaf, since $j'_{!*}F$ is a perverse sheaf and π_* sends perverse sheaves to perverse sheaves (since it is exact for the perverse t-structure [KW, III-6.1]). We verify that that $\pi_* j'_{!*}F$ satisfies conditions (1), (2).

Condition (1) is immediate since $\pi|_U$ is an isomorphism and both $j'_{!*}F$ and $j_{!*}F$ restrict to F over U.

To verify condition (2), we see that since π_* is exact for the perverse t-structure we have

$${}^p\mathscr{H}^0(i^*\pi_* j'_{!*}F) \simeq {}^p\mathscr{H}^0(\pi_* i'^* j'_{!*}F) \simeq \pi_* {}^p\mathscr{H}^0(i'^* j'_{!*}F) = 0,$$

$${}^p\mathscr{H}^0(i^!\pi_* j'_{!*}F) \simeq {}^p\mathscr{H}^0(\pi_* i'^! j'_{!*}F) \simeq \pi_* {}^p\mathscr{H}^0(i'^! j'_{!*}F) = 0.$$

□

Therefore if X_0 is a separated finite type $\mathbb{F}_q$-scheme and $\pi_0 : X'_0 \to X_0$ is its normalization, we can identify $\pi_{0*} IC_{X'_0}$ and IC_{X_0}. Base changing to the algebraic closure, we get an isomorphism $\pi_* IC_{X'} \simeq IC_X$ and an isomorphism $IH^*(X, \mathbb{Q}_\ell) \simeq IH^*(X', \mathbb{Q}_\ell)$ respecting the action of Frobenius. Therefore it suffices to prove 9.2 in the case when X_0 is normal, which we assume henceforth.

9.5. Let X_0 be a normal, 2-dimensional separated scheme of finite type over $\mathbb{F}_q$. To prove independence of ℓ for the intersection cohomology of X_0 we can assume that X_0 is connected, since $IH^*(X, \mathbb{Q}_\ell)$ splits into a direct sum of terms coming from each connected component of X_0, and the Frobenius action on $IH^*(X, \mathbb{Q}_\ell)$ leaves these summands invariant. So we assume X_0 is connected. Since X_0 is normal and $\mathbb{F}_q$ is perfect, X_0 is geometrically normal, i.e., X is normal. Therefore in particular the connected components of X are equidimensional. Since X_0 is assumed to be connected, the Galois group $Gal(k/\mathbb{F}_q)$ acts transitively on the connected components of X, which implies that X is equidimensional of dimension 2.

9.6. Thus we may assume that X_0 is connected and its geometric irreducible components have dimension 2; hence $IC_{X_0} = j_{0!*}(\mathbb{Q}_{\ell,U_0}[2])$ where $j_0 : U_0 \hookrightarrow X_0$ is the inclusion of the smooth locus. By [BBD, 2.1.11], $IC_{X_0} = \tau_{\leq -1} j_{0*}(\mathbb{Q}_{\ell,U_0}[2])$, where τ denotes truncation with respect to the standard t-structure on $D^b_c(X_0, \mathbb{Q}_\ell)$. Therefore we have

$$IH^*(X, \mathbb{Q}_\ell) \simeq H^*(X, \tau_{\leq 1} j_* \mathbb{Q}_{\ell,U})$$

(after taking account of the shift in the definition of intersection cohomology).

9.7. Next we want to describe the complex $\tau_{\leq 1} j_* \mathbb{Q}_{\ell,U}$ in terms of a resolution of singularities for X_0. This is very similar to the argument in [W 1, Thm 1.1] computing the intersection cohomology of a proper surface over $\mathbb{C}$. Let $X_0' \to X_0$ be a resolution of singularities of X_0, by which we mean (as usual) a commutative diagram

$$\begin{array}{ccccc} U_0 & \xrightarrow{j_0'} & X_0' & \xleftarrow{i_0'} & S_0' \\ \| & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0|_{S_0'}} \\ U_0 & \xrightarrow{j_0} & X_0 & \xleftarrow{i_0} & S_0 \end{array}$$

where U_0 is the smooth locus of X_0, X_0' is smooth, S_0' is a simple normal crossings divisor on X_0', and π_0 is proper birational and an isomorphism over U_0. Now consider the exact triangle in $D^b_c(X', \mathbb{Q}_\ell)$

$$i'_* i'^! \mathbb{Q}_{\ell,X'} \to \mathbb{Q}_{\ell,X'} \to j'_* \mathbb{Q}_{\ell,U} \to i'_* i'^! \mathbb{Q}_{\ell,X'}[1]$$

and apply π_*. Using commutativity of the diagrams above defining X_0' and S_0', we get an exact triangle

$$i_*(\pi|_{S'})_* i'^! \mathbb{Q}_{\ell,X'} \to \pi_* \mathbb{Q}_{\ell,X'} \to j_* \mathbb{Q}_{\ell,U} \to i_*(\pi|_{S'})_* i'^! \mathbb{Q}_{\ell,X'}[1] \tag{9.7.1}$$

On applying truncation, we get an exact triangle

$$G_\ell \to \tau_{\leq 1} \pi_* \mathbb{Q}_{\ell,X'} \to \tau_{\leq 1} j_* \mathbb{Q}_{\ell,U} \to G_\ell[1].$$

where $G_\ell[-1]$ is defined to be the cone of the map $\tau_{\leq 1} \pi_* \mathbb{Q}_{\ell,X'} \to \tau_{\leq 1} j_* \mathbb{Q}_{\ell,U}$. We claim that G_ℓ is in fact isomorphic to a sheaf sitting in degree 2: by definition of truncation, G_ℓ has no cohomology sheaves in dimensions ≥ 3, while it has no cohomology sheaves in degrees 0 or 1 since by purity $i'^! \mathbb{Q}_{\ell,X'}$ has no cohomology sheaves in degrees 0 or 1. Therefore the only non-zero cohomology sheaf of G_ℓ is in degree 2, implying that G_ℓ is isomorphic in $D^b_c(X, \mathbb{Q}_\ell)$ to its second-degree cohomology sheaf $\mathscr{H}^2(G_\ell)$ sitting in degree 2. If we identify G_ℓ with this cohomology sheaf sitting in degree 2, we have an exact sequence

$$0 \to R^1 \pi_* \mathbb{Q}_{\ell,X'} \to R^1 j_* \mathbb{Q}_{\ell,U} \to G_\ell \to 0,$$

which (on comparing with the long exact sequence in cohomology obtained from 9.7.1) implies that

$$G_\ell[2] \cong \ker(i_* R^2((\pi|_{S'})_* i'^!) \mathbb{Q}_{\ell,X'} \to R^2 \pi^* \mathbb{Q}_{\ell,X'})$$

(Here we identify $\mathscr{H}^2(G_\ell)$ with $G_\ell[2]$). One can simplify this description as follows. Let $S' = \cup_{i \in I} S_i$ be the decomposition of the simple normal crossings divisor S' into its smooth components. Then by the purity theorem

$$R^2 i'^! \mathbb{Q}_{\ell,X'} \cong \bigoplus_{i \in I} \mathbb{Z}_{S_i} \otimes \mathbb{Q}_\ell(-1),$$

where $\mathbb{Z}_{S_i}$ denotes the constant sheaf with group $\mathbb{Z}$ on the closed subset S_i. Moreover, $R^j i'^! \mathbb{Q}_{\ell,X'} = 0$ for $j = 0, 1$. Therefore the spectral sequence

$$R^p \pi_* R^q i'^! \mathbb{Q}_{\ell,X'} \Rightarrow R^{p+q}(\pi_* i'^!) \mathbb{Q}_{\ell,X'}$$

implies that

$$R^2(\pi_* i'^!)\mathbb{Q}_{\ell,X'} \simeq \pi_* R^2 i'^! \mathbb{Q}_{\ell,X'} \simeq \bigoplus_{i\in I} \pi_*(\mathbb{Z}_{S_i} \otimes \mathbb{Q}_\ell(-1)). \tag{9.7.2}$$

9.8. Now we use the fact that X' is a normal surface, so that S_0 (and hence $S := S_0 \times_{\mathbb{F}_q} k$) is a finite point set. Then we have

$$\Gamma(X, G_\ell[2]) \cong \bigoplus_{s\in S} (G_\ell)_s.$$

This allows us to interpret $\Gamma(X, G_\ell[2](1))$ in terms of the intersection matrix of the divisors S_i as follows. Let $s \in S$ be a point. Then $\pi^{-1}(s)$ is a union of smooth divisors S_i contained in S'; let $I_s \subset I$ be the subset of I such that $\pi^{-1}(s) = \cup_{i\in I_s} S_i$. Then using 9.7.2 we have

$$(R^2(\pi_* i'^!)\mathbb{Q}_{\ell,X'}(1))_s \cong \bigoplus_{i\in I_s} \mathbb{Z} \otimes \mathbb{Q}_\ell,$$

and by the base change theorem for proper morphisms we have

$$(R^2\pi_*\mathbb{Q}_{\ell,X'}(1))_s \cong H^2(\pi^{-1}(s), \mathbb{Q}_\ell(1)) \cong \bigoplus_{i\in I_s} H^2(S_i, \mathbb{Q}_\ell(1)).$$

Moreover, the map $(R^2(\pi^* i'^!)\mathbb{Q}_{\ell,X'}(1))_s \to (R^2\pi_*\mathbb{Q}_{\ell,X'}(1))_s$ corresponds to the map

$$\bigoplus_{i\in I_s} \mathbb{Z} \otimes \mathbb{Q}_\ell \xrightarrow{cycle} H^2(X, \mathbb{Q}_\ell(1)) \xrightarrow{restr} \bigoplus_{i\in I_s} H^2(S_i, \mathbb{Q}_\ell(1)).$$

More concretely, if we let $[S_i] \in H^2(S_i, \mathbb{Q}_\ell(1))$ denote the fundamental class of S_i, and let $e_i \in \oplus_{i\in I}\mathbb{Z} \otimes \mathbb{Q}_\ell$ the basis element corresponding to S_i, then this map is defined by

$$e_i \mapsto \sum_j (S_i \cap S_j)[S_j],$$

with the intersection taking place in X' (or any open subset of X' containing both S_i and S_j). This is simply the intersection matrix of the divisors S_i, so $\Gamma(X, G_\ell[2](1))$ can be interpreted as the kernel of the intersection matrix (tensored with $\mathbb{Q}_\ell$) on the divisors $S_i \in S'$. In particular, we have the following:

Lemma 9.9. *The system $(\Gamma(X, G_\ell[2])_{\ell\neq p}$ is compatible.*

Remark 9.10. If X is projective then the intersection matrix of the divisors in S' is known to be non-degenerate [Mum, p.6], so $G_\ell = 0$ in this case.

9.11. Now return to the exact triangle

$$G_\ell \to \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'} \to \tau_{\leq 1} j_*\mathbb{Q}_{\ell,X'} \to G_\ell[1].$$

Recall that $IH^i(X, \mathbb{Q}_\ell) = H^i(X, \tau_{\leq 1} j_*\mathbb{Q}_{\ell,X'})$. We just showed that $(\Gamma(X, G_\ell[2]))_{\ell\neq p}$ is a compatible system, and this is the only non-zero cohomology group of G_ℓ since G_ℓ is supported on the finite point set S. Therefore by Lemma 2.6, to prove that $(IH^i(X, \mathbb{Q}_\ell))_{\ell\neq p}$ is a compatible system, it suffices to show that the actions of Frobenius on

(1) $H^i(X, \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'})$, and
(2) $\ker(\Gamma(X, G_\ell[2]) \to H^2(X, \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'}))$

have characteristic polynomial independent of ℓ (for all i in the first case). We start with (2). By the description of G_ℓ in (9.7) we have an injection

$$G_\ell[2] \hookrightarrow \oplus_{i\in I}\pi_*(\mathbb{Z}_{S_i} \otimes \mathbb{Q}_\ell(-1)).$$

Let C_ℓ be the cokernel of this injection. Then we have a commutative diagram in $D^b_c(X, \mathbb{Q}_\ell)$

$$\begin{array}{ccccccc} G_\ell & \longrightarrow & \oplus_{i\in I}\pi_*(\mathbb{Z}_{S_i} \otimes \mathbb{Q}_\ell(-1) & \longrightarrow & C_\ell & \longrightarrow & G_\ell[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'} & \longrightarrow & \pi_*\mathbb{Q}_{\ell,X'} & \longrightarrow & \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'} & \longrightarrow & \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'} \end{array}$$

where the rows are exact triangles. Taking cohomology gives a commutative diagram

$$\begin{array}{ccc} \Gamma(X, G_\ell[2]) & \hookrightarrow & \mathbb{Z}^I \otimes \mathbb{Q}_\ell(-1) \\ \downarrow & & \downarrow \\ H^2(X, \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'}) & \hookrightarrow & H^2(X', \mathbb{Q}_\ell), \end{array}$$

where the rows are injections because C_ℓ and $\tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}$ are complexes concentrated in degrees ≥ 2. This implies that $\ker(\Gamma(X, G_\ell[2]) \to H^2(X, \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'})$ can be described as follows. Let $B \subset \mathbb{Z}_I$ consist of the cycles on X' supported in $S' = \cup_{i\in I} S_i$ such that

- $b \in B$ is in the kernel of the intersection pairing $\mathbb{Z}^I \to \mathbb{Z}^I$ (so it defines an element of $G_\ell[2](1)$), and
- $cl(b) = 0$ in $H^2(X', \mathbb{Q}_\ell(1))$, which is equivalent to $cl(b) = 0$ in $NS(X')$ by 4.10.

Then

$$\ker(\Gamma(X, G_\ell[2]) \to H^2(X, \tau_{\leq 1}\pi_*\mathbb{Q}_\ell)) = B \otimes_{\mathbb{Z}} \mathbb{Q}_\ell(-1).$$

This shows that this kernel has Frobenius action independent of ℓ.

9.12. Therefore we are left with showing that the Frobenius action on $H^i(X, \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'})$ has characteristic polynomial independent of ℓ for each i. Start with the truncation exact triangle

$$\tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'} \to \pi_*\mathbb{Q}_{\ell,X'} \to \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'} \to \tau_{\leq 1}\pi_*\mathbb{Q}_{\ell,X'}[1].$$

By Lemma 2.6, it suffices to show that the actions of Frobenius on

(a) $H^i(X, \pi_*\mathbb{Q}_{\ell,X'}) = H^i(X', \mathbb{Q}_\ell)$,
(b) $H^i(X, \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'})$, and
(c) $\ker(H^i(X', \mathbb{Q}_\ell) \to H^i(X, \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}))$

all have characteristic polynomials with rational coefficients independent of ℓ. For (a) this follows since X' is smooth. For (b), note that $\tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}$ is supported on the finite point set $Z \subset X$. Moreover, by proper base change $\pi_*\mathbb{Q}_{\ell,X'}$ has non-zero cohomology sheaves only in degrees 0, 1 and 2, so $\tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}$ is the sheaf $R^2\pi_*\mathbb{Q}_{\ell,X'}$ sitting in degree 2. Therefore $H^i(X, \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}) = 0$ for $i \neq 2$, while (again by proper base change)

$$H^2(X, \tau_{\geq 2}\pi_*\mathbb{Q}_{\ell,X'}) = \bigoplus_{s\in S}(R^2\pi_*\mathbb{Q}_{\ell,X'})_s = H^2(S', \mathbb{Q}_\ell).$$

This proves (b). Moreover, we see that $\ker(H^i(X', \mathbb{Q}_\ell) \to H^i(X, \tau_{\geq 2}\pi_*\mathbb{Q}_\ell)) = H^i(X', \mathbb{Q}_\ell)$ for $i \neq 2$, while

$$\ker(H^2(X', \mathbb{Q}_\ell) \to H^2(X, \tau_{\geq 2}\pi_*\mathbb{Q}_\ell)) = \ker(H^2(X', \mathbb{Q}_\ell) \to H^2(S', \mathbb{Q}_\ell)).$$

So to show (c) has Frobenius characteristic polynomial independent of ℓ it suffices to show that the kernel of the map $H^2(X', \mathbb{Q}_\ell) \to H^2(S', \mathbb{Q}_\ell)$ induced by $S' \hookrightarrow X'$ has Frobenius characteristic polynomial independent of ℓ. Recalling that U is the complement $X' \setminus S'$ and $j' : U \hookrightarrow X'$ is the inclusion, we have an exact triangle

$$j'_!\mathbb{Q}_{\ell,U} \to \mathbb{Q}_{\ell,X'} \to i'_*\mathbb{Q}_{\ell,S'} \to j'_!\mathbb{Q}_{\ell,U}[1].$$

Notice that the map $H^2(X', \mathbb{Q}_\ell) \to H^2(S', \mathbb{Q}_\ell)$ appears as one of the maps in the long exact sequence of cohomology associated to this triangle. Therefore, by Lemma 2.6, to show that $\ker(H^2(X', \mathbb{Q}_\ell) \to H^2(S', \mathbb{Q}_\ell))$ has Frobenius characteristic polynomial independent of ℓ it suffices to show that $H^i(X, j_!\mathbb{Q}_{\ell,U})$, $H^i(X', \mathbb{Q}_\ell)$ and $H^i(S', \mathbb{Q}_\ell)$ all have characteristic polynomials independent of ℓ for each i. This is clear for X' and S since X' is smooth and S is one-dimensional. If we compactify X' to a proper smoooth surface $\overline{X'}$ with complement a normal crossings divisor E, then the cohomology of $j'_!\mathbb{Q}_{\ell,U}$ is $H^i_{D,E}(\overline{X'}, \mathbb{Q}_\ell)$, which was shown to form a compatible system for each i in Theorem 5.1. Thus we conclude that (c) has Frobenius characteristic polynomial independent of ℓ.

This completes the proof of Theorem 9.2.

10. Proof of theorem 1.8

As in 1.8, let k be a separably closed field and X/k a smooth 2-dimensional k-scheme. Let p denote the characteristic of k. In what follows ℓ denotes a prime not equal to p.

Lemma 10.1. *Suppose $H^3(X, \mathbb{Z}_\ell(1))_{\mathrm{tors}} = 0$. Then for any $n \geq 1$, if $x \in \mathrm{Br}(X)[\ell^n]$, then x is ℓ-divisible.*

Proof. From the morphism of Kummer sequences

(10.1.1)
$$\begin{CD}
0 @>>> \mu_{\ell^{n+1}} @>>> \mathbb{G}_m @>{\cdot\ell^{n+1}}>> \mathbb{G}_m @>>> 0\\
@. @VV{\cdot\ell}V @VV{\cdot\ell}V @VV{\mathrm{id}}V @.\\
0 @>>> \mu_{\ell^n} @>>> \mathbb{G}_m @>{\cdot\ell^{n}}>> \mathbb{G}_m @>>> 0,
\end{CD}$$

we get a commutative diagram

$$\begin{CD}
H^2(X, \mu_{\ell^{n+1}}) @>>> \mathrm{Br}(X)[\ell^{n+1}] @>>> 0\\
@VVV @VV{\cdot\ell}V @.\\
H^2(X, \mu_{\ell^n}) @>>> \mathrm{Br}(X)[\ell^n] @>>> 0.
\end{CD}$$

It therefore suffices to show that the map

$$H^2(X, \mu_{\ell^{n+1}}) \to H^2(X, \mu_{\ell^n})$$

is surjective. This follows from noting that in fact the map

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n})$$

is surjective. Indeed from the short exact sequence

$$0 \longrightarrow \mathbb{Z}_\ell(1) \xrightarrow{\cdot \ell^n} \mathbb{Z}_\ell(1) \longrightarrow \mu_{\ell^n} \longrightarrow 0$$

we get an exact sequence

$$H^2(X, \mathbb{Z}_\ell(1)) \to H^2(X, \mu_{\ell^n}) \to H^3(X, \mathbb{Z}_\ell(1))_{\text{tors}},$$

and

$$H^3(X, \mathbb{Z}_\ell(1))_{\text{tors}} = 0$$

by assumption. □

Lemma 10.2. *Poincare duality defines an isomorphism*

$$\text{Ext}^1_{\mathbb{Z}_\ell}(H^3(X, \mathbb{Z}_\ell(1)), \mathbb{Z}_\ell) \simeq H^2_c(X, \mathbb{Z}_\ell(1))_{\text{tors}}.$$

In particular, the group $H^2_c(X, \mathbb{Z}_\ell(1))$ is torsion free if and only if the group $H^3(X, \mathbb{Z}_\ell(1))$ is torsion free.

Proof. To ease notation, write simply K for the complex $R\Gamma(X, \mathbb{Z}_\ell(1))$. Poincare duality defines an isomorphism

$$R\mathcal{H}om(K, \mathbb{Z}_\ell) \simeq R\Gamma_c(X, \mathbb{Z}_\ell(1))[4].$$

In particular we have

$$H^2_c(X, \mathbb{Z}_\ell(1)) \simeq \text{Ext}^{-2}(K, \mathbb{Z}_\ell).$$

From the distinguished triangle

$$\tau_{\leq 1} K \to K \to \tau_{\geq 2} K$$

we obtain a long exact sequence

$$\cdots \to \text{Ext}^{-3}(\tau_{\leq 1} K, \mathbb{Z}_\ell) \to \text{Ext}^{-2}(\tau_{\geq 2} K, \mathbb{Z}_\ell) \to \text{Ext}^{-2}(K, \mathbb{Z}_\ell) \to \text{Ext}^{-2}(\tau_{\leq 1} K, \mathbb{Z}_\ell) \to \cdots .$$

The complex

$$R\mathcal{H}om(\tau_{\leq 1} K, \mathbb{Z}_\ell)$$

is concentrated in degrees ≥ -1, and therefore

$$\text{Ext}^{-2}(\tau_{\geq 2} K, \mathbb{Z}_\ell) \to \text{Ext}^{-2}(K, \mathbb{Z}_\ell)$$

is an isomorphism. Considering the distinguished triangle

$$\tau_{\leq 3}\tau_{\geq 2} K \to \tau_{\geq 2} K \to H^4(K)[-4]$$

and the fact that (like any $\mathbb{Z}_\ell$-module) $H^4(K)$ has projective dimension ≤ 1, we then get an isomorphism

$$\text{Ext}^{-2}(\tau_{\geq 2} K, \mathbb{Z}_\ell) \simeq \text{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2} K, \mathbb{Z}_\ell).$$

Finally from

$$H^2(K)[-2] \to \tau_{\leq 3}\tau_{\geq 2} K \to H^3(K)[-3]$$

we get an exact sequence

$$\text{Ext}^{-3}(H^2(K)[-2], \mathbb{Z}_\ell) \to \text{Ext}^{-2}(H^3(K)[-3], \mathbb{Z}_\ell) \to \text{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2} K, \mathbb{Z}_\ell) \to \text{Ext}^{-2}(H^2(K)[-2], \mathbb{Z}_\ell).$$

Since

$$R\mathcal{H}om(H^2(K)[-2], \mathbb{Z}_\ell)$$

is concentrated in degrees ≥ -2, this gives an exact sequence

$$0 \to \text{Ext}^1(H^3(K), \mathbb{Z}_\ell) \to \text{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2} K, \mathbb{Z}_\ell) \to \text{Hom}(H^2(K), \mathbb{Z}_\ell).$$

In particular we get an isomorphism

$$\mathrm{Ext}^1(H^3(K), \mathbb{Z}_\ell) \simeq \mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell)_{\mathrm{tors}}.$$

Combining this with the isomorphism

$$\mathrm{Ext}^{-2}(\tau_{\leq 3}\tau_{\geq 2}K, \mathbb{Z}_\ell) \simeq H^2_c(X, \mathbb{Z}_\ell(3))$$

we get the result. □

10.3. The group $H^2_c(X, \mathbb{Z}_\ell(1))$ can be analyzed as follows. Choose a compactification

$$j : X \hookrightarrow \overline{X},$$

where $\overline{X}/k$ is proper and smooth, and the complement $D := \overline{X} - X$ is divisor with simple normal crossings on $\overline{X}$. Let

$$i : D \hookrightarrow \overline{X}$$

be the inclusion, and as in 4.9 set

$$\mathscr{H} := \mathrm{Ker}(\mathbb{G}_{m,\overline{X}} \to i_*\mathbb{G}_{m,D})$$

so that for every n we have an exact sequence (see (4.9.1))

$$0 \longrightarrow j_!\mu_{\ell^n} \longrightarrow \mathscr{H} \xrightarrow{\cdot \ell^n} \mathscr{H} \longrightarrow 0.$$

Taking cohomology over $\overline{X}$ we obtain an exact sequence

$$0 \to H^1(\overline{X}, \mathscr{H})/\ell^n H^1(\overline{X}, \mathscr{H}) \to H^2_c(X, \mu_{\ell^n}) \to H^2(\overline{X}, \mathscr{H})[\ell^n] \to 0. \tag{10.3.1}$$

As in (4.9.2) we have

$$H^1(\overline{X}, \mathscr{H})/\ell^n H^1(\overline{X}, \mathscr{H}) \simeq \mathrm{Pic}(\overline{X}, D)/\ell^n \mathrm{Pic}(\overline{X}, D).$$

Set

$$\widetilde{NS}(\overline{X}, D) := \mathrm{Pic}(\overline{X}, D)/\mathrm{Pic}^{00}(\overline{X}, D).$$

Then $\widetilde{NS}(\overline{X}, D)$ is an extension of $NS(\overline{X}, D)$ by a finite group (namely the component group of $\mathrm{Pic}^0(\overline{X}, D)$), and therefore is finitely generated, and we have

$$\mathrm{Pic}(\overline{X}, D)/\ell^n \mathrm{Pic}(\overline{X}, D) \simeq \widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D).$$

We can therefore rewrite (10.3.1) as

$$0 \to \widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D) \to H^2_c(X, \mu_{\ell^n}) \to H^2(\overline{X}, \mathscr{H})[\ell^n] \to 0. \tag{10.3.2}$$

Passing to the inverse limit with respect to the morphisms obtained from the diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & j_!\mu_{\ell^{n+1}} & \longrightarrow & \mathscr{H} & \xrightarrow{\cdot \ell^{n+1}} & \mathscr{H} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cdot \ell} & & \downarrow{\scriptstyle \cdot \ell} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & j_!\mu_{\ell^n} & \longrightarrow & \mathscr{H} & \xrightarrow{\cdot \ell^n} & \mathscr{H} & \longrightarrow & 0
\end{array}$$

we get an exact sequence

$$0 \to \widetilde{NS}(\overline{X}, D) \otimes \mathbb{Z}_\ell \to H^2_c(X, \mathbb{Z}_\ell(1)) \to T_\ell H^2(\overline{X}, \mathscr{H}) \to 0.$$

Here the exactness on the right follows from the fact that the system

$$\{\widetilde{NS}(\overline{X}, D)/\ell^n \widetilde{NS}(\overline{X}, D)\}$$

satisfies the Mittag-Leffler condition.

Since $T_\ell H^2(\overline{X}, \mathscr{H})$ is ℓ-torsion free, we get an isomorphism

$$(\widetilde{NS}(\overline{X}, D) \otimes \mathbb{Z}_\ell)_{\text{tors}} \simeq H^2_c(X, \mathbb{Z}_\ell(1))_{\text{tors}}.$$

Since $\widetilde{NS}(\overline{X}, D)$ is a finitely generated abelian group, this yields the following corollary:

Corollary 10.4. *For all but finitely many primes ℓ, the group $H^2_c(X, \mathbb{Z}_\ell(1))$ is torsion free.*

10.5. Let $\text{Br}(X)(\ell) \subset \text{Br}(X)$ (resp. $\text{Br}(X)_{\text{div}}(\ell) \subset \text{Br}(X)_{\text{div}}$) be the subgroup of elements whose order is a power of ℓ.

Notice that for an element $x \in \text{Br}(X)(\ell)$, the condition of being divisible is equivalent to the condition of being ℓ-divisible.

Combining 10.1, 10.2, and 10.4 we get that for all but a finite number of primes ℓ we have

$$\text{Br}(X)_{\text{div}}(\ell) = \text{Br}(X)(\ell).$$

Set

$$W_\ell := \text{Br}(X)(\ell)/\text{Br}(X)_{\text{div}}(\ell).$$

Since the quotient

$$\widetilde{\text{Br}}(X)/\widetilde{\text{Br}}(X)_{\text{div}} \tag{10.5.1}$$

is isomorphic to the group

$$\oplus_{\ell \neq p} W_\ell,$$

to prove that 10.5.1 is finite it suffices to show that each of the groups W_ℓ is finite.

For this notice that from the Kummer sequence

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\cdot \ell^n} \mathbb{G}_m \longrightarrow 0$$

we obtain compatible maps

$$\text{Br}(X) \to H^3(X, \mu_{\ell^n})$$

and therefore a map

$$\text{Br}(X) \to H^3(X, \mathbb{Z}_\ell(1)).$$

The kernel of this map is precisely the ℓ-divisible elements of $\text{Br}(X)$, so we get from this an inclusion

$$W_\ell \hookrightarrow H^3(X, \mathbb{Z}_\ell(1)).$$

Since $H^3(X, \mathbb{Z}_\ell(1))$ is a finitely generated $\mathbb{Z}_\ell$-module this implies that W_ℓ is finite.

10.6. It remains to analyze the divisible part $\widetilde{\text{Br}}(X)_{\text{div}}$. Let S_ℓ denote the ℓ-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$. We have

$$F_p \simeq \oplus_{\ell \neq p} S_\ell,$$

where F_p is defined as in 1.8. Moreover, we have

$$\widetilde{\text{Br}}(X)_{\text{div}} \simeq \oplus_{\ell \neq p} \text{Br}(X)_{\text{div}}(\ell).$$

Therefore it suffices to show that there exists an integer r such that for every $\ell \neq p$ we have

$$S_\ell^{\oplus r} \simeq \text{Br}(X)_{\text{div}}(\ell).$$

Lemma 10.7. *Let A be a divisible ℓ-torsion group, and assume that $A[\ell]$ is a finite dimensional $\mathbb{F}_\ell$-vector space of some rank r. Then*
$$A \simeq S_\ell^{\oplus r}.$$

Proof. Choose a basis $x_1, \dots, x_r \in A[\ell]$ for $A[\ell]$ as a $\mathbb{F}_\ell$-vector space, and for each $1 \leq i \leq r$ choose a sequence of elements $\{x_i^{(n)}\}_{n \geq 1}$ with
$$x_i^{(1)} = x_i, \quad \ell x_i^{(n)} = x_i^{(n-1)}.$$
This is possible since A is divisible. Let
$$\pi_i : S_\ell \to A$$
be the map sending $1/\ell^n$ to $x_i^{(n)}$, and let
$$\pi : S_\ell^{\oplus r} \to A$$
be the direct sum of the π_i. We claim that π is an isomorphism. Since both sides are torsion groups, to verify that π is an isomorphism it suffices to show that π induces an isomorphism on ℓ^n-torsion for every n. This we show by induction on n. For $n = 1$ the result is immediate. For the inductive step consider the commutative diagrams
$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_\ell^{\oplus r}[\ell] & \longrightarrow & S_\ell^{\oplus r}[\ell^n] & \xrightarrow{\cdot \ell} & S_\ell^{\oplus r}[\ell^{n-1}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A[\ell] & \longrightarrow & A[\ell^n] & \xrightarrow{\cdot \ell} & A[\ell^{n-1}] & \longrightarrow & 0. \end{array}$$

□

10.8. Now observe that from the Kummer sequence we have a surjection
$$H^2(X, \mu_\ell) \to \mathrm{Br}(X)[\ell],$$
and therefore $\mathrm{Br}(X)[\ell]$ is a finite dimensional $\mathbb{F}_\ell$-vector space. Hence $\mathrm{Br}(X)_{\mathrm{div}}[\ell]$ is also a finite dimensional $\mathbb{F}_\ell$-vector space. It follows that there exists an integer r_ℓ such that
$$\mathrm{Br}(X)_{\mathrm{div}}(\ell) \simeq S_\ell^{\oplus r_\ell}.$$

10.9. We now show that all the r_ℓ are equal.

Let $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the subgroup of divisible elements, and set
$$NS(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X).$$
Notice that if $X \hookrightarrow \overline{X}$ is a compactification, with $\overline{X}/k$ a smooth proper surface, then we have a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}^0(\overline{X}) & \longrightarrow & \mathrm{Pic}(\overline{X}) & \longrightarrow & NS(\overline{X}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & NS(X) & \longrightarrow & 0. \end{array}$$
Since $\overline{X}$ is smooth, the restriction map
$$\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(X)$$

is surjective. Therefore $NS(X)$ is a quotient of $NS(\overline{X})$ and in particular is a finitely generated abelian group.

Note also that since $\mathrm{Pic}^0(X)$ is a divisible group, we have

$$\mathrm{Pic}(X)/\ell^n\mathrm{Pic}(X) \simeq NS(X)/\ell^n NS(X).$$

From the Kummer sequences

$$0 \longrightarrow \mu_{\ell^n} \longrightarrow \mathbb{G}_m \xrightarrow{\cdot \ell^n} \mathbb{G}_m \longrightarrow 0$$

we get short exact sequences

$$0 \to NS(X)/\ell^n NS(X) \to H^2(X, \mu_{\ell^n}) \to \mathrm{Br}(X)[\ell^n] \to 0,$$

which upon passing to the inverse limit gives a short exact sequence

$$0 \to NS(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)) \to T_\ell \mathrm{Br}(X) \to 0.$$

Since

$$T_\ell \mathrm{Br}(X) \simeq T_\ell \mathrm{Br}(X)_{\mathrm{div}}(\ell) \simeq \mathbb{Z}_\ell^{r_\ell},$$

we conclude that

$$r_\ell = \dim_{\mathbb{Q}_\ell} H^2(X, \mathbb{Q}_\ell) - \dim_{\mathbb{Q}_\ell}(NS(X) \otimes \mathbb{Q}_\ell),$$

which is independent of ℓ by 1.2. This completes the proof of 1.8. □

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