MODULI OF SPHERICAL VARIETIES AND LOG GEOMETRY

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ABSTRACT. We give moduli interpretations using log geometry of the main components of the moduli spaces of polarized spherical varieties defined by Alexeev and Brion. The key new insight is that logarithmic structures alone do not suffice to give a good moduli interpretation. Instead we introduce a notion of an *enriched log structure* which is a logarithmic structure together with some extra data related to the group action on the varieties being parametrized.

CONTENTS

1. INTRODUCTION

This paper is a continuation of our earlier work [7, 8] on isolating main components of moduli spaces using log geometry in the sense of Fontaine, Illusie, and Kato. In this paper we extend our logarithmic interpretation of the main component in Alexeev's moduli spaces for broken toric varieties, studied in [7], to the main components in the moduli spaces for stable spherical varieties introduced by Alexeev and Brion in [1].

The basic moduli problem considered in [1] can loosely be described as follows. Throughout we work over an algebraically closed field k of characteristic 0. Let G/k be a connected reductive algebraic group over k and fix a representation V of G. Alexeev and Brion then

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consider moduli of pairs (X, f), where X is a projective variety with action of G and

$$f : X \to \mathbb{P}V$$

is a G-equivariant finite morphism satisfying some conditions (it is a *stable spherical variety with weight set* Γ; see [1, §4]). In [1, 4.2] it is shown that this moduli problem is representable by a proper algebraic stack with finite diagonal $\mathscr{M}_{\Gamma, \mathbb{P}V}$.

Fix a pair (X_0, f_0) over k with X_0 irreducible, and let L_0 denote the ample invertible sheaf $f_0^* \mathscr{O}_{\mathbb{P}V}(1)$. For suitably chosen (X_0, L_0), there is an open substack $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}^{\circ} \subset \mathscr{M}_{\Gamma, \mathbb{P}V}$ classifying pairs (X, f) such that the corresponding polarized scheme with G-action $(X, f^* \mathscr{O}_{\mathbb{P}V}(1))$ is isomorphic to (X_0, L_0). We let $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)} \subset \mathscr{M}_{\Gamma, \mathbb{P}V}$ denote the scheme-theoretic closure of $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}^{\circ}$.

An analysis of 1-parameter degenerations exhibits toric phenomena very similar to the case of toric varieties and polarized abelian varieties studied in [7], and this suggests there may be a moduli interpretation using log geometry of $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}$. The aim of this paper is to explain such a moduli interpretation.

A degenerating family $\mathcal{X}$ of stable spherical varieties (say over a trait S) does have a natural log structure $M_{\mathcal{X}}$, but the induced morphism of log schemes $(\mathcal{X}, M_{\mathcal{X}}) \to (S, M_S)$, where M_S denotes the standard log structure on S, is not log smooth. Perhaps more significantly, the deformation theory of the pair $(\mathcal{X}, M_{\mathcal{X}})$ is not well-behaved and one does not get a good moduli stack using just log structures. To remedy this situation we introduce a new notion of an *enrichment* on the log structure $M_{\mathcal{X}}$. The definition is technical but the basic idea can be understood as follows (for full details see section 8).

Fix a Borel subgroup B of G, a maximal torus $T \subset B \subset G$, let $U \subset B$ be the unipotent radical of B, and let (X_0, L_0) be an irreducible polarized spherical variety (in the setting of (X, f) as above the line bundle is $f^* \mathscr{O}_{\mathbb{P}V}(1)$). One can then consider the ring with $\widetilde{G} := G \times \mathbb{G}_m$-action $A := \oplus_{n \geq 0} \Gamma(X_0, L_0^{\otimes n})$ and the corresponding cone $C_{X_0} := \mathrm{Spec}(A)$, an affine spherical variety in the sense of [1, 1.3]. The ring of U-invariants A^U with its natural T-action then defines a toric variety. Roughly speaking the log structure alluded to above only sees information about A^U and its degenerations, but this is not enough to understand the entire ring A. Our notion of an enriched log structure is a way to understand the degeneration of the multiplicative structure on A and not just A^U.

More precisely, let Λ be the character group of T and let $\Lambda^+ \subset \Lambda$ be the dominant weights. Then the character group of $\widetilde{T} := T \times \mathbb{G}_m \subset \widetilde{G}$ is $\Lambda \times \mathbb{Z}$ and the dominant weights $\widetilde{\Lambda}^+ \subset \widetilde{\Lambda}$ are given by $\Lambda^+ \times \mathbb{Z}$. Let $P \subset \widetilde{\Lambda}^+$ be the dominant weights of the irreducible representations which occur in A. We then have

$$A = \oplus_{p \in P} A_p,$$

where A_p is a representative for the irreducible representation corresponding to p. To understand the component of the moduli space of polarized spherical varieties containing (X_0, L_0) we essentially have to understand how to deform the multiplicative structure on A. For each $p \in P$ we have a line $L_p \subset A_p$ of highest weight vectors, and

$$A^U = \oplus_{p \in P} L_p.$$

Using the toric structure we can therefore find an isomorphism $A^U \simeq k[P]$ compatible with the torus actions, or equivalently highest weight vectors $v_p \in A_p$ such that $v_p \cdot v_q = v_{p+q}$

(multiplication in A). For each p, q, r with A_r occurring in $A_p \cdot A_q$ let

$$m_{p,q}^r : A_p \otimes A_q \to A_r$$

be the r-th component of the multiplication map. To get families of spherical varieties over a ring R we can then change the multiplicative structure by multiplying the $m_{p,q}^r$ by elements $\eta_{p,q}^r \in R^*$ (or if we want to consider degenerations we take $\eta_{p,q}^r \in R$). These elements have to satisfy a number of compatibilities to make the resulting R-module $A \otimes_k R$ an algebra. Note that the variation in algebra structure on A^U obtained in this way is determined by the elements $\eta_{p,q}^{p+q}$, but to work with the whole algebra A we need also the elements $\eta_{p,q}^r$.

To globalize this discussion we consider logarithmic structures. We consider proper integral morphisms of log schemes $(f, f^b) : (X, M_X) \to (T, M_T)$ together with a relatively ample line bundle L on X and an isomorphism $\alpha : P \simeq f_* M_{X/T}$, where $M_{X/T}$ is the cokernel in the category of sheaves of integral monoids of the map $f^b : f^* M_T \to M_X$. This data is required to satisfy a number of conditions, among them that the map $f_* M_X \to f_* M_{X/T}$ is surjective. For each $p \in P$ the set of liftings L_p (the notation here is consistent with the preceding paragraph; L_p is a global version of the torsor of highest weight vectors) of $\alpha(p)$ to $f_* M_X$ has a natural action of M_T and in fact is what we call an *invertible M_T-set* (see section 2). These invertible M_T-sets behave in many ways like torsors, and in particular for two invertible M_T-sets L and L' we can form a third $L \wedge L'$, and more generally for an invertible M_T-set L and an arbitrary M_T-set N we can form $L \wedge N$. A *logarithmic enrichment* is a collection of maps

$$\eta_{p,q}^r : L_p \wedge L_q \to M_T \wedge L_r$$

for each $p, q, r \in P$ with A_r occurring in $A_p \cdot A_q$. These maps $\eta_{p,q}^r$ are required to satisfy a number of natural compatibilities (see 8.6).

In the case when G is a torus, the enrichment on the log structure is no additional information and our theory reduces to the logarithmic interpretation of the main components of moduli spaces for broken toric varieties studied in [7].

Using the notion of enriched log structures we relate the main components $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}$ to a suitable moduli problem using enriched log structures. The main result is theorem 10.6. We construct a moduli problem classifying enriched polarized log schemes with G-action and show that it is representable by a proper algebraic stack $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}^{\log}$. Furthermore, there is a natural morphism

$$\tau' : \mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}^{\log} \to \mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}$$

which is a universal homeomorphism.

1.1. **Conventions.** All sheaves of monoids are assumed commutative with unit and integral.

In section 3 onward we work over an algebraically closed field k of characteristic 0.

2. Background: Invertible M-sets

2.1. Let C be a site and M a sheaf of monoids on C.

Following [6, 1.4.5], an *invertible M-set* is a sheaf of sets S with an action of M such that for every object $U \in C$ there exists a covering $\{U_i \to U\}_{i \in I}$ and sections $u_i \in S(U_i)$ such that the maps $M|_{C/U_i} \to S|_{C/U_i}$ induced by $m \mapsto m \cdot u_i$ are isomorphisms. An isomorphism of invertible M-sets $S \to S'$ is an isomorphism of sheaves of sets compatible with the M-action.

Note that if M is a sheaf of groups then the notion of invertible M-set is equivalent to the notion of M-torsor.

If $f : M \to N$ is a morphism of sheaves of monoids and S is an invertible M-set, then we can form the *pushout* $N \wedge^M S$ by taking the quotient of $N \times S$ by the equivalence relation generated by $(f(m) + n, s) \sim (n, m \cdot s)$ for local sections $n \in N$, $m \in M$, and $s \in S$. The action of N induced by the action on the first factor makes $N \wedge^M S$ an invertible N-set.

In particular, if $M^* \subset M$ denotes the subsheaf of invertible elements then there is a functor

$$(2.1.1) \qquad\qquad (M^*\text{-torsors}) \to (\text{invertible } M\text{-sets}).$$

Proposition 2.2 (See also [6, p. 205]). *The functor (2.1.1) is an equivalence of categories.*

Proof. Both sides of (2.1.1) form stacks over C and every object of the right side is locally in the image by definition. So it suffices to show that the functor is fully faithful, which is immediate. $\qquad\square$

2.3. Invertible M-sets will arise for us in the following way. Let (X, M_X) be a fine log scheme, and let $M_X \to N$ be an injective integral morphism of sheaves of integral monoids on X. Assume that the quotient N/M_X is isomorphic to a constant sheaf P, and that for every specialization of geometric points $\bar{\eta} \to \bar{x}$ the induced diagram

$$
\begin{array}{ccc}
M_{X,\bar{x}} & \longrightarrow & N_{\bar{x}} \\
\downarrow & & \downarrow \\
M_{X,\bar{\eta}} & \longrightarrow & N_{\bar{\eta}}
\end{array}
$$

is cocartesian. Let $\pi : N \to P$ be the projection, and for $p \in P$ let $S_p \subset N$ be the sheaf $\pi^{-1}(p)$.

Proposition 2.4. *The natural action of M_X on S_p makes S_p an invertible M_X-set.*

Proof. Let $\bar{x} \to X$ be a geometric point. Let us first show that the $M_{X,\bar{x}}$-set $S_{p,\bar{x}}$ is invertible. Indeed by [6, 2.1.5] the set $S_{p,\bar{x}}$ contains a minimal element $q \in S_{p,\bar{x}}$. Let $t \in S_{p,\bar{x}}$ be another element. Then since t and q have the same image in P, there exists elements $m_1, m_2 \in M_{X,\bar{x}}$ such that $t + m_1 = q + m_2$. Since the map $M_{X,\bar{x}} \to N_{\bar{x}}$ is integral we can find an element $t_0 \in N_{\bar{x}}$ and elements $m_3, m_4 \in M_{X,\bar{x}}$ such that $t = t_0 + m_3$, $q = t_0 + m_4$ and $m_1 + m_3 = m_2 + m_4$. Since q is a minimal element the element m_4 must be a unit, whence $t = q + (m_3 - m_4)$.

Fix a geometric point $\bar{x} \to X$ and let $U \to X$ be an étale neighborhood of $\bar{x}$ over which there exists a section $s \in S_p(U)$ whose image $s_{\bar{x}}$ in $S_{p,\bar{x}}$ generates $S_{p,\bar{x}}$. We claim that after possibly further shrinking on U the section s defines an isomorphism $M_X \to S_p$. Let $\overline{S}_p$ denote the quotient $S_p/\mathscr{O}_X^*$, so $\overline{S}_p$ is a $\overline{M}_X$-set. A section s of S_p generates S_p if and only if

its image in $\overline{S}_p$ generates $\overline{S}_p$. Since the sheaf $\overline{S}_p$ is constructible it therefore suffices to show that the set of points $\bar{\eta} \to U$ for which the section $\bar{s} \in \overline{S}_p$ induced by s generates $\overline{S}_{p,\bar{\eta}}$ is stable under generization.

To prove this let $\bar{\eta} \to U$ be a geometric point specializing to a point $\bar{y} \to U$ for which $\bar{s}_y \in \overline{S}_{p,\bar{y}}$ is a generator. By our assumptions the map $\overline{S}_{p,\bar{y}} \to \overline{S}_{p,\bar{\eta}}$ is surjective. Let $\rho \in \overline{S}_{p,\bar{\eta}}$ be a minimal element and let $\tilde{\rho} \in \overline{S}_{p,\bar{y}}$ be a lifting of ρ. Since $\bar{s}_{p,\bar{y}}$ is minimal there exists an element $m \in \overline{M}_{X,\bar{y}}$ such that $\tilde{\rho} = \bar{s}_{p,\bar{y}} + m$. Therefore $\bar{s}_{p,\bar{\eta}}$ is also in the sub-$\overline{M}_{X,\bar{\eta}}$-set of $S_{p,\bar{\eta}}$ generated by $\bar{s}_{p,\bar{\eta}}$, whence $\bar{s}_{p,\bar{\eta}}$ generates $\overline{S}_{p,\bar{\eta}}$. $\qquad\square$

2.5. For $p \in P$ let L_p denote the $\mathscr{O}_X^*$-torsor corresponding to S_p under the equivalence (2.1.1). The sheaf L_p is the subsheaf of S_p of sections which generate S_p. For two sections $e_p \in L_p$ and $e_q \in L_q$ their sum $e_p + e_q$ inside N is an element of S_{p+q}. We can therefore write $e_p + e_q$ as $m + e_{p+q}$ for some element $m \in M_X$ and $e_{p+q} \in L_{p+q}$. The pair (m, e_{p+q}) is not unique but any other such pair is of the form $(m + u, e_{p+q} - u)$ for some element $u \in \mathscr{O}_X^*$. We therefore get a well-defined element of the quotient $M_X \wedge^{\mathscr{O}_X^*} L_{p+q}$ of the product $M_X \times L_{p+q}$ by the action of $\mathscr{O}_X^*$ given by $u * (m, e) = (m + u, e - u)$. In this way we obtain a map of $\mathscr{O}_X^*$-sets

$$\theta_{p,q} : L_p \wedge^{\mathscr{O}_X^*} L_q \to M_X \wedge^{\mathscr{O}_X^*} L_{p+q}.$$

3. Spherical varieties: The basics

In this paper we follow the conventions and notation of [1]. For convenience of the reader and to fix notation we review some of the notation and definitions from loc. cit.

3.1. For the remainder of the paper we fix the following notation. Let k be an algebraically closed field of characteristic 0. Fix a connected reductive group G/k, let $B \subset G$ be a Borel subgroup with maximal torus $T \subset B$ and unipotent radical U. We write Λ for the character group of T and $\Lambda^+ \subset \Lambda$ for the dominant weights.

If V is a finite dimensional irreducible G-representation then there exists a unique B-invariant line $L \subset V$ on which B acts through a character $\chi \in \Lambda^+$. This establishes a bijection between the set of isomorphism classes of irreducible representations of G and Λ^+. We fix for each $\lambda \in \Lambda^+$ a corresponding irreducible representation $V(\lambda)$.

Definition 3.2. (i) A *G-variety over k* is a connected, reduced, separated, finite type k-scheme X with an action of G over k.

(ii) A *polarized G-variety over k* is a pair (X, L) consisting of a projective G-variety X and an ample G-linearized invertible sheaf L.

(iii) An affine G-variety X over k is *multiplicity free* if the multiplicity of any irreducible representation occuring in $\Gamma(X, \mathscr{O}_X)$ is ≤ 1.

(iv) A polarized G-variety (X, L) is *multiplicity free* if for every n and irreducible representation V of G the multiplicity of V in the representation $\Gamma(X, L^{\otimes n})$ is ≤ 1.

3.3. Let $\widetilde{G}$ denote the group $\mathbb{G}_m \times G$ which has a Borel subgroup $\widetilde{B} := \mathbb{G}_m \times B$ and maximal torus $\widetilde{T} := \mathbb{G}_m \times T$. Let $\widetilde{\Lambda} = \mathbb{Z} \times \Lambda$ be the corresponding character group and let $\widetilde{\Lambda}^+ = \mathbb{Z} \times \Lambda^+$ be the dominant weights. If (X, L) is a polarized G-variety then we can form the $\widetilde{G}$-representation

$$R(X, L) := \oplus_{n \geq 0} \Gamma(X, L^{\otimes n})$$

and the $\widetilde{G}$-variety $\widetilde{X} := \mathrm{Spec}(R(X,L))$. The scheme $\widetilde{X}$ is called the *cone* of (X,L).

An affine irreducible G-variety X is *spherical* if it is normal and multiplicity free. A polarized G-variety (X,L) is *spherical* if X is normal and (X,L) is multiplicity free.

3.4. For an affine G-variety X define the *weight set* $\Lambda^+(X) \subset \Lambda^+$ to be the set of $\lambda \in \Lambda^+$ for which $V(\lambda)$ occurs in the representation $\Gamma(X, \mathscr{O}_X)$, and the *weight cone* $C(X)$ to be the cone $\mathbb{R}_{\geq 0}\Lambda^+(X) \subset \Lambda_{\mathbb{R}}$ generated by $\Lambda^+(X)$.

For a polarized G-variety define the *weight set* $\widetilde{\Lambda}^+(X,L) \subset \widetilde{\Lambda}^+$ to be $\Lambda^+(\widetilde{X})$, and the weight cone $\widetilde{C}(X,L) \subset \widetilde{\Lambda}_{\mathbb{R}}$ to be the cone $C(\widetilde{X})$. Also define the *moment set* $Q(X,L)$ to be

$$Q(X,L) := \widetilde{C}(X,L) \cap (\{1\} \times \Lambda^+_{\mathbb{R}}) \subset \widetilde{\Lambda}_{\mathbb{R}} = \mathbb{R} \times \Lambda_{\mathbb{R}}.$$

Then $\widetilde{C}(X,L)$ is the cone over $Q(X,L)$. If X is irreducible then $Q(X,L)$ is a rational convex polytope in $\Lambda^+_{\mathbb{R}}$ called the *moment polytope*.

3.5. Fix a subgroup $\Gamma \subset \Lambda$. A *stable affine spherical variety with weights in* Γ is an affine G-variety X satisfying the following:

    (i) X is seminormal.
    (ii) X contains only finitely many G-orbits and they are spherical.
    (iii) For any G-orbit closure Y the weight group $\Lambda(Y)$ (the subgroup of Λ generated by the weight set of Y) is contained in Γ as a direct summand.

Similarly for a subgroup $\Gamma \subset \widetilde{\Lambda}$ a *polarized stable spherical variety with weights in* Γ is a polarized G-variety (X,L) satisfying the following:

    (i) X is seminormal.
    (ii) X contains only finitely many G-orbits and they are spherical.
    (iii) For any G-orbit closure Y the weight group $\widetilde{\Lambda}(Y,L)$, which is defined to be the subgroup of $\widetilde{\Lambda}$ generated by $\widetilde{\Lambda}^+(Y,L)$, is contained in Γ as a direct summand.

3.6. These definitions extend to families as follows. For a scheme S/k and subgroup $\Gamma \subset \Lambda$ a *stable affine spherical variety over S with weights in* Γ is an S-scheme $f : X \to S$ with action of G such that the following hold:

    (i) f is flat and locally of finite presentation.
    (ii) For every $\lambda \in \widetilde{\Lambda}^+$, with corresponding representation $V(\lambda)$, the sheaf $\mathscr{H}om_G(V(\lambda), f_*\mathscr{O}_X)$ is a locally free sheaf of rank 0 or 1.
    (iii) Every geometric fiber of f is a stable affine spherical variety with weights in Γ.

Similarly for a subgroup $\Gamma \subset \widetilde{\Lambda}$ a *polarized stable spherical variety over S with weights in* Γ is a pair $(f : X \to S, L)$, such that the following hold:

    (i) f is proper, flat, and locally of finite presentation and G acts on X over S.
    (ii) L is a relatively ample invertible sheaf on X.
    (iii) Every geometric fiber of f is a polarized stable spherical variety with weights in Γ.

If $(X/S, L)$ is a polarized stable spherical variety over a connected scheme S with weights in Γ then the weight sets of all geometric fibers are the same.

4. Pavings

4.1. With notation as in section 3, fix a subgroup $\Gamma \subset \widetilde{\Lambda}$ and a polytope $Q \subset \Gamma_{\mathbb{R}}$, and let (X, L) be a polarized stable spherical variety with moment set Q and weights in Γ.

Let $Y \subset X$ be the closure of a G-orbit in X. Then by [1, 2.1.3] the pair $(Y, L|_Y)$ is again a spherical variety, and by [1, 2.2.3 (iii)] the restriction map

$$H^0(X, L^{\otimes n}) \to H^0(Y, L^{\otimes n}|_Y)$$

is surjective for all n. This implies that if R (resp. R_Y) denotes the coordinate ring of the cone $\widetilde{X}$ (resp. $\widetilde{Y}$) then the induced map $R \to R_Y$ is surjective. Let $Q_Y \subset Q$ be the moment polytope of $(Y, L|_Y)$. Then it follows from [1, 2.3.1] that the set of subpolytopes $S = \{Q_Y\}$ obtained from the closures Y of G-orbits in X form a paving of Q in the sense of [7, 3.1.1].

4.2. Pavings that arise from closed fibers of 1-parameter degenerations of irreducible spherical varieties can be described as follows. Let V be a complete discrete valuation ring over k with field of fractions K, residue field F, and uniformizer $\pi \in \mathfrak{m}_V$. Let $(\mathscr{X}, \mathscr{L})/V$ be a stable spherical variety over V with irreducible generic fiber and with moment polytope Q. Let $\mathcal{R}$ be the coordinate ring of $\widetilde{\mathscr{X}}$, and let $\mathcal{R}_\eta$ (resp. $\mathcal{R}_s$) denote the coordinate ring of the generic (resp. closed) fiber of $\widetilde{\mathscr{X}}$. We have a G-equivariant inclusion $\mathcal{R} \hookrightarrow \mathcal{R}_\eta$.

By [1, 3.1.2] we can, after replacing V by a finite extension, find a stable spherical variety (X_0, L_0) over k such that $(\mathscr{X}_\eta, \mathscr{L}_\eta)$ is isomorphic to $(X_0, L_0) \otimes_k K$. We fix one such isomorphism. Let R_0 be the coordinate ring of the cone of (X_0, L_0) so $\mathcal{R}_\eta \simeq R_0 \otimes_k K$. For λ in the weight set of (X, L) with corresponding representation $V(\lambda)$ let $R_{0,\lambda} \subset R_0$ (resp. $\mathcal{R}_\lambda \subset \mathcal{R}$) denote the image of the natural map

$$V(\lambda) \otimes \mathrm{Hom}_{\widetilde{G}}(V(\lambda), R_0) \to R_0 \quad (\text{resp. } V(\lambda) \otimes \mathrm{Hom}_{\widetilde{G}}(V(\lambda), \mathcal{R}) \to \mathcal{R}).$$

Then there exists a unique integer $h(\lambda)$ for which

$$\mathcal{R}_\lambda = \pi^{h(\lambda)} R_{0,\lambda} \otimes_k V$$

as submodules of $\mathcal{R}_\eta$.

Let $P \subset \Gamma$ (resp. $P_Y \subset \Gamma$) denote the intersection $\widetilde{C}(X, L) \cap \Gamma$ (resp. $\widetilde{C}(Y, L|_Y) \cap \Gamma$). By [1, 1.3.1 (v)] the weight set of (X, L) (resp. $(Y, L|_Y)$) is equal to P (resp. P_Y).

Lemma 4.3. *The function h is linear on each P_Y.*

Proof. Let $\lambda, \eta \in P_Y$ be two elements and let v_λ (resp. v_η, $v_{\lambda+\eta}$) be a highest weight vector in $R_{0,\lambda}$ (resp. $R_{0,\eta}$, $R_{0,\lambda+\eta}$). Then the product

$$\left(\pi^{h(\lambda)} v_\lambda\right) \cdot \left(\pi^{h(\eta)} v_\eta\right)$$

has $(\eta + \lambda)$-component given by

$$\pi^{h(\lambda)+h(\eta)-h(\lambda+\eta)} \cdot \left(\pi^{h(\lambda+\eta)}\right) \cdot v_{\lambda+\eta}.$$

Since this product reduces to something nonzero modulo π this implies that

$$h(\lambda) + h(\eta) = h(\lambda + \eta).$$

$\square$

4.4. In particular we get a function $\bar{h}$ on the rational points of Q by the formula

$$\bar{h}(p/\deg(p)) = \frac{1}{\deg(p)} h(p).$$

Lemma 4.5. *The function $\bar{h}$ is piecewise linear on the rational points of Q, convex, with domains of linearity precisely the polytopes $\{Q_Y\}$.*

Proof. We have already seen that $\bar{h}$ is linear on each of the polytopes Q_Y. Furthermore, for two elements $\lambda, \eta \in P$ we must have

$$h(\lambda) + h(\eta) - h(\lambda + \eta) \geq 0,$$

since for highest weight vectors v_λ and v_η as above we must have

$$(\pi^{h(\lambda)} v_\lambda) \cdot (\pi^{h(\eta)} \cdot v_\eta) \subset V \cdot \pi^{h(\lambda+\eta)} v_{\lambda+\eta}.$$

It remains to see that the domains of linearity are precisely the Q_Y. For this let $\lambda, \eta \in P$ be two elements which do not lie in a common Q_Y and let $v_\lambda \in R_{0,\lambda}$ and $v_\eta \in R_{0,\eta}$ be highest weight vectors. Then the product

$$(\pi^{h(\lambda)} v_\lambda) \cdot (\pi^{h(\eta)} v_\eta) \in \mathcal{R}$$

maps to zero in the coordinate ring of each irreducible component of the closed fiber. It follows that this product lies in $\pi^{h(\lambda+\eta)+1} V \cdot R_{0,\lambda+\eta}$, whence $h(\lambda) + h(\eta) > h(\lambda + \eta)$. This implies that the polytopes $\{Q_Y\}$ are exactly the domains of linearity of the function $\bar{h}$. $\square$

Definition 4.6. Let (X_0, L_0) be an irreducible polarized spherical variety over k.

(i) A piecewise linear function $\bar{h} : Q \to \mathbb{R}$ is *admissible with respect to* (X_0, L_0) if there exists a family of polarized spherical varieties $(\mathscr{X}, \mathscr{L})$ over a discrete valuation ring such that $(\mathscr{X}_\eta, \mathscr{L}_\eta) \simeq (X_0, L_0) \otimes_k \operatorname{Frac}(V)$ such that $\bar{h}$ is the function obtained by the above construction.

(ii) A paving of Q is *regular with respect to* (X_0, L_0) if the polytopes of the paving are precisely the domains of linearity of an admissible function with respect to (X_0, L_0).

Remark 4.7. In 5.2 below we give a more combinatorial characterization of the condition that a piecewise linear function be admissible with respect to (X_0, L_0).

5. Encoding multiplication maps

5.1. Let K be a field of characteristic 0 (there is no harm in considering just $K = k$, but later we will also want to consider extensions of k) and fix an irreducible polarized spherical G-variety (X, L) over K. Let R denote the ring $R(X, L)$, and let P denote the weight set $\widetilde{\Lambda}(X, L)$. Let Q denote the moment polytope of (X, L). As in 4.2, for $a \in P$ let $R_a \subset R$ denote the image of $\operatorname{Hom}_{\widetilde{G}}(V(a), R) \otimes V(a) \to R$, so R_a is noncanonically isomorphic to $V(a)$ as $\widetilde{G}$-representations. For $a, b, c \in P$ let

$$m_{a,b}^c : R_a \otimes R_b \to R_c$$

denote the composite of the multiplication map $R_a \otimes R_b \to R$ with the projection onto R_c.

For $a, b, c \in P$ write $(a, b) \succ c$ (or $c \prec (a, b)$) if R_c occurs with nonzero multiplicity in $R_a \cdot R_b \subset R$.

Lemma 5.2. *A piece-wise linear function $\bar{h} : Q \to \mathbb{R}$ is admissible with respect to (X, L) if and only if the corresponding function $h : P \to \mathbb{R}$ defined by $h(a) := \deg(a) \cdot \bar{h}(a/\deg(a))$ has values in the natural numbers $\mathbb{N}$ and for every $a, b, c \in P$ with $(a, b) \succ c$ we have*

$$h(a) + h(b) - h(c) \geq 0.$$

Proof. It follows from the construction that an admissible function has the indicated properties. Conversely let $\mathcal{R} \subset R \otimes_K K((t))$ denote the $K[[t]]$-submodule given by $\oplus_{a \in P} t^{h(a)} K[[t]] \otimes R_a$. The assumptions on h imply that $\mathcal{R}$ is in fact a subalgebra of $R \otimes_K K((t))$. It follows from [1, 2.1.5] that $\mathcal{R}$ defines a stable spherical variety over $K[[t]]$, whence $\bar{h}$ is admissible with respect to (X, L). $\square$

5.3. Let F be the free abelian group with generators $\eta_{a,b}^c$, one for each triple $a, b, c \in P$ with $(a, b) \succ c$, and let $\widetilde{E}$ be the quotient of F by the relations

(i) $\eta_{a,b}^{a+b} = 0$ for all $a, b \in P$;
(ii) $\eta_{a,b}^c = \eta_{b,a}^c$ for all $(a, b) \succ c$;
(iii) For all $a, b, c, d, e, f \in P$ (some of them possibly equal) with

$$e \prec (a, b), \ d \prec (e, c), \ f \prec (b, c), \ \text{and} \ d \prec (a, f)$$

we have

$$\eta_{e,c}^d + \eta_{a,b}^e = \eta_{a,f}^d + \eta_{b,c}^f.$$

(iv) For $a, b, c, d, e, f \in P$ with

$$e \prec (a, b), \ d \prec (e, c), \ f \prec (a, b), \ d \prec (f, c)$$

we have

$$\eta_{a,b}^f + \eta_{c,f}^d = \eta_{a,b}^e + \eta_{c,e}^d.$$

There is a map

$$\tilde{\epsilon} : \widetilde{E} \to P^{\mathrm{gp}}$$

sending $\eta_{a,b}^c$ to $a + b - c$.

Proposition 5.4. *Let E be the quotient of $\widetilde{E}$ by its torsion subgroup. Then the map $\epsilon : E \to P^{\mathrm{gp}}$ induced by $\tilde{\epsilon}$ is injective.*

Proof. Note that the formation of $\widetilde{E}$, E, P, etc. is invariant under field extensions of K. For the proof we may therefore without loss of generality assume that K is algebraically closed.

Consider the group algebra $K[E]$, and let S denote $\mathrm{Spec}(K[E])$. Define a graded $K[E]$-algebra with $\widetilde{G}$-action $\mathcal{R}$ as follows. The underlying graded module with $\widetilde{G}$-action is just $R \otimes_K K[E]$. The multiplication map, however, is given by summing the maps

(5.4.1) $$\eta_{a,b}^c \cdot m_{a,b}^c : (R_a \otimes K[E]) \otimes_{K[E]} (R_b \otimes K[E]) \to R_c \otimes K[E]$$

over all triples a, b, c with $(a, b) \succ c$.

Lemma 5.5. *This definition gives $\mathcal{R}$ the structure of a commutative $K[E]$-algebra.*

Proof. The only nontrivial point is that this multiplication map is associative. To ease notation, for $a \in P$ write $\mathcal{V}_a$ for $R_a \otimes K[E]$ and let

$$\chi_{a,b}^c : \mathcal{V}_a \otimes \mathcal{V}_b \to \mathcal{V}_c$$

be the c-th factor of the multiplication map, so we have $\chi_{a,b}^c = \eta_{a,b}^c \cdot m_{a,b}^c$ if $c \prec (a,b)$ and 0 otherwise. The associativity condition amounts to the statement that for $a, b, c, d \in P$ we have

$$\sum_{e \prec (a,b), d \prec (c,e)} \chi_{e,c}^d \circ (\chi_{a,b}^e \otimes \mathrm{id}) = \sum_{f \prec (b,c), d \prec (a,f)} \chi_{a,f}^d \circ (\mathrm{id} \otimes \chi_{b,c}^f).$$

Now observe that relations (iii) and (iv) in the definition of $\widetilde{E}$ implies that there exists a single element $\eta_{a,b,c}^d \in E$ such that

$$\sum_{e \prec (a,b), d \prec (c,e)} \chi_{e,c}^d \circ (\chi_{a,b}^e \otimes \mathrm{id}) = \eta_{a,b,c}^d \cdot \sum_{e \prec (a,b), d \prec (c,e)} m_{e,c}^d \circ (m_{a,b}^e \otimes \mathrm{id}),$$

and

$$\sum_{f \prec (b,c), d \prec (a,f)} \chi_{a,f}^d \circ (\mathrm{id} \otimes \chi_{b,c}^f) = \eta_{a,b,c}^d \cdot \sum_{f \prec (b,c), d \prec (a,f)} m_{a,f}^d \circ (\mathrm{id} \otimes m_{b,c}^f).$$

The associativity then follows from the fact that R is an associative algebra. $\qquad\square$

The spectrum of R^U, where as in 3.1 U denotes the unipotent radical of B, is an affine toric variety with torus $D(P^{\mathrm{gp}})$, and therefore is isomorphic to $\mathrm{Spec}(K[P])$. Fix one such isomorphism. Then by construction we have an induced isomorphism $\mathcal{R}^U \simeq \mathrm{Spec}(K[E \rtimes P])$, where $E \rtimes P$ is defined as in [7, 3.1.9].

Let Ω denote the field of fractions of $K[E]$, and consider the $\widetilde{G}$-variety $\mathrm{Spec}(\mathcal{R}_\Omega)$. It comes equipped with an isomorphism $\sigma : \mathcal{R}_\Omega^U \simeq \Omega[P]$ (since $\eta_{a,b}^{a+b} = 0$ for all $a, b \in P$).

Lemma 5.6. *There exists an affine $\widetilde{G}$-variety $\mathrm{Spec}(R')$ over K such that R'^U is isomorphic to $K[P]$ and a finite extension Ω' of Ω over which we have an isomorphism of $\widetilde{G}$-varieties $\delta : \mathrm{Spec}(R' \otimes_K \Omega') \simeq \mathrm{Spec}(\mathcal{R}_\Omega \otimes_\Omega \Omega')$.*

Proof. Let

$$M_{K[P]} : (K\text{-schemes})^{\mathrm{op}} \to \mathrm{Set}$$

be the functor, defined in [2, 1.11], sending a K-scheme S to the set of equivalence classes of pairs (X, ι), where X is an affine $\widetilde{G}$-scheme over S and $\iota : X//U \to \mathrm{Spec}(K[P]) \times_{\mathrm{Spec}(K)} S$ is a T-equivariant isomorphism (see loc. cit. for more details). There is an action of the torus $D(P^{\mathrm{gp}})$ on $M_{K[P]}$: For a scheme S an element $\alpha \in D(P^{\mathrm{gp}})(S)$ sends an element $[(X, \iota)] \in M_{K[P]}(S)$ to $[(X, \alpha \circ \iota)] \in M_{K[P]}$, where we write also α for the corresponding automorphism of $\mathrm{Spec}(K[P]) \times_{\mathrm{Spec}(K)} S$. By [2, 1.12] the functor $M_{K[P]}$ is representable by a quasi-projective K-scheme, and by [2, 3.4] the action of $D(P^{\mathrm{gp}})$ on $M_{K[P]}$ has only finitely many orbits.

The pair $(\mathrm{Spec}(\mathcal{R}_\Omega), \sigma)$ then defines a point of $M_{K[P]}(\Omega)$. Since the action of $D(P^{\mathrm{gp}})$ on $M_{K[P]}$ has only finitely many orbits we can find a finite extension Ω' of Ω over which there exists an element $\alpha \in D(P^{\mathrm{gp}})(\Omega')$ such that $(\mathrm{Spec}(\mathcal{R}_{\Omega'}), \alpha \circ \sigma)$ defines a K-point of $M_{K[P]}$. The corresponding $\widetilde{G}$-variety $\mathrm{Spec}(R')$ over K then has the desired properties. $\qquad\square$

With notation as in 5.6 fix also an isomorphism of toric varieties $\lambda : \mathrm{Spec}(R'^U) \simeq \mathrm{Spec}(K[P])$. The isomorphism δ induces by taking U-invariants an isomorphism of toric varieties

$$\mathrm{Spec}(\Omega'[P]) \xrightarrow{\ \sigma\ } \mathrm{Spec}(\mathcal{R}^U_{\Omega'}) \xrightarrow{\ \delta^{-1}\ } \mathrm{Spec}(R'^U \otimes_K \Omega') \xrightarrow{\ \lambda\ } \mathrm{Spec}(\Omega'[P]),$$

which corresponds to an element $u \in D(P^{\mathrm{gp}})(\Omega')$.

With notation as in 5.6 observe that the natural map $\iota : E \to \Omega$, induced by the standard map $E \to K[E]^*$, is injective and that $E \cap K^* = \{1\}$. We can read off this map ι from $\mathcal{R}_\Omega$ and the given identifications $\sigma_a : (\mathcal{R}_\Omega)_a \simeq R_a \otimes_K \Omega$ using the multiplicative structure on $\mathcal{R}_\Omega$ (5.4.1). Similarly we get a map $\iota' : E \to K$ from the pair $(\mathrm{Spec}(R'), \lambda)$. The existence of the isomorphism δ implies that the composition of ι' with the inclusion $K \hookrightarrow \Omega'$ is equal to the map

$$E \to \Omega', \quad e \mapsto u(e) \cdot \iota(e).$$

In particular, ι and ι' agree on the kernel κ of the map $E \to P^{\mathrm{gp}}$ so $\iota(\kappa) \subset E \cap K^* = 0$. It follows that $\kappa = 0$. $\qquad\square$

6. The standard construction

In this section we construct certain families of degenerating spherical varieties with log structures. The construction is a generalization of the toric case [7, §3.1].

6.1. Fix an irreducible polarized spherical G-variety (X, L) over k, which we will refer to as the *starting variety*, and let $\widetilde{X}$ be its cone, an affine $\widetilde{G}$-variety. Let $Q = Q(X, L) \subset \Lambda^+_{\mathbb{R}}$ be the corresponding rational convex polytope. Let R be the affine coordinate ring of $\widetilde{X}$ so we have a decomposition

$$R \simeq \oplus_{\lambda \in \widetilde{\Lambda}^+} R_\lambda,$$

where each $R_\lambda = F_\lambda \otimes V(\lambda)$ for a k-vector space F_λ of dimension 0 or 1.

Let S be a paving of Q regular with respect to (X, L).

6.2. For $\omega \in S$ let P_ω denote the intersection $P \cap \mathrm{Cone}(\omega)$. For $\eta \subset \omega$ we have a natural inclusion $P_\eta \subset P_\omega$. Let

$$P_S := \varinjlim_{\omega \in S} P_\omega$$

be the colimit in the category of integral monoids, and let P^{gp}_S be its associated group. By the universal property of the group associated to a monoid we have

$$P^{\mathrm{gp}}_S = \varinjlim_{\omega \in S} P^{\mathrm{gp}}_\omega.$$

For $\omega \in S$ let $\rho_\omega : P_\omega \to P^{\mathrm{gp}}_S$ be the natural map, and define $\rho : P \to P^{\mathrm{gp}}_S$ to be the set map sending $p \in P_\omega$ to $\rho_\omega(p)$ (as in [7, 3.1.4] the map ρ is well-defined). Define $K_S \subset P^{\mathrm{gp}}_S$ to be the submonoid generated by elements of the form

$$\rho(\lambda) + \rho(\mu) - \rho(\eta)$$

where $\lambda, \mu, \eta \in P$ are such that $(\lambda, \mu) \succ \eta$.

Remark 6.3. We will see below that the monoid K_S is finitely generated, but this does not follow immediately from the definition.

Remark 6.4. In [7, 3.1.4] we considered the submonoid $H_S \subset K_S$ generated by elements of the form $\rho(\lambda) + \rho(\mu) - \rho(\lambda + \mu)$. In the case when G is a torus we have $H_S = K_S$ but in general K_S is bigger than H_S.

Remark 6.5. There is a map $P_S^{\mathrm{gp}} \to P^{\mathrm{gp}}$ induced by the inclusions $P_\omega^{\mathrm{gp}} \hookrightarrow P^{\mathrm{gp}}$. In [5], the image of an element $\rho(\lambda) + \rho(\mu) - \rho(\eta)$ in P^{gp} is called a *tail* of $\widetilde{X}$, and the real cone generated by the image of K_S in $\widetilde{\Lambda}_{\mathbb{R}}$ is called the *tail cone*.

6.6. Let $\mathcal{R}$ denote the $k[K_S]$-module $R \otimes_k k[K_S]$ and consider the map

$$\mathcal{R} \otimes_{k[K_S]} \mathcal{R} \to \mathcal{R}$$

sending $f \otimes g \in R_\lambda \otimes R_\mu$ to the element whose η-component is given by the η-component of fg (multiplication in R) multiplied by the element $\rho(\lambda) + \rho(\mu) - \rho(\eta) \in K_S$.

Lemma 6.7. *(i) This gives $\mathcal{R}$ the structure of a commutative $k[K_S]$-algebra.*

(ii) There exists a finite dimensional k-representation V of G and a G-equivariant surjection

(6.7.1) $$k[K_S] \otimes \mathrm{Sym}^{\cdot}(V) \to \mathcal{R}.$$

In particular, $\mathcal{R}$ is a $k[K_S]$-algebra of finite type.

Proof. For (i) it suffices to show that for every homomorphism $g : k[K_S] \to A$ to a discrete valuation ring sending all elements of K_S to nonzero elements of A we get an algebra structure on $\mathcal{R} \otimes_{k[K_S]} A$. Fix a uniformizer $\pi \in A$. The homomorphism g corresponds to two morphisms of monoids $u : K_S \to A^*$ and $h : K_S \to \mathbb{N}$ defined by the formula

$$g(m) = u(m) \cdot \pi^{h(m)}, \quad m \in K_S.$$

After replacing A by a finite extension (which is harmless for the purposes of verifying that $\mathcal{R}$ is an algebra) we can extend these morphisms to morphisms

$$\tilde{u} : \varinjlim_{\omega} P_\omega^{\mathrm{gp}} \to A^*, \quad \tilde{h} : \varinjlim_{\omega} P_\omega^{\mathrm{gp}} \to \mathbb{Z}.$$

We then get an injective morphism of A-modules with G-action

$$\mathcal{R} \otimes_{k[K_S]} A \to R \otimes_k \mathrm{Frac}(A)$$

which on $R_\lambda \otimes_k A \subset \mathcal{R} \otimes_{k[K_S]} A$ is given by $\tilde{u}(\lambda)\pi^{\tilde{h}(\lambda)}$ times the standard inclusion $R_\lambda \otimes_k A \hookrightarrow R_\lambda \otimes_k \mathrm{Frac}(A)$. This inclusion is compatible with the multiplication operations, from which it follows that $\mathcal{R} \otimes_{k[K_S]} A$ is a commutative A-algebra.

To prove (ii) consider first a single $\omega \in S$ and choose a set of generators $\lambda_1, \ldots, \lambda_r \in P_\omega$. Choosing isomorphisms $R_{\lambda_i} \simeq V(\lambda_i)$ we obtain a morphism

(6.7.2) $$k[K_S] \otimes_k \mathrm{Sym}^{\cdot}(V(\lambda_1) \oplus \cdots \oplus V(\lambda_r)) \to \mathcal{R}.$$

If $v_{\lambda_i} \in V(\lambda_i)$ denotes a highest weight vector, then the image of $v_{\lambda_1}^{\otimes n_1} \otimes \cdots \otimes v_{\lambda_r}^{\otimes n_r}$ (for integers $n_i \geq 0$) maps to a highest weight vector in $k[K_S] \otimes_k V(n_1\lambda_1 + \cdots + n_r\lambda_r) \subset \mathcal{R}$. It follows that the image of (6.7.2) contains $\mathcal{R}_\lambda$ for all $\lambda \in P_\omega$. Repeating the above construction for each ω we then find a finite-dimensional representation V of G and a surjective map (6.7.1) as desired. $\square$

6.8. A more subtle issue is to show that the algebra $\mathcal{R}$ is finitely presented as a $k[K_S]$-algebra. Since $k[K_S]$ is not a priori noetherian this is not immediate from the above. In fact, we will show that $\mathcal{R}$ is indeed finitely presented and then deduce from this that K_S is a finitely generated monoid.

Fix a G-representation V and a surjective map (6.7.1) as in (ii). Let $\mathbb{A}(V)$ denote $\mathrm{Spec}(\mathrm{Sym}^{\cdot}(V))$, and let $h : \widetilde{\Lambda}^+ \to \mathbb{N}$ be the function sending all elements of the weight set $\widetilde{\Lambda}^+(X, L)$ to 1 and other elements to 0. We can then consider the equivariant Hilbert functor $\mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V))$ defined in [2, 1.5], which assigns to a k-scheme W the set of finitely presented $\widetilde{G}$-equivariant closed subschemes $Z \hookrightarrow \mathbb{A}(V)_W$ flat over T and with Hilbert function h. By [2, 1.7] the functor $\mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V))$ is a quasi-projective scheme.

Let $\mathbb{A}(V)//U$ denote the spectrum of $\mathrm{Sym}^{\cdot}(V)_\lambda^U$, an affine scheme with action of the torus $\widetilde{T}$. We can then also consider the $\widetilde{T}$-equivariant Hilbert scheme $\mathrm{Hilb}_h^{\widetilde{T}}(\mathbb{A}(V)//U)$. By [2, 1.7] there is a closed embedding

$$q : \mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V)) \hookrightarrow \mathrm{Hilb}_h^{\widetilde{T}}(\mathbb{A}(V)//U)$$

sending $Z \hookrightarrow \mathbb{A}(V)_W$ to $Z//U \hookrightarrow (\mathbb{A}(V)//U)_W$.

The restriction of $\mathcal{R}$ to $k[K_S^{\mathrm{gp}}]$ is finitely presented as $k[K_S^{\mathrm{gp}}]$ is noetherian. The surjection (6.7.1) restricted to $k[K_S^{\mathrm{gp}}]$ therefore defines a morphism

$$b : \mathrm{Spec}(k[K_S^{\mathrm{gp}}]) \to \mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V)).$$

Lemma 6.9. *The morphism b extends to a morphism $b' : \mathrm{Spec}(k[K_S]) \to \mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V))$.*

Proof. Fix an isomorphism $R^U \simeq k[P]$ compatible with the $\widetilde{T}$-actions. Let $H_S \subset K_S$ be the submonoid defined in [7, 3.1.4]. Then with notation as in [7, 3.1.9] we have

$$\mathcal{R}^U \simeq k[P \rtimes H_S] \otimes_{k[H_S]} k[K_S].$$

By [7, 3.1.8] the monoid H_S is finitely generated, from which it follows that $\mathcal{R}^U$ is finitely presented over $k[K_S]$. The algebra $\mathcal{R}^U$ therefore defines a morphism

$$a : \mathrm{Spec}(k[K_S]) \to \mathrm{Hilb}_h^{\widetilde{T}}(\mathbb{A}(V)//U)$$

fitting into a commutative diagram of solid arrows

$$
\begin{array}{ccc}
\mathrm{Spec}(k[K_S]) & \xrightarrow{\ a\ } & \mathrm{Hilb}_h^{\widetilde{T}}(\mathbb{A}(V)//U) \\
\downarrow & \overset{a'}{\dashrightarrow} & \uparrow q \\
\mathrm{Spec}(k[K_S^{\mathrm{gp}}]) & \xrightarrow{\ b\ } & \mathrm{Hilb}_h^{\widetilde{G}}(\mathbb{A}(V)).
\end{array}
$$

Since q is a closed embedding it follows that the indicated morphism a' exists. $\square$

Corollary 6.10. *The $k[K_S]$-algebra $\mathcal{R}$ is finitely presented.*

Proof. The morphism a' corresponds to a surjection

$$k[K_S] \otimes \mathrm{Sym}^{\cdot}(V) \twoheadrightarrow \mathcal{R}'$$

which since $q \circ a' = a$ gives a factorization of (6.7.1) as

$$k[K_S] \otimes \mathrm{Sym}^{\cdot}(V) \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R}$$

with $\mathcal{R}' \to \mathcal{R}$ an isomorphism over $k[K_S^{\mathrm{gp}}]$. Since $\mathcal{R}'$ and $\mathcal{R}$ are both flat over $k[K_S]$ this implies that $\mathcal{R}' \simeq \mathcal{R}$. $\qquad\square$

Corollary 6.11. *The monoid K_S is finitely generated.*

Proof. Since $\mathcal{R}$ is finitely presented over $k[K_S]$ there exists a finitely generated submonoid $M \subset K_S$ such that $M^{\mathrm{gp}} = K_S^{\mathrm{gp}}$ and a model $\mathcal{R}_0$ over $k[M]$ with an isomorphism $\mathcal{R}_0 \otimes_{k[M]} k[K_S] \simeq \mathcal{R}$. This gives an embedding of $k[M]$-algebras with $\widetilde{G}$-action $\mathcal{R}_0 \hookrightarrow R \otimes_k k[K_S^{\mathrm{gp}}]$. Now if $V(\eta)$ occurs in $R_\lambda \cdot R_\mu$, then the η-component of $\mathcal{R}_{0,\lambda} \cdot \mathcal{R}_{0,\mu}$ must be contained in $R_\eta \otimes k[M]$, which implies that $K_S \subset M$, whence $M = K_S$. $\qquad\square$

6.12. There is a log structure on $\mathrm{Spec}(\mathcal{R})$ defined as follows. The ring of invariants R^U is a toric variety with torus $D(P^{\mathrm{gp}})$. Choose an isomorphism $R^U \simeq k[P]$. If $e^p \in R^U$ is the "monomial" corresponding to $p \in P$, then the image of e^p in R_λ is a highest weight vector $v_\lambda \in R_\lambda$ (which is noncanonically isomorphic to $V(\lambda)$). These highest weight vectors v_λ have the property that $v_\lambda \cdot v_\mu$ equal $v_{\lambda+\mu}$ in R.

Let $K_S \rtimes P$ be the monoid whose underlying set is the product $K_S \times P$ but with additive structure given by

$$(h, p) + (h', p') := ((\rho(p) + \rho(p') - \rho(p + p')) + h + h', p + p').$$

See [7, 3.1.9] for a further discussion of this construction. We then get a morphism of monoids

(6.12.1) $$K_S \rtimes P \to \mathcal{R}$$

induced by the set map $P \to \mathcal{R}$ sending p to v_p and the natural map $K_S \to k[K_S]$. This defines a log structure $M_{\mathcal{R}}$ on $\mathrm{Spec}(\mathcal{R})$. Furthermore if M_{K_S} denotes the log structure on $\mathrm{Spec}(k[K_S])$ defined by $K_S \to k[K_S]$ then we get a morphism of fine log schemes

(6.12.2) $$(\tilde{f}, \tilde{f}^b) : (\mathrm{Spec}(\mathcal{R}), M_{\mathcal{R}}) \to (\mathrm{Spec}(k[K_S]), M_{K_S}).$$

There is an action of $\widetilde{B}$ on $(\mathrm{Spec}(\mathcal{R}), M_{\mathcal{R}})$ over $(\mathrm{Spec}(k[K_S]), M_{K_S})$. In particular, removing the origin in $\mathrm{Spec}(\mathcal{R})$ and taking the quotient by $\mathbb{G}_m$ we obtain a family of polarized log schemes with B-action

(6.12.3) $$(f, f^b) : (\mathscr{X}, M_{\mathscr{X}}, L_{\mathscr{X}}) \to (\mathrm{Spec}(k[K_S]), M_{K_S}),$$

as well as an extension of the B-action to a G-action on the underlying scheme $\mathscr{X}$.

Lemma 6.13. *Let* $\deg : P \to \mathbb{N}$ *be the map corresponding to the grading on P. Then for every $n \geq 0$ the natural map*

$$\oplus_{\lambda \in P, \deg(\lambda)=n} \mathcal{R}_\lambda \to f_* L_{\mathscr{X}}^{\otimes n}$$

is an isomorphism, and the formation of $f_* L_{\mathscr{X}}^{\otimes n}$ commutes with arbitrary base change $S \to \mathrm{Spec}(k[K_S])$.

Proof. This follows from the same argument as in [1, Proposition 2.2.3 and discussion following Definition 3.2.1]. $\qquad\square$

6.14. Let $\mathscr{C}^\circ$ denote the complement of the origin in $\mathrm{Spec}(\mathcal{R})$, and let $M_{\mathscr{C}^\circ}$ denote the pullback of the log structure $M_{\mathscr{X}}$ to $\mathscr{C}^\circ$. The chart $K_S \rtimes P \to M_{\mathcal{R}}$ on $\mathrm{Spec}(\mathcal{R})$ induces a map $K_S \rtimes P \to \overline{M}_{\mathscr{C}^\circ}$ which is $\widetilde{B}$-equivariant, and hence descends to a morphism $\gamma : K_S \rtimes P \to \overline{M}_{\mathscr{X}}$, which locally lifts to a chart.

6.15. There is a morphism $k[K_S] \to k$ defined as follows. Let $K_S^{\mathrm{sat},\times} \subset K_S^{\mathrm{sat}}$ denote the units in the saturation of K_S. Then the quotient $\overline{K}_S^{\mathrm{sat,gp}} := K_S^{\mathrm{gp}}/K_S^{\mathrm{sat},\times}$ is torsion free, and therefore there exists a (non-canonical) splitting $K_S^{\mathrm{sat}} \simeq K_S^{\mathrm{sat},\times} \oplus \overline{K}_S^{\mathrm{sat}}$. This implies that there exists a unique homomorphism $K_S^{\mathrm{sat}} \to k$ (viewed as a multiplicative monoid) sending $K_S^{\mathrm{sat},\times}$ to 1 and the nonunits of K_S^{sat} to 0. Let $v : k[K_S] \to k$ denote the induced morphism, and let $k[[K_S]]$ denote the completion of $k[K_S]$ with respect to this maximal ideal.

6.16. Let $\beta : K_S \to k$ be an arbitrary morphism. Define a new paving S' of Q which is coarser than S as follows. By blowing up $k[K_S]$ along β and looking at a generic point of the exceptional fiber, we can find a field extension L of k, an inclusion $\tilde{\beta} : k[K_S] \hookrightarrow A$ into a discrete valuation ring A with residue field L such that the diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(L) & \hookrightarrow & \mathrm{Spec}(A) \\
\downarrow & & \downarrow{\tilde{\beta}} \\
\mathrm{Spec}(k) & \stackrel{\beta}{\hookrightarrow} & \mathrm{Spec}(k[K_S])
\end{array}
$$

commutes. Let $\rho : K_S \to \mathbb{N}$ be the map induced by the map $\tilde{\beta}$ and the valuation on A, and write also $\rho : K_S^{\mathrm{gp}} \to \mathbb{Z}$ for the induced map on groups. Let $\mathcal{R}_A$ denote the base change of $\mathcal{R}$ to A, so $\mathcal{R}_A$ is a subring of $R \otimes_k K$, where K denotes the field of fractions of A. Then the image of $\mathcal{R}_{A,\lambda}$ in $R \otimes_k K$ is equal to $\pi^{h'(\lambda)} \cdot V(\lambda) \otimes_k A \subset V(\lambda) \otimes_k K$, for some function $h' : P \to \mathbb{Z}$.

Lemma 6.17. *The function h' is admissible with respect to (X, L) and defines a regular paving S' of Q which is coarser than S.*

Proof. Note that h' is linear on P_ω for each $\omega \in S$. Furthermore, if $V(\eta)$ occurs in $R_\lambda \cdot R_\mu$ then we must have $h'(\lambda) + h'(\mu) \geq h'(\eta)$ since $\mathcal{R}$ is an algebra. It follows that the function h' is admissible and also that the associated paving S' is coarser than S. $\qquad\square$

6.18. Define P_S^{gp} and $P_{S'}^{\mathrm{gp}}$ as in 6.2. Since S' is coarser than S there is a surjection

$$
P_S^{\mathrm{gp}} \to P_{S'}^{\mathrm{gp}}
$$

which induces a surjection $\pi : K_S \to K_{S'}$.

6.19. Let

$$
(\mathscr{X}', M_{\mathscr{X}'}, L_{\mathscr{X}'}) \to (\mathrm{Spec}(k[K_{S'}]), M_{K_{S'}})
$$

be the polarized log scheme with B-action obtained from the construction of (6.12.3) using S' instead of S, and let $(Y', M_{Y'}, L_{Y'}) \to (\mathrm{Spec}(k), M_k')$ be the base change along the vertex morphism $k[K_{S'}] \to k$.

Also let $(Y_\beta, M_{Y_\beta}, L_{Y_\beta})/(\mathrm{Spec}(k), M_\beta)$ be the polarized log scheme obtained by pullback along β of the family (6.12.3).

Proposition 6.20. *There exists a commutative diagram*

$$
\begin{array}{ccc}
(Y_\beta, M_{Y_\beta}) & \xrightarrow{\;(g,g^b)\;} & (Y', M_{Y'}) \\
\downarrow & & \downarrow \\
(\mathrm{Spec}(k), M_\beta) & \xrightarrow{\;(\mathrm{id},\sigma)\;} & (\mathrm{Spec}(k), M'_k)
\end{array}
$$

where the horizontal morphisms are isomorphisms, (g, g^b) is B-equivariant and g is G-equivariant, and a G-equivariant isomorphism $g^* L_{Y'} \simeq L_{Y_\beta}$.

Proof. Fix a map $\tilde{\beta} : K_S \to A$ to a discrete valuation ring A, as in 6.16. After possibly replacing A by a finite extension we can extend this to a map $K_S^{\mathrm{sat}} \to A$, which we will again denote by $\tilde{\beta}$. Let $F \subset K_S$ denote the face generated by elements $\rho(\lambda) + \rho(\mu) - \rho(\lambda + \mu)$ where $\lambda, \mu \in P$ lie in a single P_ω for some $\omega \in S'$, and let $K_{S,F}$ denote the localization of K_S with respect to the face F.

Lemma 6.21. *The surjection $K_S \to K_{S'}$ induces an isomorphism $K_{S,F}/F^{\mathrm{gp}} \simeq K_{S'}$.*

Proof. Let P_S and $P_{S'}$ be as in 6.2. Then by the same argument as in [7, Proof of 3.1.21] the kernel of the surjection $P_S^{\mathrm{gp}} \to P_{S'}^{\mathrm{gp}}$ is generated by elements $\rho(\lambda) + \rho(\mu) - \rho(\lambda + \mu)$, where λ and μ lie in a single $\omega \in S'$. Since these elements already lie in F this implies that the induced map

$$
K_{S,F}/F^{\mathrm{gp}} \to K_{S'}
$$

is injective. Since it is evidently surjective we obtain the lemma. $\square$

Note that we have $\tilde{\beta}(F) \subset A^*$. After possibly replacing A by a finite extension once more, we can find a homomorphism $\gamma : K_S^{\mathrm{gp}} \to A^*$ whose restriction to F^{gp} is equal to the restriction of β. Let $\beta' : K_S \to A$ be the map sending $h \in K_S$ to $\gamma(h)^{-1} \cdot \beta(h)$. Let $\mathcal{R}'_A$ denote the base change along β' of $\mathcal{R}$.

Fix an extension $\tilde{\gamma} : (K_S \rtimes P)^{\mathrm{gp}} \to A^*$ of γ. Then the map

$$
\mathcal{R}_A \to \mathcal{R}_{A'}
$$

induced by multiplication by $\tilde{\gamma}(\lambda)$ on the λ-components is an isomorphism of $k[K_S]$-algebras. Furthermore, we have a commutative diagram

where the maps labelled κ are the charts (6.12.1). From this it follows that it suffices to prove 6.20 after replacing β by β'. We may therefore assume that $\beta(F) = 1$, in which case

the map $\tilde{\beta}$ factors through a map $\tilde{\beta}' : K_{S'} \to A$. Similarly, we may assume that the induced map $\tilde{\beta}' : K_{S'}^{\mathrm{sat}} \to A$ sends $K_{S'}^{\times}$ to 1, in which case the induced map $k[K_{S'}] \to k$ is the vertex. In this case proposition 6.20 follows immediately from the construction. $\qquad\square$

7. SOME CALCULATIONS FOR THE STANDARD FAMILY

In this section we collect together various calculations of global sections that will be used in what follows.

7.1. Fix an irreducible polarized spherical variety (X_0, L_0) over k with moment polytope Q and let S be a paving of Q admissible with respect to (X_0, L_0). Let $(f, f^b) : (X, M_X, L) \to (\mathrm{Spec}(k), M_k)$ be the polarized log scheme obtained by base change from the standard construction along a morphism $K_S \to k$ factoring through a morphism $K_S^{\mathrm{sat}} \to k$ sending all nonunits of K_S^{sat} to 0. Let R (resp. R_0) be the coordinate ring of the cone of (X, L) (resp. (X_0, L_0)). So if $(\widetilde{X}, M_{\widetilde{X}})/(\mathrm{Spec}(k), M_k)$ denotes the base change of the log scheme (6.12.2) then $\widetilde{X}$ is affine with coordinate ring R.

To ease notation let E denote the log scheme $(\mathrm{Spec}(k), M_k)$. Let $M_{X/E}$ be the relative characteristic of $(X, M_X)/E$. By definition this is the cokernel in the category of sheaves of integral monoids on X of the morphism $f^b : f^* M_E \to M_X$. Similarly define the relative characteristic $M_{\widetilde{X}/E}$ of $(\widetilde{X}, M_{\widetilde{X}})/(\mathrm{Spec}(k), M_k)$.

Proposition 7.2. *The natural map $P^{\mathrm{gp}} \to H^0(\widetilde{X}, M_{\widetilde{X}/E}^{\mathrm{gp}})$ is an isomorphism.*

The proof occupies the following 7.3–7.9.

7.3. For $\omega \in S$ we have a natural inclusion $P_\omega \hookrightarrow K_S \rtimes P$ which induces a morphism of log structures $M_\omega \hookrightarrow M_X$ on X, where M_ω is the log structure induced by the composition $P_\omega \to K_S \rtimes P \to R$. Let $R \twoheadrightarrow R_\omega$ be the quotient corresponding to the irreducible $\widetilde{G}$-subscheme of $\mathrm{Spec}(R)$ defined by ω. So $R_{\omega,a}$ is nonzero if and only if a lies in the integral points P_ω of the cone over ω. Let $i_\omega : \widetilde{X}_\omega \hookrightarrow \widetilde{X}$ be the spectrum of R_ω.

Note that
$$\widetilde{X}_\omega // U := \mathrm{Spec}(R_\omega^U)$$
is an irreducible affine toric variety, and $M_\omega|_{\widetilde{X}_\omega}$ is the pullback of the standard log structure on this toric variety.

We have an exact sequence of sheaves of groups on $\widetilde{X}_\omega$

(7.3.1) $$0 \to \overline{M}_\omega^{\mathrm{gp}}|_{\widetilde{X}_\omega} \to M_{\widetilde{X}/E}^{\mathrm{gp}}|_{\widetilde{X}_\omega} \to P^{\mathrm{gp}}/P_\omega^{\mathrm{gp}} \to 0,$$

where $P^{\mathrm{gp}}/P_\omega^{\mathrm{gp}}$ denotes the constant sheaf. To prove 7.2 it suffices to show that the map
$$P_\omega^{\mathrm{gp}} \to H^0(\widetilde{X}_\omega, \overline{M}_\omega^{\mathrm{gp}}|_{\widetilde{X}_\omega})$$
is an isomorphism. For then using (7.3.1) we get that the map
$$P^{\mathrm{gp}} \to H^0(\widetilde{X}_\omega, M_{\widetilde{X}/E}^{\mathrm{gp}}|_{\widetilde{X}_\omega})$$

is an isomorphism. Now choose an ordering $\{\omega_1, \ldots, \omega_r\}$ of the top dimensional simplices of S and for two indices i, j let ω_{ij} denote the intersection $\omega_i \cap \omega_j$. Then we have an exact sequence of sheaves on $\widetilde{X}$

$$0 \to M^{\mathrm{gp}}_{\widetilde{X}/E} \to \prod_{j=1}^{r} i_{\omega_j *} M^{\mathrm{gp}}_{\widetilde{X}/E}|_{\widetilde{X}_{\omega_j}} \to \prod_{j<k} i_{\omega_{jk}*} M^{\mathrm{gp}}_{\widetilde{X}/E}|_{\widetilde{X}_{\omega_{jk}}}.$$

Taking global sections we then get 7.2 in general.

It therefore suffices to prove 7.2 in the case when X is irreducible, which we assume henceforth. Note that in this case $M_{X/E} = \overline{M}_X$.

Lemma 7.4. *Let X be a finite type k-scheme and let Λ be a torsion Gorenstein local ring of dimension 0. Let $r \geq 0$ be an integer and let $f : X \times \mathbb{A}^r \to X$ be the first projection. Then for any F in the derived category $D^b_c(X, \Lambda)$ of bounded complexes of Λ-modules on the étale site of X with constructible cohomology sheaves the adjunction map*

$$F \to Rf_* f^* F$$

is an isomorphism.

Proof. Using Verdier duality it suffices to show the dual statement: For any $F \in D^b_c(X, \Lambda)$ the adjunction map

$$Rf_! f^! F \to F$$

is an isomorphism. Note that $f^! F \simeq f^* F(r)[2r]$ since f is smooth of relative dimension r. By the base change theorem for $Rf_!$ we can verify this after base changing to a point. Applying duality again over a point this then reduces the proof to the case when $X = \mathrm{Spec}(k)$ where the result is immediate. $\square$

Remark 7.5. Lemma 7.4 remains true (with the same proof) over fields of positive characteristic invertible in Λ.

For a k-scheme T, let $\widetilde{X}_T$ denote the base change of $\widetilde{X}$ to T and let $M_{\widetilde{X}_T}$ denote the pullback of the log structure $M_{\widetilde{X}}$ to $\widetilde{X}_T$.

Lemma 7.6. *For any k-scheme T the action of $U(T)$ on $H^0(\widetilde{X}_T, \overline{M}^{\mathrm{gp}}_{\widetilde{X}_T})$ is trivial.*

Proof. For an integer N let F_N denote the sheaf $\overline{M}^{\mathrm{gp}}_{\widetilde{X}_T} / N\overline{M}^{\mathrm{gp}}_{\widetilde{X}_T}$. It suffices to show that the $U(T)$ action on each $H^0(\widetilde{X}_T, F_N)$ is trivial. Consider the diagram

$$\widetilde{X}_T \times_T U_T \xrightarrow{\mu} \widetilde{X}_T$$
$$\downarrow f$$
$$\widetilde{X}_T,$$

where the map f is the projection and the map μ is the action. This is a diagram of schemes over $(X/\!/U)_T$, and since F_N is pulled back from $(X/\!/U)_T$ we get an isomorphism $\epsilon : \mu^* F_N \to f^* F_N$.

For a point $u \in U(T)$ the action on $H^0(\widetilde{X}_T, F_N)$ is then the composite map
(7.6.1)
$$H^0(\widetilde{X}_T, F_N) \xrightarrow{\mu^*} H^0(\widetilde{X}_T \times_T U_T, \mu^* F_N) \xrightarrow{\epsilon} H^0(\widetilde{X}_T \times_T U_T, f^* F_N) \xrightarrow{(\mathrm{id} \times u)^*} H^0(\widetilde{X}_T, F_N).$$

Now observe that the map

$$H^0(\widetilde{X}_T \times_T U_T, f^* F_N) \xrightarrow{(\mathrm{id} \times u)^*} H^0(\widetilde{X}_T, F_N)$$

is inverse to the pullback map $f^* : H^0(\widetilde{X}_T, F_N) \to H^0(\widetilde{X}_T \times_T U_T, f^* F_N)$, which is an isomorphism by 7.4. In particular, the composite map (7.6.1) is independent of the choice of u. Taking u the identity element of U we obtain the lemma. $\qquad\square$

Remark 7.7. It follows that we get an action of U on $H^0(\widetilde{X}, M_{\widetilde{X}}^{\mathrm{gp}})$. Indeed for any $m \in H^0(\widetilde{X}, M_{\widetilde{X}}^{\mathrm{gp}})$ and a point $u \in U(T)$ over some scheme T, there exists a unique unit $v_m(u) \in \Gamma(\widetilde{X}_T, \mathscr{O}_{\widetilde{X}_T}^*) = \Gamma(T, \mathscr{O}_T^*)$ for which in $H^0(\widetilde{X}_T, M_{\widetilde{X}_T}^{\mathrm{gp}})$ we have $u^* m = v_m(u) + m$. This in fact defines a morphism of group schemes $v_m : U \to \mathbb{G}_m$ which must be trivial since U is unipotent. It follows that the action of U on $H^0(\widetilde{X}, M_{\widetilde{X}}^{\mathrm{gp}})$ is also trivial.

Let $j : V \hookrightarrow \widetilde{X}//U$ be the locus of triviality of the log structure on $\widetilde{X}//U$ (so V is isomorphic to a torus), and let $\tilde{j} : \widetilde{X}_V \hookrightarrow \widetilde{X}$ be the preimage.

Lemma 7.8. *The natural maps*

$$k^* \oplus P^{\mathrm{gp}} \to H^0(\widetilde{X}_V, \mathscr{O}_{\widetilde{X}_V}^*)^{U(k)}, \quad P^{\mathrm{gp}} \to H^0(\widetilde{X}, (\tilde{j}_* \mathscr{O}_{\widetilde{X}_V}^* / k^*))^{U(k)}$$

are isomorphisms.

Proof. Note first of all that since k^* is a constant sheaf and $\widetilde{X}$ is irreducible the map

$$H^0(\widetilde{X}_V, \mathscr{O}_{\widetilde{X}_V}^*) \to H^0(\widetilde{X}, (\tilde{j}_* \mathscr{O}_{\widetilde{X}_V}^* / k^*))$$

is surjective. We claim that it remains surjective after taking U-invariants. To see let $\bar{m} \in H^0(\widetilde{X}, (\tilde{j}_* \mathscr{O}_{\widetilde{X}_V}^* / k^*))$ be a $U(k)$-invariant section, and let $T_{\bar{m}}$ be the set of liftings of $\bar{m}$ to $H^0(\widetilde{X}_V, \mathscr{O}_{\widetilde{X}_V}^*)$. The set $T_{\bar{m}}$ has a simply transitive action of k^* and a commuting $U(k)$-action. To prove the lemma it suffices to show that $T_{\bar{m}}$, with its k^* and $U(k)$ actions, is the k-points of a $\mathbb{G}_m$-torsor with U-action. For such torsors are classified by homomorphisms $U \to \mathbb{G}_m$, and all such homomorphisms are trivial since U is unipotent.

Consider the inclusion of sets with k^* and $U(k)$-action

$$H^0(\widetilde{X}_V, \mathscr{O}_{\widetilde{X}_V}^*) \hookrightarrow H^0(\widetilde{X}_V, \mathscr{O}_{\widetilde{X}_V}).$$

Let R_V denote the coordinate ring of $\widetilde{X}_V$. We can then consider the sheaf $\mathscr{A}$ on the big étale site of $\mathrm{Spec}(k)$ associating to any k-algebra B the ring $R_V \otimes_k B$. This is a sheaf with U-action and we also have an action of $\mathbb{G}_m$ on this sheaf. Let $\mathscr{Q}$ denote the quotient, in the category of sheaves on the big étale site of $\mathrm{Spec}(k)$, of $\mathscr{A}$ by the $\mathbb{G}_m$-action. The section $\bar{m}$ then defines a global section $q_{\bar{m}} \in H^0(\mathrm{Spec}(k), \mathscr{Q})$ and $T_{\bar{m}}$ is the k-points of the $\mathbb{G}_m$-torsor of liftings of $q_{\bar{m}}$ to $\mathscr{A}$.

It follows that it suffices to show that the map $k^* \oplus P^{\mathrm{gp}} \to \Gamma(\widetilde{X}_V, \mathscr{O}^*_{\widetilde{X}_V})^{U(k)}$ is an isomorphism. This follows from noting that $\Gamma(\widetilde{X}_V, \mathscr{O}^*_{\widetilde{X}_V})^{U(k)} = \Gamma(V, \mathscr{O}^*_V)$ and that V is isomorphic to the torus associated to P^{gp}. $\qquad\square$

7.9. To complete the proof of 7.2 observe that the map $M_X \to \mathscr{O}_{\widetilde{X}}$ induces an inclusion of sheaves

$$\overline{M}^{\mathrm{gp}}_X \hookrightarrow (\tilde{j}_* \mathscr{O}^*_{\widetilde{X}_V})/k^*,$$

and therefore also an inclusion

$$\theta : H^0(\widetilde{X}, \overline{M}^{\mathrm{gp}}_{\widetilde{X}}) \hookrightarrow H^0(\widetilde{X}, (\tilde{j}_* \mathscr{O}^*_{\widetilde{X}_V})/k^*)^{U(k)} = P^{\mathrm{gp}}.$$

Since the composite of θ with the map $\rho : P^{\mathrm{gp}} \to H^0(\widetilde{X}, \overline{M}^{\mathrm{gp}}_{\widetilde{X}})$ is the identity on P^{gp} this implies that ρ is an isomorphism.

This completes the proof of 7.2. $\qquad\square$

Let $\widetilde{X}^\circ$ denote the complement of the vertex in $\widetilde{X}$.

Lemma 7.10. *(i) The natural map $P^{\mathrm{gp}} \to H^0(\widetilde{X}^\circ, M^{\mathrm{gp}}_{\widetilde{X}/E})$ is an isomorphism.*

(ii) The map $P^{\mathrm{gp}} \to H^0(X, M^{\mathrm{gp}}_{X/E})$ is an isomorphism.

Proof. Statement (i) follows from the same reasoning proving 7.2, replacing $\widetilde{X}$ by $\widetilde{X}^\circ$. Statement (ii) follows from (i) noting that there is an inclusion $H^0(X, M^{\mathrm{gp}}_{X/E}) \hookrightarrow H^0(\widetilde{X}^\circ, M^{\mathrm{gp}}_{\widetilde{X}/E})$ compatible with the maps from P^{gp}. $\qquad\square$

Remark 7.11. A variant of the argument proving 7.2 also shows that the restriction map

$$H^0(\widetilde{X}, M_{\widetilde{X}/E}) \to H^0(\widetilde{X}^\circ, M_{\widetilde{X}^\circ/E})$$

is an isomorphism.

Remark 7.12. By the same argument as in [7, 2.3.7] the restriction map

$$\Gamma(\widetilde{X}, \mathscr{O}_{\widetilde{X}}) \to \Gamma(\widetilde{X}^\circ, \mathscr{O}_{\widetilde{X}^\circ})$$

is an isomorphism. The isomorphism $P^{\mathrm{gp}} \to H^0(X, M^{\mathrm{gp}}_{X/E})$ is characterized by the property that for any $p \in P$ the image of p in $H^0(X, M^{\mathrm{gp}}_{X/E})$ admits a lifting to $H^0(X, M_X)$ whose image in $\Gamma(\widetilde{X}^\circ, \mathscr{O}_{\widetilde{X}^\circ}) \simeq \Gamma(\widetilde{X}, \mathscr{O}_{\widetilde{X}})$ maps to a highest weight vector for the representation corresponding to p.

Lemma 7.13. *Let $\alpha \in P^{\mathrm{gp}}$ be an element not in P, and let K'_S denote the image of K_S in $K^{\mathrm{sat}}_S/K^{\mathrm{sat}*}_S$. Then there exist morphisms of monoids $h : K'_S \rtimes P \to \mathbb{N}$ and $g : P \to \mathbb{N}$ such that the following hold:*

  (i) $h(K'_S - \{0\}) \subset \mathbb{N}_{>0}$.
  (ii) *There exists a nonzero element $p \in P$ such that $h(0, p) = 0$ and $g(p) = 0$.*
  (iii) $g^{\mathrm{gp}}(\alpha) < 0$, *where $g^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathbb{Z}$ is the map induced by g.*

Proof. Recall [4, 5.8 (1)] that we have

$$P = \bigcap_F P_F,$$

where the intersection runs over the faces of codimension 1 of P. We can therefore find a face F of the polytope Q such that α does not lie in P_F. By duality for monoids [6, 2.2.3] we can then find a homomorphism $g : P \to \mathbb{N}$ sending F to 0 and such that $g^{\mathrm{gp}}(\alpha) < 0$. Note that the subset $\{0\} \times F \subset K_S' \rtimes P$ is also a face. We can therefore find a homomorphism

$$h : K_S' \rtimes P \to \mathbb{N}$$

sending $\{0\} \times F$ to 0 and $K_S' - \{0\}$ to $\mathbb{N}_{>0}$. . $\qquad\square$

Lemma 7.14. *The diagram*

$$\begin{array}{ccc}
P & \longrightarrow & P^{\mathrm{gp}} \\
\downarrow & & \downarrow{\scriptstyle (7.10\ (\mathrm{ii}))} \\
H^0(X, M_{X/E}) & \longrightarrow & H^0(X, M_{X/E}^{\mathrm{gp}})
\end{array}$$

is cartesian, so in particular the natural map $P \to H^0(X, M_{X/E})$ is an isomorphism.

Proof. Let $\alpha \in P^{\mathrm{gp}}$ be an element not in P whose image in $H^0(X, M_{X/E}^{\mathrm{gp}})$ lies in $H^0(X, M_{X/E})$. We show that this gives a contradiction. Choose $h : K_S \rtimes P \to \mathbb{N}$ and $g : P \to \mathbb{N}$ as in 7.13. Let N denote the log structure $k^* \oplus \mathbb{N} \oplus \mathbb{N}$ on $\mathrm{Spec}(k)$ sending all nonzero elements of $\mathbb{N}^2$ to 0 in k. We then have a commutative diagram of monoids

$$\begin{array}{ccc}
K_S \rtimes P & \xrightarrow{h \oplus g} & \mathbb{N} \oplus \mathbb{N} \\
\uparrow & & \uparrow{\scriptstyle a \mapsto (a,0)} \\
K_S & \xrightarrow{h|_{K_S}} & \mathbb{N}
\end{array}$$

such that the induced morphism on relative characteristics $P \to \mathbb{N}$ is the map g. This diagram induces a morphism of log schemes

$$(\mathrm{Spec}(k), N) \to (\mathrm{Spec}(R^U), M_{R^U})$$

over $(\mathrm{Spec}(k), M_k)$. As discussed in [2, proof of 1.2] the map $\mathrm{Spec}(R) \to \mathrm{Spec}(R^U)$ is surjective. We can therefore lift this point to a morphism of log schemes

$$(\mathrm{Spec}(k), N) \to (\widetilde{X}, M_{\widetilde{X}}).$$

The image of this point is not the vertex of $\widetilde{X}$ since g sends a nonzero face to 0 in $\mathbb{N}$. We therefore get a morphism of log schemes

$$(t, t^b) : (\mathrm{Spec}(k), N) \to (X, M_X)$$

such that the induced map

$$P^{\mathrm{gp}} \simeq H^0(X, M_{X/E}^{\mathrm{gp}}) \to (N/M_k)^{\mathrm{gp}} \simeq \mathbb{Z}$$

is the map g and sends $H^0(X, M_{X/E})$ to $\mathbb{N}$. This is a contradiction since $g(\alpha) < 0$. $\qquad\square$

8. ENRICHED LOG STRUCTURES

8.1. Let T be a k-scheme and consider a collection of data

(8.1.1) $$(M_T, (f, f^b) : (X, M_X) \to (T, M_T), (\rho, \rho^b), L)$$

as follows:

(i) M_T is a fine log structure on T.

(ii) $(f, f^b) : (X, M_X) \to (T, M_T)$ is a morphism of fine log schemes whose underlying morphism of schemes f is proper, locally of finite presentation, and flat.

(iii) ρ is an action of G on X/T and ρ^b is an action of B on (X, M_X) over (T, M_T) such that the underlying action of B on X is equal to the action obtained by restriction from G.

(iv) L is a G-linearized ample invertible sheaf on X such that (X, L) is a stable spherical variety over T.

(v) For every geometric point $\bar{t} \to T$ the pullback of the collection (8.1.1) to $\bar{t}$ is isomorphic to that obtained by the standard construction from some starting variety (X_0, L_0) and some paving regular with respect to (X_0, L_0).

Let $\mathcal{R}$ denote the quasi-coherent sheaf of $\mathcal{O}_T$-algebras given by the relative cone of (X, L), and for $a \in P$ let $\mathcal{R}_a$ denote the corresponding $\mathcal{O}_T$-submodule of $\mathcal{R}$. Let $\tilde{f} : \widetilde{X} \to T$ denote the relative spectrum of $\mathcal{R}$, and let $\tilde{f}^\circ : \widetilde{X}^\circ \to T$ denote the complement of the vertex so that X is the quotient of $\widetilde{X}^\circ$ by the natural $\mathbb{G}_m$-action. Let $M_{\widetilde{X}^\circ}$ denote the pullback of M_X.

By the proper base change theorem and 7.14, the stalk of $f_* M_{X/T}$ is isomorphic to P at every geometric point. Assume in addition to the above that the following conditions hold:

(vi) There exists an isomorphism $P \to f_* M_{X/T}$ inducing the isomorphism 7.14 in every geometric fiber.

(vii) The natural map $\tilde{f}^\circ_* M_{\widetilde{X}^\circ} \to f_* M_{X/T}$ is surjective, and the same remains true after arbitrary base change $T' \to T$.

Remark 8.2. Note that it follows from (vi) that the pullback map $f_* M_{X/T} \to \tilde{f}^\circ_* M_{\widetilde{X}^\circ/T}$ is an isomorphism. The map $\tilde{f}^\circ_* M_{\widetilde{X}^\circ} \to f_* M_{X/T}$ in (vii) is the one induced from this and the map $\tilde{f}^\circ_* M_{\widetilde{X}^\circ} \to \tilde{f}^\circ_* M_{\widetilde{X}^\circ/T}$ arising from the quotient map $M_{\widetilde{X}^\circ} \to M_{\widetilde{X}^\circ/T}$.

8.3. The discussion of 2.5 then applies and we get for every element $p \in P$ an $\mathcal{O}_T^*$-torsor L_p and for any two elements $p, q \in P$ a map

$$\theta_{p,q} : L_p \wedge L_q \to M_T \wedge L_{p+q}.$$

Lemma 8.4. *The restriction map*

$$\mathcal{R} \to \tilde{f}^\circ_* \mathcal{O}_{\widetilde{X}^\circ}$$

is an isomorphism.

Proof. Since $X = [\widetilde{X}^\circ/\mathbb{G}_m]$ we have

$$\tilde{f}^\circ_* \mathcal{O}_{\widetilde{X}^\circ} = \oplus_{n \in \mathbb{Z}} f_* L^{\otimes n}.$$

By [1, paragraph following 3.2.1] we have

$$\mathcal{R} \simeq \oplus_{n \geq 0} f_* L^{\otimes n},$$

so to prove the lemma it suffices to show that $f_* L^{\otimes n} = 0$ for $n < 0$ which follows from 7.12 which shows it holds in every fiber. $\qquad \square$

8.5. For $a \in P$ let

$$\alpha_a : L_a \to \mathcal{R}_a$$

denote the map induced by the map

$$\tilde{f}_*^\circ M_{\widetilde{X}^\circ} \xrightarrow{\alpha_{M_{\widetilde{X}^\circ}}} \tilde{f}_*^\circ \mathcal{O}_{\widetilde{X}^\circ} \simeq \mathcal{R}.$$

Definition 8.6. A *logarithmic enrichment* $\underline{\eta} = \{\eta_{a,b}^c\}$ on the family (8.1.1) is a collection of maps

$$\eta_{a,b}^c : L_a \wedge L_b \to M_T \wedge L_c,$$

one for each triple $a, b, c \in P$ with $(a, b) \succ c$, such that the following conditions hold:

(i) For $a, b \in P$ we have $\eta_{a,b}^{a+b} = \theta_{a,b}$.
(ii) $\eta_{a,b}^c = \eta_{b,a}^c$ for all $(a, b) \succ c$.
(iii) For all $a, b, c, d, e, f \in P$ (some of them possibly equal) with

$$e \prec (a, b), \ d \prec (e, c), \ f \prec (b, c), \ \text{and } d \prec (a, f)$$

the diagram

$$
\begin{array}{ccccc}
L_a \wedge L_b \wedge L_c & \xrightarrow{\eta_{a,b}^e} & M_T \wedge L_e \wedge L_c & \xrightarrow{\eta_{e,c}^d} & M_T \wedge M_T \wedge L_d \\
\downarrow{\scriptstyle \eta_{b,c}^f} & & & & \downarrow{\scriptstyle +_{M_T}} \\
L_a \wedge M_T \wedge L_f & \xrightarrow{\eta_{a,f}^d} & M_T \wedge M_T \wedge L_d & \xrightarrow{+_{M_T}} & M_T \wedge L_d
\end{array}
$$

commutes.
(iv) For $a, b, c, d, e, f \in P$ with

$$e \prec (a, b), \ d \prec (c, e), \ f \prec (a, b), \ d \prec (c, f)$$

the diagram

$$
\begin{array}{ccccc}
L_a \wedge L_b \wedge L_c & \xrightarrow{\eta_{a,b}^e} & M_T \wedge L_e \wedge L_c & \xrightarrow{\eta_{e,c}^d} & M_T \wedge M_T \wedge L_d \\
\downarrow{\scriptstyle \eta_{a,b}^f} & & & & \downarrow{\scriptstyle +_{M_T}} \\
M_T \wedge L_f \wedge L_c & \xrightarrow{\eta_{c,f}^d} & M_T \wedge M_T \wedge L_d & \xrightarrow{+_{M_T}} & M_T \wedge L_d
\end{array}
$$

commutes.
(v) For every $c \prec (a, b)$ the diagram

$$
\begin{array}{ccc}
L_a \times L_b & \xrightarrow{\eta_{a,b}^c} & M_T \wedge L_c \\
\downarrow{\scriptstyle \alpha_a \times \alpha_b} & & \downarrow{\scriptstyle \alpha_{M_T} \wedge \alpha_c} \\
\mathcal{R}_a \otimes \mathcal{R}_b \xrightarrow{\text{mult}} \mathcal{R} & \xrightarrow{\text{proj}} & \mathcal{R}_c
\end{array}
$$

commutes, where $\text{proj} : \mathcal{R} \to \mathcal{R}_c$ is the projection.

8.7. With notation as in 6.1, the standard family (6.12.3) over $k[K_S]$ has a canonical logarithmic enrichment η. Indeed in this case the torsor L_p is the $\mathbb{G}_m$-torsor of highest weight vectors in $\mathcal{R}_p$ (notation as in 6.12). In particular, we have a trivialization v_p for $\mathcal{R}_p$. For $c \prec (a, b)$ the map

$$\eta_{a,b}^c : L_a \wedge L_b \to M_{K_S} \wedge L_c$$

is given by sending $v_a \wedge v_b$ to the image of $a + b - c \in K_S$ in M_{K_S} times v_c. The fact that this defines a logarithmic enrichment follows immediately from the construction.

8.8. Assume now that $T = \mathrm{Spec}(A)$ is the spectrum of an artinian local ring A with residue field κ, and that the closed fiber

(8.8.1) $$(M_\kappa, (f_\kappa, f_\kappa^b) : (X_\kappa, M_{X_\kappa}) \to (\mathrm{Spec}(\kappa), M_\kappa), (\rho_\kappa, \rho_\kappa^b), L_\kappa)$$

of (8.1.1) is isomorphic to the base change of the standard family associated to a paving S of Q and a morphism $\beta_\kappa : K_S \to \kappa$ induced by a morphism $K_S^{\mathrm{sat}} \to \kappa$ sending all non-units to 0. Assume further given a logarithmic enrichment $\eta = \{\eta_{a,b}^c\}$ on (8.1.1) reducing to the standard enrichment on the closed fiber (8.8.1) arising from its description as a standard family.

Proposition 8.9. *There exists a homomorphism $\beta : K_S \to A$ such that the family (8.1.1) together with its enrichment η is isomorphic, as a deformation of the closed fiber, to the base change along β of the standard family.*

The proof occupies the remainder of this section. In fact we will prove a slightly stronger statement that we will need later.

8.10. Let $A \to A_0$ be a surjective map of Artinian local rings with kernel $J \subset A$ annihilated by the maximal ideal of A, so J can be viewed as a κ-vector space. Assume given a morphism $\beta_{A_0} : K_S \to A_0$ lifting β_κ so that the family over A_0 is isomorphic to the base change of the standard family along β_{A_0}, and fix such an isomorphism.

Proposition 8.11. *There exists a morphism $\beta_A : K_S \to A$ lifting β_{A_0} such that the family over A is isomorphic to the base change of the standard family along β_A.*

Remark 8.12. Note that proceeding by induction on the length of A proposition 8.11 implies 8.9.

The proof of 8.11 is in the following steps (8.13)–(8.17).

Lemma 8.13. *(i) The map*

(8.13.1) $$H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A^\circ}) \to H^0(\widetilde{X}_{A_0}^\circ, M_{\widetilde{X}_{A_0}^\circ})$$

is surjective.

(ii) The action of U on $H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A^\circ})$ is trivial.

Proof. It suffices to prove the analogous results with $M_{\widetilde{X}_A}$ (resp. $M_{\widetilde{X}_{A_0}}$) replaced by $M_{\widetilde{X}_A}^{\mathrm{gp}}$ (resp. $M_{\widetilde{X}_{A_0}}^{\mathrm{gp}}$). For this note that by assumption 8.1 (vii) we have a commutative diagram

with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\widetilde{X}_A^\circ, M_A^{\mathrm{gp}}) & \longrightarrow & H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A}^{\mathrm{gp}}) & \longrightarrow & H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A/A}^{\mathrm{gp}}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \simeq & & \\
0 & \longrightarrow & H^0(\widetilde{X}_{A_0}^\circ, M_{A_0}^{\mathrm{gp}}) & \longrightarrow & H^0(\widetilde{X}_{A_0}^\circ, M_{\widetilde{X}_{A_0}}^{\mathrm{gp}}) & \longrightarrow & H^0(\widetilde{X}_{A_0}^\circ, M_{\widetilde{X}_{A_0}/A_0}^{\mathrm{gp}}) & \longrightarrow & 0.
\end{array}
$$

From this and the snake lemma statement (i) follows.

Statement (ii) follows from a similar argument as in 7.7. $\qquad\square$

Let $\gamma_{A_0} : K_S \rtimes P \to H^0(\widetilde{X}_{A_0}^\circ, M_{\widetilde{X}_{A_0}^\circ})$ be the chart obtained from the identification of the family over A_0 with the standard family.

Lemma 8.14. *The chart γ_{A_0} lifts to a chart $\gamma_A : K_S \rtimes P \to H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A^\circ})$.*

Proof. It suffices to show that the map γ_{A_0} lifts to a map $K_S \rtimes P \to H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A^\circ}^{\mathrm{gp}})$.

From the exact sequence of sheaves

$$
0 \to 1 + J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ} \to M_{\widetilde{X}_A^\circ}^{\mathrm{gp}} \to M_{\widetilde{X}_{A_0}^\circ}^{\mathrm{gp}} \to 0
$$

we get a diagram

$$
\begin{array}{ccccccc}
0 \longrightarrow H^0(\widetilde{X}_\kappa^\circ, J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ}) \longrightarrow H^0(\widetilde{X}_A^\circ, M_{\widetilde{X}_A^\circ}^{\mathrm{gp}}) \longrightarrow H^0(\widetilde{X}_{A_0}^\circ, M_{\widetilde{X}_{A_0}^\circ}^{\mathrm{gp}}) \longrightarrow H^1(\widetilde{X}_\kappa^\circ, J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ})
\end{array}
$$

$$
\begin{array}{c}
\nwarrow \quad \gamma_{A_0} \uparrow \quad \epsilon \nearrow \\
K_S \rtimes P.
\end{array}
$$

Since $H^0(\widetilde{X}_\kappa^\circ, J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ})$ is a $\mathbb{Q}$-vector space, to prove the existence of the dotted arrow it suffices to show that the map ϵ is 0. Let $z \in K_S \rtimes P$ be an element. The class $\epsilon(z)$ is the class of the torsor of liftings of $\gamma_{A_0}(z)$ to $M_{\widetilde{X}_A^\circ}$. Considering the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 1 + J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ} & \longrightarrow & M_{\widetilde{X}_A^\circ} & \longrightarrow & M_{\widetilde{X}_{A_0}^\circ} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 1 + J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ} & \longrightarrow & \mathscr{O}_{\widetilde{X}_A^\circ} & \longrightarrow & \mathscr{O}_{\widetilde{X}_{A_0}^\circ} & \longrightarrow & 0
\end{array}
$$

we get that the torsor corresponding to $\epsilon(z)$ is equal to the torsor of liftings of $\alpha(\gamma_{A_0}(z)) \in H^0(\widetilde{X}_{A_0}^\circ, \mathscr{O}_{\widetilde{X}_{A_0}^\circ})$ to $\mathscr{O}_{\widetilde{X}_A^\circ}$. Since the section $\alpha(\gamma_{A_0}(z))$ extends to $\widetilde{X}_{A_0}$ by 8.4, it follows that the class $\epsilon(z)$ is in the image of the restriction map

$$
H^1(\widetilde{X}_\kappa, J \otimes \mathscr{O}_{\widetilde{X}_\kappa}) \to H^1(\widetilde{X}_\kappa^\circ, J \otimes \mathscr{O}_{\widetilde{X}_\kappa^\circ}),
$$

whence 0 since $\widetilde{X}_\kappa$ is affine. $\qquad\square$

8.15. Fix a lifting γ_A of γ_{A_0}. By 8.13 (ii) this lifting induces a morphism $K_S \rtimes P \to R_A^U$ and therefore a T-equivariant log structure on $\mathrm{Spec}(R_A^U)$ lifting the log structure on $\mathrm{Spec}(R_{A_0}^U)$. Let $\beta_A : K_S \to M_A$ be the restriction of γ_A, so β_A is a lifting of the chart β_{A_0}.

For $a \in P$ let $L_{a,A}$ be the torsor defined as in 8.3. The map γ_A defines trivialization of these torsors $L_{a,A}$ lifting the standard trivializations over A_0. In particular the maps $\eta_{a,b}^c$ for $c \prec (a,b)$ become identified with elements $\tilde{\eta}_{a,b}^c \in \Gamma(\mathrm{Spec}(A), M_A)$, lifting the corresponding elements over A_0. It follows that $\tilde{\eta}_{a,b}^c$ can be written uniquely as $u_{a,b}^c$ plus the image under β of the element $\rho(a) + \rho(b) - \rho(c) \in K_S$, where $u_{a,b}^c$ is an element of

$$\mathrm{Ker}(A^* \to A_0^*) = 1 + J.$$

Lemma 8.16. *The elements $u_{a,b}^c$ defines a homomorphism $u : T \to J$, where T is defined as in 5.3.*

Proof. Indeed the conditions on the $\eta_{a,b}^c$ imply that the conditions in 5.3 hold for the elements $u_{a,b}^c$, so we get a homomorphism $\widetilde{T} \to J$. Since J is torsion free this descends to a homomorphism $T \to J$. $\qquad\square$

8.17. After changing our choice of γ_A, we can arrange that $u = 0$. The map γ_A is also compatible with the T-action by construction.

Using 8.4 the map γ_A defines a morphism

$$K_S \rtimes P \to H^0(\widetilde{X}_A, \mathcal{O}_{\widetilde{X}_A}),$$

compatible with the T-action and whose image is contained in the U-invariants. Let $\mathcal{R}_A$ (resp. $\mathcal{R}_A^{std}$) denote the coordinate ring of $\widetilde{X}_A$ (resp. the base change of the cone over the standard family to A). Then it follows that there exists a unique $\widetilde{G}$-invariant map of A-modules

$$(8.17.1) \qquad\qquad\qquad \mathcal{R}_A^{std} \to \mathcal{R}_A$$

which on the highest weight vectors is given by γ_A. In fact, it follows from the construction that this is a map of A-algebras. Indeed for any $a, b, c \in P$ with $(a, b) \succ c$ the $\mathcal{R}_{A,c}$-component of $\mathcal{R}_{A,a} \cdot \mathcal{R}_{A,b}$ is given by the image of $\eta_{a,b}^c$, viewed as an element of K_S as in 8.7, under γ_A. This implies that the map (8.17.1) is a ring homomorphism. Since both algebras are flat over A it follows that (8.17.1) is an isomorphism. From this proposition 8.11 follows. $\qquad\square$

9. Stable spherical log varieties

Fix a subgroup $\Gamma \subset \widetilde{\Lambda}$ and an irreducible polarized spherical variety (X_0, L_0) with weights in Γ.

Definition 9.1. Let T be a scheme. A *stable spherical log variety over T with weights in Γ adapted to (X_0, L_0)* is a collection of data

$$(M_T, (f, f^b) : (X, M_X) \to (T, M_T), (\rho, \rho^b), L, \underline{\eta}),$$

where $(M_T, (f, f^b) : (X, M_X) \to (T, M_T), (\rho, \rho^b), L)$ satisfies conditions 8.1 (i)–(vii) and $\underline{\eta}$ is a logarithmic enrichment, such that in every geometric fiber the isomorphism with the standard family occurring in 8.1 (v) can be chosen to be compatible with the logarithmic enrichments and with starting variety the given (X_0, L_0).

9.2. For two stable spherical log varieties with weights in Γ and adapted to (X_0, L_0) over a scheme T

$$(9.2.1) \quad (M_T, f : (X, M_X) \to (T, M_T), \rho, L_X, \underline{\eta}), \quad (M'_T, f' : (X', M_{X'}) \to (T, M'_T), \rho', L_{X'}, \underline{\eta}')$$

an isomorphism between them is a triple $(\sigma, (g, g^b), r)$, where $\sigma : M'_T \to M_T$ is an isomorphism of log structures on T, $(g, g^b) : (X, M_X) \to (X', M_{X'})$ is an isomorphism of log schemes, and $r : g^* L_{X'} \to L_X$ is an isomorphism of line bundles on X, such that the following conditions hold:

(i) The isomorphism g is compatible with the G-action, and the pair (g, g^b) is compatible with the B-action.
(ii) The diagram

$$
\begin{array}{ccc}
(X, M_X) & \xrightarrow{(g, g^b)} & (X', M_{X'}) \\
\downarrow & & \downarrow \\
(T, M_T) & \xrightarrow{(\mathrm{id}, \sigma)} & (T, M'_T)
\end{array}
$$

commutes.
(iii) The isomorphisms $(\sigma, (g, g^b))$ are compatible with the logarithmic enrichments $\underline{\eta}$ and $\underline{\eta}'$.
(iv) The isomorphism r is compatible with the G-linearizations.

Let $\mathscr{S}^{\mathrm{log}}_{\Gamma, (X_0, L_0)}$ denote the fibered category over the category of k-schemes whose fiber over T is the groupoid of stable spherical log varieties over T with weights in Γ and adapted to (X_0, L_0).

9.3. We show in 9.6 below that $\mathscr{S}^{\mathrm{log}}_{\Gamma, (X_0, L_0)}$ is an algebraic stack. The key ingredient is to understand the isomorphisms between two objects of $\mathscr{S}^{\mathrm{log}}_{\Gamma, (X_0, L_0)}$.

So fix two objects

$$(9.3.1) \qquad (M_T, (X, M_X), \rho, L_X, \underline{\eta}), \quad (M'_T, (X', M_{X'}), \rho', L_{X'}, \underline{\eta}') \in \mathscr{S}^{\mathrm{log}}_{\Gamma, (X_0, L_0)}(T)$$

over a scheme T. Let I^{log} denote the functor on the category of T-schemes which to any T'/T associates the set of isomorphisms

$$(M_T, (X, M_X), \rho, L_X, \underline{\eta})_{T'} \to (M'_T, (X', M_{X'}), \rho', L_{X'}, \underline{\eta}')_{T'}$$

in $\mathscr{S}^{\mathrm{log}}_{\Gamma, (X_0, L_0)}(T')$, where we write $(M_T, (X, M_X), \rho, L_X, \underline{\eta})_{T'}$ etc. for the base change to T'. Let I denote the functor sending T'/T to the set of pairs (a, b), where $a : X_{T'} \to X'_{T'}$ is a G-equivariant isomorphism over T' and $b : a^* L_{X'_{T'}} \to L_{X_{T'}}$ is an isomorphism of line bundles on $X_{T'}$.

Lemma 9.4. *The functor I is representable by a T-scheme which is Zariski locally on T quasi-projective over T.*

Proof. We may work Zariski locally on T, and can therefore assume that T is an affine scheme. Let J be the functor sending T'/T to the set of G-equivariant isomorphisms $X_{T'} \to X'_{T'}$. Then J is a quasi-projective scheme represented by a locally closed subscheme of the Hilbert scheme of $X \times X'$. In particular J is a quasi-projective scheme. Over J there is a tautological

isomorphism $a_J : X_J \to X'_J$. The two invertible sheaves L_X and $a_J^* L_{X'}$ therefore define two morphisms $J \to \mathrm{Pic}_{X/T}$ to the relative Picard scheme of X/T. Let $H \subset J$ be the fiber product of the diagram

$$
\begin{array}{ccc}
 & & \mathrm{Pic}_{X/T} \\
 & & \downarrow{\scriptstyle \Delta} \\
J & \longrightarrow & \mathrm{Pic}_{X/T} \times \mathrm{Pic}_{X/T},
\end{array}
$$

where the horizontal morphism is given by $[L_X] \times [a_J^* L_{X'}]$. Then H is again a quasi-projective scheme over T. Let a_H denote the restriction of a_J to H. Then the two invertible sheaves L_X and $a_H^* L_{X'}$ are fppf-locally on H isomorphic. Let M denote $L_X \otimes a_H^* L_{X'}^{-1}$. Then $f_* M$ is a locally free sheaf of rank 1 on H. The functor I is then isomorphic to the functor over H of G-equivariant trivializations of M which is quasi-projective over T. $\square$

Proposition 9.5. *The forgetful map $i : I^{\log} \to I$ is an isomorphism. In particular $I^{\log}$ is a scheme, which is Zariski locally on T quasi-projective over T.*

Proof. Let T' be a T-scheme, and let

$$
a : X_{T'} \to X'_{T'}, \quad b : a^* L_{X'_{T'}} \to L_{X_{T'}}
$$

be a G-equivariant isomorphism. We have to show that this extends uniquely to an isomorphism in $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$. For this we may as well replace T by T', so may assume that $T = T'$.

Consider first the case when $T = \mathrm{Spec}(k)$ is an algebraically closed field. In this case we have pavings S and S' of Q such that the family

$$
(M_k, (X, M_X), \rho, L_X) \quad (\text{resp. } (M'_k, (X', M_{X'}), \rho', L_{X'}))
$$

is induced by base change along the vertex map $k[K_S] \to k$ (resp. $k[K_{S'}] \to k$) from the standard family. We claim that $S = S'$.

Let R (resp. R') denote the affine cone of (X, L_X) (resp. $(X', L_{X'})$) so we get a $\widetilde{G}$-equivariant isomorphism $\sigma : R \to R'$. This induces an isomorphism $\sigma_\lambda : R_\lambda \to R'_\lambda$ for all λ. Now by the construction of the standard family, we have isomorphisms $R_\lambda \simeq V(\lambda)$ and $R'_\lambda \simeq V(\lambda)$, and the map σ_λ is identified with multiplication by an element $u_\lambda \in k^*$. Let $v_\lambda \in V(\lambda)$ be a highest weight vector as before. The paving S (resp. S') is characterized by the condition that $v_\lambda \cdot v_\mu = 0$ in R (resp. R') if and only if λ and μ lie in a common element of S. It follows that $S = S'$.

Therefore our two objects are isomorphic, and the proof in the case when $T = \mathrm{Spec}(k)$ is reduced to showing that an automorphism (a, b) of the underlying polarized G-scheme (X, L_X) extends uniquely to an automorphism in $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}(k)$. Let $H_S \subset K_S$ be as in 6.4 and let $N \subset M_T$ be the sub log structure defined by H_S. By the same argument as in the previous paragraph, the map $H_S \to k$ is induced by a map $H_S^{\mathrm{sat}} \to k$ sending all non-units to 0. Let M_R denote the log structure on the cone $\mathrm{Spec}(R)$ induced by the map $K_S \rtimes P \to R^U$, and let $N_R \subset M_R$ be the sub log structure obtained from the submonoid $P \rtimes H_S \subset K_S \rtimes P$.

We then have a cartesian diagram of log schemes with $\widetilde{B}$-action

(9.5.1)
$$\begin{array}{ccc} (\mathrm{Spec}(R), M_R) & \longrightarrow & (\mathrm{Spec}(R), N_R) \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(k), M_T) & \longrightarrow & (\mathrm{Spec}(k), N), \end{array}$$

where the horizontal morphisms are the identity maps on underlying schemes. The log structure N_R is by construction obtained from a log structure N_{R^U} on $\mathrm{Spec}(R^U)$, and the morphism

$$(\mathrm{Spec}(R^U), N_{R^U}) \to (\mathrm{Spec}(k), N)$$

is a standard family in the sense of [7, 3.2.1]. By [7, 3.2.2] the pair (a, b) therefore is induced by a unique $\widetilde{B}$-equivariant automorphism of $(N, (\mathrm{Spec}(R), N_R))$. Let

(9.5.2)
$$\sigma' : N \to N, \quad (\tilde{a}, \tilde{a}^b) : (\mathrm{Spec}(R), N_R) \to (\mathrm{Spec}(R), N_R)$$

denote this automorphism. It follows from the proof of [7, 3.2.2] that the induced map $\bar{\sigma}' : \overline{N} \to \overline{N}$ is the identity, and therefore σ is induced by a homomorphism $\rho' : H_S^{\mathrm{gp}} \to k^*$. Since H_S^{gp} is included into K_S^{gp} we can extend this to a homomorphism $\rho : K_S^{\mathrm{gp}} \to k^*$ to get an automorphism $\sigma : M_T \to M_T$ extending σ. Since the square (9.5.1) is cartesian we therefore get automorphisms

$$\sigma : M_T \to M_T, \quad (\tilde{a}, \tilde{a}^b) : (\mathrm{Spec}(R), M_R) \to (\mathrm{Spec}(R), M_R)$$

inducing the pair (a, b), where we abusively also write $(\tilde{a}, \tilde{a}^b)$ for the automorphism of log schemes obtained from the automorphism in (9.5.2) by base change to $(\mathrm{Spec}(k), M_T)$. Note that while the data $(\sigma', \tilde{a}, \tilde{a}^b)$ is uniquely determined by (a, b), this is not the case for σ. But we claim that there exists a unique choice of ρ extending ρ' such that $(\sigma, \tilde{a}, \tilde{a}^b)$ is compatible with the logarithmic enrichments. Indeed choosing any extension ρ as above, and for $c \in P$ let $\alpha_c : L_c \to L_c$ be the induced isomorphism. The failure of $(\sigma, \tilde{a}, \tilde{a}^b)$ to be compatible with the logarithmic enrichment is measured by the squares

(9.5.3)
$$\begin{array}{ccc} L_c \wedge L_d & \xrightarrow{\eta_{c,d}^e} & M_T \wedge L_e \\ \downarrow{\scriptstyle \alpha_c \wedge \alpha_d} & & \downarrow{\scriptstyle \sigma \wedge \alpha_e} \\ L_c \wedge L_d & \xrightarrow{\eta_{c,d}^e} & M_T \wedge L_e \end{array}$$

defined for each $c, d, e \in P$ with $(c, d) \succ e$. The map α_c is given by a unit $\gamma_c \in k^*$ and the condition that (9.5.3) commutes amounts to the equality

$$\gamma_e \cdot \rho(\eta_{c,d}^e) = \gamma_c \cdot \gamma_d$$

for all $(c, d) \succ e$. Now observe that the γ_c define a homomorphism

$$\lambda : \varinjlim_{\omega \in S} P_\omega^{\mathrm{gp}} \to k^*,$$

so changing the choice of ρ by the homomorphism

$$K_S^{\mathrm{gp}} \to k^*, \quad x \mapsto \rho(x)^{-1} \cdot \lambda(x)$$

we get the desired extension of ρ'. From this it is also clear that the choice of ρ is unique subject to the condition that it be compatible with the enrichments.

Next we consider the case when $T = \mathrm{Spec}(A')$ is an artinian local ring with residue field k, and $J \subset A'$ is an ideal annihilated by the maximal ideal of A' and such that the result holds over $A := A'/J$. Let $(\sigma, (g, g^b), r) \in I^{\log}(A)$ be an isomorphism over A. We show how to lift this to an isomorphism $(\sigma', (g', g'^b), r') \in I^{\log}(A')$ inducing the given isomorphism $(a, b) \in I(A')$. By descent theory (using the uniqueness) we may assume that k is algebraically closed.

By 8.11 and using the case of an algebraically closed field, there exists maps $\beta, \beta' : K_S \to A'$ such that the two objects (9.3.1) are obtained by base change along these maps from the standard family. Furthermore, the isomorphism σ over A is induced by a homomorphism $u : K_S^{\mathrm{gp}} \to A^*$ such that for every $m \in K_S$ we have $\bar{\beta}'(m) = u(m) \cdot \bar{\beta}(m)$, where $\bar{\beta}, \bar{\beta}' : K_S \to A$ are the maps induced by β and β'.

Let $R_{A'}$ (resp. $R'_{A'}$) denote the coordinate ring of the cone of (X, L_X) (resp. $(X', L_{X'})$) and let $\tilde{a} : R_{A'} \to R'_{A'}$ be the isomorphism induced by (a, b). For $c \in P$, let $R_{A',c}$ (resp. $R'_{A',c}$) denote the corresponding subrepresentation of $R_{A'}$ (resp. $R'_{A'}$). The presentation of the objects (9.3.1) as induced by base change from the standard family defines trivializations of each $R_{A',c}$ and $R'_{A',c}$, and using these trivializations the map $\tilde{a}$ is induced by a map

$$\gamma : \varinjlim_{\omega \in S} P_\omega^{\mathrm{gp}} \to A'^*.$$

Let $u' : K_S^{\mathrm{gp}} \to A'^*$ be the restriction of this map to K_S^{gp}. Then u' restricts to u over A. By construction for any $c, d, e \in P$ with $(c, d) \succ e$ the diagram

$$
\begin{array}{ccc}
R_{A',c} \otimes R_{A',d} & \longrightarrow & R_{A',e} \\
\downarrow{\scriptstyle \tilde{a}} & & \downarrow{\scriptstyle \tilde{a}} \\
R'_{A',c} \otimes R'_{A',d} & \longrightarrow & R'_{A',e}
\end{array}
$$

commutes, where the horizontal morphisms are induced by the multiplication map and the vertical morphisms induced by $\tilde{a}$. From this it follows that for every $m \in K_S$ we have $\beta'(m) = u'(m) \cdot \beta(m)$, and therefore u' defines a lifting $M_{A'} \to M_{A'}$ of σ. This also shows that the morphism $\tilde{a}$ extends to an element of $I^{\log}(A')$. The uniqueness of this extension also follows from the preceding discussion.

Having treated the case of an artinian local ring, we get the extension of (a, b) to an element of $I^{\log}$ in the case of a complete local ring by passing to the limit as in [7, Proof of 3.7.6].

From this we deduce that if T is a k-scheme then the map

(9.5.4) $I^{\log}(T) \to I(T)$

is bijective as follows. Both I and $I^{\log}$ are sheaves for the étale topology on the category of k-schemes, so to prove that (9.5.4) is bijective we may replace T by an étale cover which reduces the proof to the case when T is affine. In this case write $T = \varprojlim_{j \in J} T_j$ as a projective limit of affine k-schemes T_j of finite type. By a standard limit argument the objects (9.3.1) are defined over some T_j and any element of $I(T)$ is induced by an element of $I(T_j)$ for some j. From this it follows that it suffices to prove the bijectivity of (9.5.4) in the case when T is an affine scheme of finite type over k. In this case the Artin approximation theorem, and the case when T is a complete noetherian local ring, implies that for any $(a, b) \in I(T)$ there exists an étale neighborhood of any point of T over which (a, b) lifts uniquely to an element

of $I^{\log}(T)$. Using again that $I^{\log}$ and I are sheaves for the étale topology this implies that $I^{\log}(T) \to I(T)$ is a bijection in general.

This completes the proof of 9.5. $\hfill\square$

Theorem 9.6. *The fibered category $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ is a quasi-separated algebraic stack.*

Proof. We have just shown that the diagonal of $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ is representable by separated schemes, so it remains to exhibit a smooth cover. Such a cover is provided by the standard families for various pavings S, by 8.11. $\hfill\square$

10. Global moduli

10.1. Fix a subgroup $\Gamma \subset \widetilde{\Lambda}$ and a finite dimensional G-representation V. Following [1, 4.1] we define $\mathscr{M}_{\Gamma,\mathbb{P}V}$ to be the stack whose fiber over a k-scheme T is the groupoid of triples (X, L, ϵ), where (X, L) is a stable polarized spherical variety over T with weights in Γ and $\epsilon : V \to \Gamma(X, L)$ is a morphism of G-representations which induces a finite map $\pi : X \to \mathbb{P}V_T$ such that the map $\pi^* \mathscr{O}_{\mathbb{P}V}(1) \to L$ (defined by ϵ) is an isomorphism.

By [1, 4.1.2 (3)] the stack $\mathscr{M}_{\Gamma,\mathbb{P}V}$ is a proper algebraic stack with finite diagonal over k.

10.2. There are certain distinguished (or main) components of $\mathscr{M}_{\Gamma,\mathbb{P}V}$ defined as follows.

First of all we have an open substack $\mathscr{M}^{\circ}_{\Gamma,\mathbb{P}V} \subset \mathscr{M}_{\Gamma,\mathbb{P}V}$ classifying triples (X, L, ϵ) with X irreducible. For any such triple, the weight set $\widetilde{\Lambda}^+(X, L) \subset \Gamma$ is a submonoid which is locally constant in families. For each submonoid $P \subset \Gamma$ we therefore have an open and closed substack $\mathscr{M}^{\circ}_{\Gamma,\mathbb{P}V,P} \subset \mathscr{M}^{\circ}_{\Gamma,\mathbb{P}V}$ classifying triples (X, L, ϵ) with weight monoid P.

For any scheme T and object $(X, L, \epsilon) \in \mathscr{M}^{\circ}_{\Gamma,\mathbb{P}V,P}(T)$, we can consider the relative cone $\widetilde{X}$ and the quotient $\widetilde{X}//U$. This quotient is a family of toric varieties with torus $D(P^{\mathrm{gp}})$ which étale locally on T is isomorphic to $Y_{P,T} := \mathrm{Spec}(k[P]) \times_{\mathrm{Spec}(k)} T$.

As in [2, 1.10], let M_{Y_P} denote the scheme classifying affine $\widetilde{G}$-varieties of type Y_P. So for a scheme T an element of $M_{Y_P}(T)$ is given by a family of affine $\widetilde{G}$-schemes $\pi : Z \to T$ and an isomorphism $\varphi : Z//U \simeq Y_{P,T}$.

Note that the automorphism group scheme of $Y_{P,T}$ (as a toric variety) is $D(P^{\mathrm{gp}})$, and therefore there is an action of $D(P^{\mathrm{gp}})$ on M_{Y_P} obtained by changing the choice of isomorphism φ. By [2, 3.4] this action has finitely many orbits.

Now let (X_0, L_0) be an irreducible polarized spherical variety over k with weight monoid P. Choosing an isomorphism $\widetilde{X}_0^U \simeq Y_P$ we get a point of M_{Y_P} and the corresponding $D(P^{\mathrm{gp}})$-orbit $O_{(X_0,L_0)}$ is independent of the choice of isomorphism. We say that (X_0, L_0) is *generic* if the orbit $O_{(X_0,L_0)}$ is open in M_{Y_P}.

10.3. Let (X_0, L_0) be a fixed irreducible polarized spherical variety over k with weight monoid $P \subset \Gamma$ and assume (X_0, L_0) is generic. Define $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ to be the fibered category over the category of k-schemes whose fiber over a scheme T is the groupoid of data

$$((M_T, (f, f^b) : (X, M_X) \to (T, M_T), (\rho, \rho^b), L, \underline{\eta}), \epsilon),$$

where
$$(M_T, (f, f^b) : (X, M_X) \to (T, M_T), (\rho, \rho^b), L, \underline{\eta}) \in \mathscr{S}^{\log}_{\Gamma, (X_0, L_0)}(T)$$
and $\epsilon : V \to \Gamma(X, L)$ is a morphism of G-representations such that the collection (X, L, ϵ) is an object of $\mathscr{M}_{\Gamma, \mathbb{P}V}(T)$.

There is a forgetful map

(10.3.1) $$\tau : \mathscr{M}^{\log}_{\Gamma, \mathbb{P}V, (X_0, L_0)} \to \mathscr{M}_{\Gamma, \mathbb{P}V}.$$

Lemma 10.4. *Let A be a discrete valuation ring over k with field of fractions K and residue field κ, and let $(X, L, \epsilon) \in \mathscr{M}_{\Gamma, \mathbb{P}V}(A)$ be an object. Suppose that the closed fiber (X_κ, L_κ) of (X, L) is irreducible with weight monoid P and generic. Then there exists a polarized spherical variety (X_0, L_0) over k with weight monoid P which is generic and such that after possibly replacing A by a finite extension we have $(X, L) \simeq (X_0, L_0) \otimes_k A$.*

Proof. Let $\mathcal{R}$ be the cone of (X, L), which is a stable affine spherical variety over A. By [1, 3.1.4], after replacing A by a finite extension, we may assume that $\mathcal{R}$ is standard in the terminology of loc. cit. This means that there exists a polarized spherical variety (X_0, L_0) over k such that if R_0 denotes the corresponding cone, then we have an inclusion $\mathcal{R} \hookrightarrow R_0 \otimes_k K$ of algebras with $\widetilde{G}$-action over A which induces an isomorphism over K. Let $\pi \in A$ be a uniformizer and let $P \subset \Gamma$ be the monoid of weights occurring in R_0. Then for each $\lambda \in P$ the image of $\mathcal{R}_\lambda$ in $R_{0,\lambda} \otimes_k K$ is equal to $\pi^{h(\lambda)} R_0 \otimes_k A$ for some function $h : P \to \mathbb{Z}$ admissible with respect to (X_0, L_0). Now observe that since the closed fiber of (X, L) is irreducible the function h is in fact linear. It follows that the T-orbit $O_{(X_\kappa, L_\kappa)}$ is in the closure of the orbit $O_{(X_K, L_K)}$. Since (X_κ, L_κ) was assumed generic this implies that $O_{(X_\kappa, L_\kappa)} = O_{(X_K, L_K)}$ which implies the result. □

Lemma 10.5. *There exists a unique open substack $\mathscr{M}^\circ_{\Gamma, \mathbb{P}V, (X_0, L_0)} \subset \mathscr{M}_{\Gamma, \mathbb{P}V}$ such that a geometric point $\bar{x} \to \mathscr{M}_{\Gamma, \mathbb{P}V}$ corresponding to an object (X, L, ϵ) is in $\mathscr{M}^\circ_{\Gamma, \mathbb{P}V, (X_0, L_0)}$ if and only if $(X, L) \simeq (X_0, L_0) \otimes_k k(\bar{x})$.*

Proof. The uniqueness is clear.

For the existence, let $(\mathscr{X}, \mathscr{L})$ be the universal polarized scheme over $\mathscr{M}_{\Gamma, \mathbb{P}V}$. By 9.4 the functor over $\mathscr{M}_{\Gamma, \mathbb{P}V}$ classifying isomorphisms between $(\mathscr{X}, \mathscr{L})$ and (X_0, L_0) is a relative scheme $\mathscr{I} \to \mathscr{M}_{\Gamma, \mathbb{P}V}$, locally quasi-projective over $\mathscr{M}_{\Gamma, \mathbb{P}V}$. Let $V \to \mathscr{M}_{\Gamma, \mathbb{P}V}$ be a smooth morphism with V an affine scheme and let $\mathscr{I}_V \to V$ be the base change of $\mathscr{I}$. Then the image $V' \subset V$ of $\mathscr{I}_V$ in V is a constructible subset and we need to show that it is open. For this it suffices to show that if A is a discrete valuation ring and $\mathrm{Spec}(A) \to V$ is a morphism sending the closed point of $\mathrm{Spec}(A)$ to V' then the generic point of $\mathrm{Spec}(A)$ also maps to V'. Equivalently, we need to show that if A is a discrete valuation ring with field of fractions K and algebraically closed residue field κ, and if (X, L, ϵ) is an object of $\mathscr{M}_{\Gamma, \mathbb{P}V}$ over A such that $(X_\kappa, L_\kappa) \simeq (X_0, L_0) \otimes_k \kappa$, then $(X_K, L_K) \simeq (X_0, L_0) \otimes_k K$, after possibly replacing A by an extension. This follows from 10.4. □

Let $\mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)} \subset \mathscr{M}_{\Gamma, \mathbb{P}V}$ be the scheme-theoretic closure of $\mathscr{M}^\circ_{\Gamma, \mathbb{P}V, (X_0, L_0)}$.

Theorem 10.6. *The map τ in (10.3.1) factors through a finite universal homeomorphism*

$$\tau' : \mathscr{M}^{\log}_{\Gamma, \mathbb{P}V, (X_0, L_0)} \to \mathscr{M}_{\Gamma, \mathbb{P}V, (X_0, L_0)}.$$

In particular, $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ is a proper algebraic stack with finite diagonal.

Proof. For an object $(M_T, (f,f^b) : (X,M_X) \to (T,M_T), (\rho,\rho^b), L, \underline{\eta})$ of $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ over a scheme T and morphism $\epsilon : V \to \Gamma(X,L)$, the condition that (X,L,ϵ) defines an object of $\mathscr{M}_{\Gamma,\mathbb{P}V}$ is an open condition on T. If $(\mathcal{X},\mathcal{L})$ is the universal polarized scheme over $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ and $f : \mathcal{X} \to \mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ is the structure morphism, then $f_*\mathcal{L}$ is a locally free sheaf on $\mathscr{S}^{\log}_{\Gamma,(X_0,L_0)}$ whose formation commutes with arbitrary base change (by [1, 2.2.3]), and $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ is realized as an open substack of the total space of $V^\vee \otimes f_*\mathcal{L}$. In particular, $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ is an algebraic stack by 9.6. Furthermore, the map τ is fully faithful by 9.5. Therefore the diagonal of $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ is finite.

The map τ' is also proper. Indeed by [3, Paragraph preceding 4.20] applied to the base change of τ' to a smooth cover of $\mathscr{M}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$, to verify this it suffices to show that for any discrete valuation ring A and morphism $g : \mathrm{Spec}(A) \to \mathscr{M}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ sending the generic point of $\mathrm{Spec}(A)$ to $\mathscr{M}^\circ_{\Gamma,\mathbb{P}V,(X_0,L_0)}$ the morphism g lifts to $\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}$. This follows from [1, 3.1.4].

Thus our morphism τ' is a finite morphism of algebraic stacks such that for any algebraically closed field κ the induced map

$$\mathscr{M}^{\log}_{\Gamma,\mathbb{P}V,(X_0,L_0)}(\kappa) \to \mathscr{M}_{\Gamma,\mathbb{P}V,(X_0,L_0)}(\kappa)$$

is an equivalence of categories (again using 9.5). This implies that τ' is a universal homeomorphism. $\qquad\square$

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