A STACKY SEMI–STABLE REDUCTION THEOREM

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ABSTRACT. We prove a stack–theoretic version of the semi–stable reduction theorem of Kempf, Knudsen, Mumford, and Saint–Donat.

1. INTRODUCTION

1.1. Let R be a discrete valuation ring of equi–characteristic 0, and suppose X/R is a proper regular scheme with smooth generic fiber for which the reduced closed fiber $X_{0,\mathrm{red}} \subset X$ is a divisor with normal crossings. The classical semi–stable reduction theorem of Mumford et. al. ([2], p. 198) asserts that after a making a finite extension R'/R, one can find a proper semi–stable scheme X'/R' with generic fiber equal to that of $X \otimes_R R'$. Here by "semi–stable," we mean that if $\pi' \in R'$ is a uniformizer, then étale locally on X' there exists a smooth map

$$X' \longrightarrow \mathrm{Spec}(R'[X_1, \ldots, X_r]/(X_1 \cdots X_r - \pi'), \tag{1.1.1}$$

for some r.

The purpose of this note is to show how one can obtain a stack–theoretic version of this theorem using a simple argument based on the techniques of ([5]), rather than the theory of toroidal embeddings developed in ([2]). The main result is the following:

Theorem 1.2. *Suppose R is a complete discrete valuation ring of equi–characteristic 0 and residue field K, and suppose X/R is a proper regular scheme with smooth generic fiber X_K and reduced closed fiber $X_{0,\mathrm{red}} \subset X$ a divisor with normal crossings. Then there exists a finite Galois extension K'/K and a proper semi–stable Deligne–Mumford stack $\mathcal{X}'$ over the integral closure R' of R in K' such that the following hold:*

(i) *The coarse moduli space of $\mathcal{X}'$ is equal to $X \otimes_R R'$, and the map $\mathcal{X}' \to X \otimes_R R'$ is an isomorphism on the generic fiber.*

(ii) *The stack comes equipped with an action of $\mathrm{Gal}(K'/K)$ compatible with the natural action on $X \otimes_R R'$.*

(iii) *Suppose R_2/R' is another extension of discrete valuation rings, and X_2/R_2 is a proper semi–stable scheme with a map $X_2 \to X \otimes_R R_2$ which is an isomorphism on the generic fiber. Then there is a map $X_2 \to \mathcal{X}'$ over $X \otimes_R R'$, which is unique up to unique isomorphism.*

Our result is of course weaker than ([2], p. 198), in that we only obtain a Deligne–Mumford stack rather than an actual scheme. For many applications, however, this is not a serious drawback. Moreover, the stack $\mathcal{X}'$ has the advantage that it is canonical and comes equipped with the additional structure in (1.2 (ii)) and (1.2 (iii)). In addition, we hope that the reader finds our proof of (1.2) significantly simpler than the proof in ([2]).

The subsequent sections of this paper are devoted to a proof of (1.2). Let us, however, also mention two other results which are proven by the same argument.

Suppose k is an algebraically closed field of characteristic 0, and define a *stacky curve* over k to be a smooth Deligne–Mumford stack $\mathcal{C}/k$ whose coarse moduli space C is a smooth curve, and such that étale locally on C there exists an integer r and a cartesian diagram

(1.2.1)
$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & [\mathbb{A}^1_k/\mu_r] \\ \downarrow & & \downarrow \tau \\ C & \xrightarrow{\epsilon} & \mathbb{A}^1_k, \end{array}$$

where $[\mathbb{A}^1_k/\mu_r]$ is the stack–theoretic quotient of $\mathbb{A}^1$ by the natural action of μ_r, τ is the map induced by the map $\mathbb{A}^1 \to \mathbb{A}^1$ sending a to a^r, and ϵ is étale. Then our argument yields the following:

Theorem 1.3. *Let $f : X \to C$ be a proper morphism between smooth k–schemes, where C is a curve, such that the inverse image in X of any closed point on C is a divisor with normal crossings. Then there exists a commutative diagram*

(1.3.1)
$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\pi_X} & X \\ \tilde{f} \downarrow & & \downarrow f \\ \mathcal{C} & \xrightarrow{\pi_C} & C, \end{array}$$

where $\mathcal{C}$ is a stacky curve, $\tilde{f} : \mathcal{X} \to \mathcal{C}$ is a proper semi–stable morphism between Deligne–Mumford stacks, and the map π_X (resp. π_C) realizes X (resp. C) as the coarse moduli space of $\mathcal{X}$ (resp. $\mathcal{C}$).

The techniques of this paper can also be used in the positive characteristic setting. For this, suppose R is either of characteristic $(0, p)$ or (p, p) with $p > 0$, and let X/R be a proper regular scheme whose reduced closed fiber is a divisor with normal crossings. Let $\mathcal{M}_R$ (resp. $\mathcal{M}_X$) be the log structure in the sense of Fontaine and Illusie ([1]) associated to the closed point (resp. closed fiber), and suppose that the induced morphism of log schemes $(X, \mathcal{M}_X) \to (\mathrm{Spec}(R), \mathcal{M}_R)$ is log smooth in the sense of ([1], 3.3) (Note: the language of log geometry is not used in the proof of (1.2) in the subsequent sections). Then we obtain the following result, which we will use in a subsequent paper about crystalline cohomology ([6]).

Theorem 1.4. *Let K denote the field of fractions of R. Then there exists a finite Galois extension K'/K and a proper Deligne–Mumford stack $\mathcal{X}'$ over the integral closure R' of R in K' such that the following hold:*

(1.4 (i)) *The multiplicities of the irreducible components of the closed fiber of $\mathcal{X}'$ are powers of p.*

(1.4 (ii)) *The coarse moduli space of $\mathcal{X}'$ is equal to $X \otimes_R R'$, and the map $\mathcal{X}' \to X \otimes_R R'$ is an isomorphism on the generic fiber.*

(1.4 (iii)) *The stack comes equipped with an action of* $\mathrm{Gal}(K'/K)$ *compatible with the natural action on $X \otimes_R R'$.*

2. The stack of local models

The key tool in the proof of (1.2) is the construction of a certain algebraic stack, which we describe in this section.

2.1. Fix an integer $r \geq 1$ and a collection $\alpha = (\alpha_1, \ldots, \alpha_r)$ of positive natural numbers. Define

$$U := \mathrm{Spec}(\mathbb{Z}[t][X_1, \ldots, X_r, V^{\pm 1}]/(X_1^{\alpha_1} \cdots X_r^{\alpha_r} V = t)),$$

where $\mathbb{Z}[t]$ is the polynomial ring in one variable. We will often think of U as representing the sheaf on the category of $\mathbb{Z}[t]$–schemes

$$T \mapsto \{(x_1, \ldots, x_r, v) | x_i \in \Gamma(T, \mathcal{O}_T), v \in \Gamma(T, \mathcal{O}_T^*), \text{ such that } x_1^{\alpha_1} \cdots x_r^{\alpha_r} v = t\}.$$

Let $H \subset \mathrm{S}_r$ be a subgroup of the symmetric group on r letters contained in the subgroup of elements $\sigma \in \mathrm{S}_r$ for which $\alpha_{\sigma(i)} = \alpha_i$ for all i, and let G be the semi-direct product $\mathbb{G}_m^r \rtimes H$ with product structure given by

$$(u_1, \ldots, u_r, h) \cdot (u_1', \ldots, u_r', h') = ((u_{h'(i)} u_i')_i, h \circ h').$$

An element $(u, h) \in G$ acts on U by

$$(x, v) \mapsto (u_{h^{-1}(1)} x_{h^{-1}(1)}, \ldots, u_{h^{-1}(r)} x_{h^{-1}(r)}, (\prod_i u_i^{-\alpha_i}) v).$$

Let $[U/G]$ denote the stack–theoretic quotient, and let $\tilde{R} = U \times_{[U/G]} U$, so that we have a groupoid in algebraic spaces ([3] 2.4.3)

$$s, b : \tilde{R} \to U, \quad m : \tilde{R} \times_U \tilde{R} \to \tilde{R}.$$

In what follows, we shall denote this (and other) groupoids simply by $\tilde{R} \rightrightarrows U$. The scheme $\tilde{R}$ represents the functor which to a $\mathbb{Z}[t]$-scheme T associates the set of triples

$$\{(x, v), (x', v'), (u, h)\},$$

where (x, v) and (x', v') are objects of $U(T)$ and $(u, h) \in G(T)$ such that

$$x'_{h(i)} = u_i x_i, \quad v' = (\prod_i u_i^{-\alpha_i}) v.$$

2.2. Define an equivalence relation Γ on $\tilde{R}$ by

$$((x, v), (x', v'), (u, h)) \sim ((y, w), (y', w'), (u^2, h^2))$$

if

(2.2 (i)) $(x, v) = (y, w)$, and $(x', v') = (y', w')$;

(2.2 (ii)) $h(i) = h^2(i)$ for all i with $x_i \notin \mathcal{O}_T^*$;

(2.2 (iii)) $u_i = u_i^2$ for all i with $x_i \notin \mathcal{O}_T^*$.

It is immediate that Γ is an equivalence relation.

Lemma 2.3. *Γ is representable by a scheme and the two projections $\Gamma \rightrightarrows \tilde{R}$ are étale.*

Proof. To see that Γ is representable note that there is a natural monomorphism

$$\Gamma \longrightarrow \tilde{R} \times_{U \times U} \tilde{R}.$$

This map identifies Γ with a locally closed subscheme of $\tilde{R} \times_{U\times U} \tilde{R}$, since condition (2.2 (ii)) is an open condition and condition (2.2 (iii)) is represented by a closed subscheme. Hence Γ is representable.

To see that the two projections are étale proceed as follows (we check the result for p_1). Suppose given a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A_0) & \longrightarrow & \Gamma \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ \mathrm{Spec}(A) & \longrightarrow & \tilde{R} \end{array}$$

where $\mathrm{Spec}(A_0) \hookrightarrow \mathrm{Spec}(A)$ is a closed immersion defined by a square zero ideal. This diagram corresponds to two equivalent elements of $\tilde{R}$

$$((\bar{x}, \bar{v}), (\bar{x}', \bar{v}'), (\bar{u}, h)), \quad ((\bar{x}, \bar{v}), (\bar{x}', \bar{v}'), (\bar{u}^2, h^2))$$

over $\mathrm{Spec}(A_0)$ together with a lift of the first one

$$((x, v), (x', v'), (u, h))$$

to $\mathrm{Spec}(A)$. We must check that there exists a unique lift of

$$((\bar{x}, \bar{v}), (\bar{x}', \bar{v}'), (\bar{u}^2, h^2))$$

to $\mathrm{Spec}(A)$ which is equivalent to $((x, v), (x', v'), (u, h))$. The uniqueness is clear: the element must be $((x, v), (x', v'), (u^2, h^2))$ where $u_i^2 = u_i$ if $x_i \notin A^*$ and $u_i^2 = u_i x'_{h^2(i)} x'^{-1}_{h(i)}$ otherwise. It is straightforward to verify that $((x, v), (x', v'), (u^2, h^2))$ really does define an element of $\tilde{R}$, and hence we get the desired lift. □

We define $R := [\tilde{R}/\Gamma]$. It is an algebraic space.

Proposition 2.4. *The groupoid structure on $\tilde{R} \rightrightarrows U$ descends to R.*

Proof. Since $\tilde{R}$ surjects onto R, it suffices to show that the maps m, i, s, and b giving the groupoid structure descend to R. Clearly s and b descend.

The map i is given by

$$((x, v), (x', v'), (u, h)) \mapsto ((x', v'), (x, v), ((u^{-1}_{h^{-1}(i)})_i, h^{-1})),$$

so to check that i descends to R it suffices to show that if

$$((x, v), (x', v'), (u, h)) \sim ((x, v), (x', v'), (u_2, h_2))$$

are equivalent elements of $\tilde{R}$ over some scheme T, then

(2.4 (i)) $h_2^{-1}(i) = h^{-1}(i)$ for every i with $x'_i \notin \mathcal{O}_T^*$,

(2.4 (ii)) $u_{2,h_2^{-1}(i)} = u_{h^{-1}(i)}$ for every i with $x'_i \notin \mathcal{O}_T^*$.

Now since (u, h) and (u_2, h_2) define equivalent elements of $\tilde{R}$, the second statement follows from the first. The first statement follows from the fact that the bijections

$$\{i | x_i \notin \mathcal{O}_T^*\} \longrightarrow \{i | x'_i \notin \mathcal{O}_T^*\}$$

induced by h and h_2 are the same.

As for m, it is enough to show that if

$$((x,v),(x',v'),(u,h)) \sim ((x,v),(x',v'),(u^2,h^2)),$$

and

$$((x',v'),(x'',v''),(u^3,h^3)),((y,w),(x,v),(u^4,h^4))$$

are sections of $\tilde{R}$ over some scheme T, then

$$m(((x,v),(x',v'),(u,h)),((x',v'),(x'',v''),(u^3,h^3))) \tag{2.4.1}$$

is equivalent to

$$m(((x,v),(x',v'),(u^2,h^2)),((x',v'),(x'',v''),(u^3,h^3))), \tag{2.4.2}$$

and

$$m(((y,w),(x,v),(u^4,h^4)),((x,v),(x',v'),(u,h))) \tag{2.4.3}$$

is equivalent to

$$m(((y,w),(x,v),(u^4,h^4)),((x,v),(x',v'),(u^2,h^2))). \tag{2.4.4}$$

But (2.4.1)–(2.4.4) are equal to

$$((x,v),(x'',v''),((u^3_{h(i)}u_i)_i,h^3\circ h))$$
$$((x,v),(x'',v''),((u^3_{h^2(i)}u_i)_i,h^3\circ h^2)).$$
$$((y,w),(x',v'),((u^4_i u_{h^4(i)})_i,h\circ h^4))$$
$$((y,w),(x',v'),((u^4_i u^2_{h^4(i)})_i,h^2\circ h^4))$$

Now as in the proof that i descends, the maps

$$h^3\circ h, h^3\circ h^2 : \{i|x_i\notin \mathcal{O}_T^*\} \longrightarrow \{i|x''_i\notin \mathcal{O}_T^*\}$$

are the same, as are

$$h\circ h^4, h^2\circ h^4 : \{i|y_i\notin \mathcal{O}_T^*\} \longrightarrow \{i|x'_i\notin \mathcal{O}_T^*\}.$$

From this the proposition follows. □

2.5. We denote by $\mathcal{S}_H(\alpha)$, or just $\mathcal{S}$ if the reference to α and H is clear, the algebraic stack associated to the groupoid $R \rightrightarrows U$. Note that the two projections from R are smooth since the two projections from $\tilde{R}$ are smooth and Γ is an étale equivalence relation. Note also that there is a natural map $[U/\mathbb{G}_m^r \rtimes H] \to \mathcal{S}_H(\alpha)$ which is étale by construction. By this we mean that for any morphism $T \to \mathcal{S}_H(\alpha)$ from a scheme T, the fiber product $T\times_{\mathcal{S}_H(\alpha)}[U/\mathbb{G}_m^r \rtimes H]$ admits an étale cover and is étale over T.

Remark 2.6. When H is the full group of elements $\sigma \in \mathrm{S}_r$ for which $\alpha_{\sigma(i)} = \alpha_i$ for all i, then the stack $\mathcal{S}_H(\alpha)$ has a natural moduli interpretation in terms of log geometry ([1]). Let $\mathcal{M}_{\mathbb{Z}[t]}$ denote the log structure on $\mathrm{Spec}(\mathbb{Z}[t])$ associated to the map $\mathbb{N}\to\mathbb{Z}[t]$, $1\mapsto t$, and let $\mathcal{L}og_{(\mathbb{Z}[t],\mathcal{M}_{\mathbb{Z}[t]})}$ denote the algebraic stack classifying log schemes over $(\mathbb{Z}[t],\mathcal{M}_{\mathbb{Z}[t]})$ defined in ([5]). Then for any $\mathbb{Z}[t]$–scheme $f : T\to \mathrm{Spec}(\mathbb{Z}[t])$, the groupoid $\mathcal{S}_H(\alpha)(T)$ is the groupoid of morphisms of log structures $f^*\mathcal{M}_{\mathbb{Z}[t]}\to\mathcal{M}$ on T such that for any geometric point $\bar{t}\to T$, the following hold:

(2.6 (i)) The stalk $\overline{\mathcal{M}}_{\bar{t}}$ is a free monoid.

(2.6 (ii)) For some (and hence all) isomorphisms $\overline{\mathcal{M}}_{\bar{t}} \simeq \mathbb{N}^{r'}$, the composite map

$$\mathbb{N} \longrightarrow \overline{\mathcal{M}}_{\mathbb{Z}[t],f(\bar{t})} \longrightarrow \overline{\mathcal{M}}_{\bar{t}} \simeq \mathbb{N}^{r'} \tag{2.6.1}$$

sends $1 \in \mathbb{N}$ to a collection of natural numbers $(\beta_1, \ldots, \beta_{r'})$ which is a subset of $(\alpha_1, \ldots, \alpha_r)$.

We leave the proofs of the above statements to the interested reader. The key techniques may be found in ([5], [4]).

3. Maps to $\mathcal{S}_H(\alpha)$

3.1. Suppose now that X/R is as in (1.1), and let $\pi \in R$ be a uniformizer. Note that our assumptions on X simply mean that étale locally on X there exists a smooth map

$$X \longrightarrow \mathrm{Spec}(R[X_1, \ldots, X_r]/(X_1^{\beta_1} \cdots X_r^{\beta_r} - \pi), \tag{3.1.1}$$

for some integers r and β_i.

We construct a canonical map from X to a suitable stack $\mathcal{S}_H(\alpha)$. For every geometric point $\bar{x} \to X$ in the closed fiber, let $\cup Z_{\bar{x},i} \subset \mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$ denote the irreducible components with the reduced structure of the divisor (π), and let $\{r_i\}$ denote the multiplicity of $Z_{\bar{x},i}$ in $(\pi) \subset \mathrm{Spec}(\mathcal{O}_{X,\bar{x}})$. Choose the set $(\alpha_1, \ldots, \alpha_r)$ to be such that for each geometric point of the closed fiber, the collection $\{r_i\}$ is a subset of $(\alpha_1, \ldots, \alpha_r)$. Let $H \subset \mathrm{S}_r$ be the subgroup of $\sigma \in \mathrm{S}_r$ for which $\alpha_{\sigma(i)} = \alpha_i$ for every i. We claim that with these choices, there is a canonical map

$$X \longrightarrow \mathcal{S}_H(\alpha)_R := \mathcal{S}_H(\alpha) \otimes_{\mathbb{Z}[t], t \mapsto \pi} R. \tag{3.1.2}$$

Remark 3.2. Here when we say that $\{r_i\}$ is a subset of $\{\alpha_i\}$, we mean that numbers which occur with multiplicity greater than 1 should be considered distinct. For example, the set $(2, 2, 3)$ is a subset of $(3, 2, 4, 2)$ but not a subset of $(2, 3, 1, 4)$.

3.3. The map (3.1.2) is constructed as follows. For any point x in the closed fiber of X, we can find an affine étale neighborhood W of x such that the irreducible components of the closed fiber of W have no self–intersection and are defined by elements $x_1, \cdots, x_s \in \Gamma(W, \mathcal{O}_W)$. Because W is regular, we can, after reordering the α_i, find a unit $v \in \Gamma(W, \mathcal{O}_W^*)$ such that

$$\pi = v x_1^{\alpha_1} \cdots x_s^{\alpha_s}. \tag{3.3.1}$$

Define a map

$$R[X_1, \ldots, X_r, V^{\pm}]/(X_1^{\alpha_1} \cdots X_r^{\alpha_r} V - \pi) \longrightarrow \Gamma(W, \mathcal{O}_W) \tag{3.3.2}$$

by sending V to v, X_i to x_i if $i \leq s$, and X_i to 1 if $i > s$. We thus obtain a map $\rho : W \to U$.

Now suppose $x'_1, \ldots, x'_r \in \Gamma(W, \mathcal{O}_W)$ is a second collection of elements and $v' \in \Gamma(W, \mathcal{O}_W^*)$ such that

$$\pi = v' x_1'^{\alpha_1} \cdots x_r'^{\alpha_r}. \tag{3.3.3}$$

Define

$$E := \{i | x'_i \notin \Gamma(W, \mathcal{O}_W^*)\}. \tag{3.3.4}$$

Then the set E is in bijection with the irreducible components of the divisor $(\pi) \subset W$, as is the set $\{1, \ldots, s\}$. Note also that there is an equality of sets

$$\{\alpha_i | i \in E\} = \{\alpha_1, \ldots, \alpha_s\}, \tag{3.3.5}$$

since this is just the set of multiplicities of the components of the closed fiber. It follows that there exists an element $h \in H$ such that for each i there exists a unit $u_i \in \Gamma(W, \mathcal{O}_W^*)$ such that $x'_{h(i)} = u_i x_i$ for each i, where we make the convention that $x_i = 1$ for $i > s$. In addition, $v' = (\prod_i u_i^{-\alpha_i})v$ since X is integral. If $\rho' : W \to U$ denotes the map obtained from the collection $(x'_1, \ldots, x'_r, v')$, it follows that there exists a map

$$\tau : W \longrightarrow \tilde{R} \tag{3.3.6}$$

so that $\mathrm{pr}_1 \circ \tau = \rho_1$ and $\mathrm{pr}_2 \circ \tau = \rho_2$. Moreover, it follows from the construction that the induced map $\tau : W \to R$ is unique with these properties. From this and descent theory it follows that we obtain a canonical map as in (3.1.2).

Proposition 3.4. *The map (3.1.2) is smooth.*

Proof. In an étale neighborhood of any point x in the closed fiber of X, we can, after reordering the α_i, find a smooth morphism

$$X \longrightarrow \mathrm{Spec}(R[Y_1, \ldots, Y_s]/(Y_1^{\alpha_1} \cdots Y_s^{\alpha_s} - \pi)), \tag{3.4.1}$$

for some $s \leq r$. It thus suffices to consider the case when X equal to the right hand side of (3.4.1). Let us denote by $\mathcal{O}$ the ring $R[Y_1, \ldots, Y_s]/(Y_1^{\alpha_1} \cdots Y_s^{\alpha_s} - \pi)$. Since the natural map

$$[U_R/\mathbb{G}_m^r \rtimes H] \longrightarrow \mathcal{S}_H(\alpha) \tag{3.4.2}$$

is étale (2.5), to verify that (3.1.2) is smooth it suffices to show that the natural map

$$\mathrm{Spec}(\mathcal{O}) \longrightarrow [U_R/\mathbb{G}_m^r \rtimes H] \tag{3.4.3}$$

is smooth. If P denotes the fiber product $\mathrm{Spec}(\mathcal{O}) \times_{[U_R/\mathbb{G}_m^r \rtimes H]} U_R$, then the map (3.1.2) is smooth if and only if the projection $P \to U_R$ is smooth. Writing out the definitions, one finds that this is equivalent to the statement that for each $h \in H$ which preserves the set $\{1, \ldots, s\}$, the map

$$R[X_1, \ldots, X_r, V^{\pm}]/(X_1^{\alpha_1} \cdots X_r^{\alpha_r} V - \pi) \longrightarrow \mathcal{O}[U_1^{\pm}, \ldots, U_r^{\pm}] \tag{3.4.4}$$

is smooth, where (3.4.4) sends V to $(\prod_i U_i^{\alpha_i})^{-1}$ and $X_{h(i)}$ to $U_i Y_i$ if $i \leq s$ and U_i if $i > s$. From this the result follows. □

Remark 3.5. From the logarithmic point of view, the map (3.1.2) is simply the map arising from the log structure $\mathcal{M}_X$ on X defined by the closed fiber and the modular interpretation of $\mathcal{S}_H(\alpha)$ given in (2.6). Moreover, it follows from ([5], 4.6) that (3.1.2) is smooth if and only if the map of log schemes $(X, \mathcal{M}_X) \to (\mathrm{Spec}(R), \mathcal{M}_R)$ is log smooth in the sense of ([1], 3.3). Hence (3.4) also follows from the theory of log geometry.

4. Proof of (1.2)

4.1. We assume given X/R as in (1.1), and fix $\alpha = (\alpha_1, \ldots, \alpha_r)$ so that we have the natural smooth map (3.1.2). Let K' be the Galois closure of $R[T]/(T^{\prod \alpha_i} - \pi)$, and let R' be its ring of integers. The element "T" defines a uniformizer of R' which we denote by π'. Let $\mathcal{S}_H(1)$ be the stack obtained by the construction of section 2 taking $\alpha = (1, \ldots, 1)$ and $H \subset \mathrm{S}_r$ the be the subgroup of h for which $\alpha_{h(i)} = \alpha_i$ for every i, and let $\mathcal{S}_H(1)_{R'}$ be the base change to R' via the map $\mathbb{Z}[t] \to R'$ sending t to π'.

4.2. There is a natural map

$$p : \mathcal{S}_H(1)_{R'} \longrightarrow \mathcal{S}_H(\alpha)_{R'} \tag{4.2.1}$$

defined as follows. Denote by $R_1 \rightrightarrows U_1$ the groupoid defining $\mathcal{S}_H(1)_{R'}$, and by $R \rightrightarrows U$ the groupoid defining $\mathcal{S}_H(\alpha)_{R'}$. The map (4.2.1) will arise from a morphism of groupoids

$$(p_R, p_U) : (R_1 \rightrightarrows U_1) \longrightarrow (R \rightrightarrows U). \tag{4.2.2}$$

Define $p_U : U_1 \to U$ to be the morphism induced by the map

$$R'[X_1, \dots, X_r, V^{\pm}]/(X_1^{\alpha_1} \cdots X_r^{\alpha_r} V - \pi) \longrightarrow R'[Y_1, \dots, Y_r, W^{\pm}]/(Y_1 \cdots Y_r W - \pi') \tag{4.2.3}$$

sending $X_i \mapsto Y_i^{\prod_{j \neq i} \alpha_j}$ and $V \mapsto W^{\prod_i \alpha_i}$, and define $p_{\tilde{R}} : \tilde{R}_1 \to \tilde{R}$ to be the map induced by the map

$$\mathbb{G}_m^r \rtimes H \longrightarrow \mathbb{G}_m^r \rtimes H, \quad (\{u_i\}, h) \mapsto (\{u_i^{\prod_{j \neq i} \alpha_j}\}, h). \tag{4.2.4}$$

Then one verifies easily that the map $p_{\tilde{R}}$ descends to a map $p_R : R_1 \to R$, and we thus obtain a morphism of groupoids. The map (4.2.1) is the map induced by this morphism of groupoids.

Proposition 4.3. *For any morphism $T \to \mathcal{S}_H(\alpha)_{R'}$, the fiber product $T \times_{\mathcal{S}_H(\alpha)_{R'}} \mathcal{S}_H(1)_{R'}$ is a Deligne–Mumford stack, whose coarse moduli space equals T. If*

$$T = \operatorname{Spec}(R'[Y_1, \dots, Y_r]/(Y_1^{\alpha_1} \cdots Y_r^{\alpha_r} - \pi), \tag{4.3.1}$$

then the fiber product is isomorphic to the stack–theoretic quotient of the scheme

$$T' := \operatorname{Spec}(R'[Z_1, \dots, Z_r, W^{\pm}]/(Z_1 \cdots Z_r W - \pi', W^{\prod \alpha_i} - 1)) \tag{4.3.2}$$

by the action of the group $G := \mu_{\prod_{j \neq 1} \alpha_j} \times \cdots \times \mu_{\prod_{j \neq r} \alpha_j}$. Here $(\zeta_1, \dots, \zeta_r) \in G$ acts on T' by $Z_i \mapsto \zeta_i Z_i$ and on W by $W \mapsto (\prod \zeta_i)^{-1} W$, and the map $T' \to T$ is the one induced by $Y_i \mapsto Z_i^{\prod_{j \neq i} \alpha_j}$.

Proof. Since any morphism $T \to \mathcal{S}_H(\alpha)_{R'}$ factors étale locally on T through a scheme as in (4.3.1), it suffices to prove the second statement in the proposition. Now note that it follows from the constructions, that the diagram

$$\begin{array}{ccc} \mathcal{S}_H(1) & \longleftarrow & [U_1/\mathbb{G}_m^r \rtimes H] \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ \mathcal{S}_H(\alpha) & \longleftarrow & [U/\mathbb{G}_m^r \rtimes H] \end{array} \tag{4.3.3}$$

is cartesian, where $\tilde{p}$ is the map induced by the morphisms (4.2.3 and 4.2.4). Thus with T as in (4.3.1), it suffices to compute $T \times_{[U/G]} [U_1/G]$, which we leave as exercise to the reader. □

4.4. It follows from (4.3), that the stack $\mathcal{X}' := X \times_{\mathcal{S}_H(\alpha)_{R'}} \mathcal{S}_H(1)_{R'}$ is a semi–stable Deligne–Mumford stack, and that (1.2 (i)) holds.

4.5. To construct the action of $\mathrm{Gal}(K'/K)$ on $\mathcal{X}'$, it suffices to construct an action of $\mathrm{Gal}(K'/K)$ on $\mathcal{S}_H(1)_{R'}$ over $\mathcal{S}_H(\alpha)_R$. Now to construct such an action, it suffices to construct an action on U_1 on U_R which commutes with the action of $\mathbb{G}_m^r \rtimes H$ and preserves the equivalence relation. Let $\chi : \mathrm{Gal}(K'/K) \to \mathbb{G}_{m,R}$ be the map arising from the action of $\mathrm{Gal}(K'/K)$ on the rank 1 R'–module $(\pi')/(\pi'^2)$. Then for each $g \in \mathrm{Gal}(K'/K)$, define the action $U_1 \to U_1$ by sending W to $\chi(g)\cdot W$ and the Y_i to themselves. This action evidently commutes with the action of $\mathbb{G}_m^r \rtimes H$, and hence induces an action of $\mathcal{S}_H(1)_{R'}$ over $\mathcal{S}_H(\alpha)_R$.

4.6. To prove (1.2 (iii)), we may work étale locally on X. We may therefore assume that X is equal to

$$\mathrm{Spec}(R[Y_1,\dots,Y_r]/(Y_1^{\alpha_1}\cdots Y_r^{\alpha_r}-\pi). \tag{4.6.1}$$

Then by the description of the stack $\mathcal{X}'$ in this case, what has to be shown is that étale locally on X_2, there exists elements $z_1,\dots,z_r \in \mathcal{O}_{X_2}$, $w \in \mu_{\prod \alpha_i}(X_2)$, such that $\prod_i z_i w = \pi'$, $(z_i^{\prod_{j\neq i}\alpha_j}) = (Y_i)$, and that for any two collections of such elements $(z_i)_i$ and $(z_i')_i$, there exist a unique elements $(\zeta_i)_i \in G(X_2)$ such that $z_i' = \zeta_i z_i$ for each i. But this is clear, for if ℓ denotes the ramification index of R_2/R', then the divisor (Y_i) on X_2 is equal to $\ell(\prod_{j\neq i}\alpha_j)$ times a reduced Cartier divisor D_i, and z_i must be a generator of I_{D_i} for which $z_i^{\prod_{j\neq i}\alpha_j} = Y_i$.

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