# SHEAVES ON ARTIN STACKS

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Abstract. We develop a theory of quasi–coherent and constructible sheaves on algebraic stacks correcting a mistake in the recent book of Laumon and Moret-Bailly. We study basic cohomological properties of such sheaves, and prove stack–theoretic versions of Grothendieck's Fundamental Theorem for proper morphisms, Grothendieck's Existence Theorem, Zariski's Connectedness Theorem, as well as finiteness Theorems for proper pushforwards of coherent and constructible sheaves. We also explain how to define a derived pullback functor which enables one to carry through the construction of a cotangent complex for a morphism of algebraic stacks due to Laumon and Moret–Bailly.

**1.1.** In the book ([LM-B]) the lisse-étale topos of an algebraic stack was introduced, and a theory of quasi–coherent and constructible sheaves in this topology was developed. Unfortunately, it was since observed by Gabber and Behrend (independently) that the lisse-étale topos is not functorial as asserted in (loc. cit.), and hence the development of the theory of sheaves in this book is not satisfactory "as is". In addition, since the publication of the book ([LM-B]), several new results have been obtained such as finiteness of coherent and étale cohomology ([Fa], [Ol]) and various other consequences of Chow's Lemma ([Ol]).

The purpose of this paper is to explain how one can modify the arguments of ([LM-B]) to obtain good theories of quasi–coherent and constructible sheaves on algebraic stacks, and in addition we provide an account of the theory of sheaves which also includes the more recent results mentioned above.

**1.2.** The paper is organized as follows. In section 2 we recall some aspects of the theory of cohomological descent ([SGA4], $\mathrm{V}^{\mathrm{bis}}$) which will be used in what follows. In section 3 we review the basic definitions of the lisse-étale site, cartesian sheaves over a sheaf of algebras, and verify some basic properties of such sheaves. In section 4 we relate the derived category of cartesian sheaves over some sheaf of rings to various derived categories of sheaves on the simplicial space obtained from a covering of the algebraic stack by an algebraic space. Loosely speaking the main result states that the cohomology of a complex with cartesian cohomology sheaves can be computed by restricting to the simplicial space obtained from a covering and computing cohomology on this simplicial space using the étale topology. In section 5 we generalize these results to comparisons between Ext–groups computed in the lisse-étale topos and Ext–groups computed using the étale topology on a hypercovering. In section 6 we specialize the discussion of sections 3-5 to quasi–coherent sheaves. We show that if $\mathcal{X}$ is an algebraic stack and $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$ denotes the structure sheaf of the lisse-étale topos, then the triangulated category $D^+_{\text{qcoh}}(\mathcal{X})$ of bounded below complexes of $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$–modules with quasi–coherent cohomology sheaves satisfies all the basic properties that one would expect from the theory for schemes. For example we show in this section that if $f : \mathcal{X} \to \mathcal{Y}$ is a quasi–compact morphism of algebraic stacks and $\mathcal{M}$ is a quasi–coherent sheaf on $\mathcal{X}$

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then the sheaves $R^if_*\mathcal{M}$ on $\mathcal{Y}$ are quasi–coherent. In section 7 we turn to the problem of defining a derived pullback functor for the derived category of complexes with quasi–coherent cohomology sheaves. It is here that we encounter the main difficulty with the non–functoriality of the lisse-étale topos, and the solution is perhaps a little unsatisfactory. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks and let $D^-_{\mathrm{qcoh}}(\mathcal{X})$ and $D^-_{\mathrm{qcoh}}(\mathcal{Y})$ be the derived categories of bounded above complexes of sheaves with quasi–coherent cohomology. Unfortunately because of the non–functoriality of the lisse-étale topos we are not able to define a functor $Lf^* : D^-_{\mathrm{qcoh}}(\mathcal{X}) \rightarrow D^-_{\mathrm{qcoh}}(\mathcal{Y})$ as would exist if the lisse-étale topos was functorial. However, for any integer $a$ we can define a functor $\tau_{\geq a}Lf^* : D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{Y}) \rightarrow D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{X})$ which is left adjoint to the functor $Rf_* : D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{X}) \rightarrow D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{Y})$. This functor $\tau_{\geq a}Lf^*$ suffices for many purposes. In particular, it enables one to define a functor $Lf^*$ not on $D^-_{\mathrm{qcoh}}(\mathcal{Y})$ but on a certain category of projective systems in $D^+_{\mathrm{qcoh}}(\mathcal{Y})$. In section 8 we use this $Lf^*$ and the method of ([LM-B], Chapitre 17) to define the cotangent complex of a morphism of algebraic stacks and show that it enjoys all the good properties discussed in (loc. cit.). In section 9 we explain how the methods of sections 3-5 can be used to develop a theory of constructible sheaves on an algebraic stack. We essentially cover the contents of ([LM-B], Chapter 18), and also prove finiteness of cohomology of constructible sheaves for proper stacks. In section 10 we compare the theory of quasi–coherent sheaves in the lisse-étale topology to the theory of quasi–coherent sheaves in the big flat and big étale topologies. Though the lisse-étale topology is often more useful than these big topologies (for example to define the cotangent complex, or to study higher direct images of quasi–coherent sheaves), the big flat and big étale topologies have the advantage that they are trivially functorial. In this section we prove results comparing cohomology of quasi–coherent sheaves in these big topologies and cohomology using the lisse-étale topology. As an application of these results, we prove finiteness of coherent cohomology for proper Artin stacks (using Chow's Lemma for stacks). In section 11 we mention three applications of Chow's Lemma which also deserve mention in the present discussion of quasi–coherent sheaves. The three results are stack–theoretic versions of Grothendieck's Fundamental Theorem for Proper morphisms ([EGA], III.4.1.5), Grothendieck's Existence Theorem ([EGA], III.5.1.4), and Zariski's Connectedness Theorem ([EGA], III.4.3.1) (which also gives a stack–theoretic generalization of Stein factorization).

There is also an appendix containing a general result of Gabber, which though not used in the text provides a very general explanation for a result used in the paper.

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**1.4** (Conventions)**.** In this paper we use a more general notion of algebraic stack than that used in ([LM-B]). By an *algebraic stack* over a scheme $S$, we mean a stack $\mathcal{X}$ over the category of $S$–schemes such that the diagonal

(1.4.1) $$\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$$

is representable, quasi–compact, and quasi–separated, and such that there exists an algebraic space $X/S$ and a smooth surjective morphism $X \rightarrow \mathcal{X}$.

In sections 6–8, unless mentioned otherwise we work throughout over a fixed base scheme $S$ the reference to which is frequently omitted.

## 2. Semisimplicial and simplicial topoi

We review some of the theory of cohomological descent from ([SGA4], $\mathrm{V}^{\mathrm{bis}}$).

**2.1.** Let $\Delta$ denote the simplicial category and $\Delta^+ \subset \Delta$ the subcategory with the same objects but morphisms only the injective maps. A *strictly simplicial topos* is a $\Delta^+$–topos in the sense of ([SGA4], $\mathrm{V}^{\mathrm{bis}}$.1.2.1). Thus a $\Delta^+$–topos $X_\bullet^+$ consists of a topos $X_n$ for each nonnegative integer $n$, and for each injective morphism $\delta : [n] \to [m]$ a morphism of topoi (which we denote by the same letter) $\delta : X_m \to X_n$ (and these morphisms of topoi have to be compatible with composition). A sheaf $F_\bullet$ on a strictly simplicial topos $X_\bullet^+$ consists of a sheaf $F_n \in X_n$ for each $n$ and for each injective $\delta : [n] \to [m]$ a morphism $\delta^* F_n \to F_m$, and these morphisms have to be compatible with compositions.

If $X_\bullet$ is a simplicial topos we denote by $X_\bullet^+$ the strictly simplicial topos obtained by restricting to the subcategory $\Delta^+$. In what follows, we denote by $Ab(X_\bullet)$ (resp. $Ab(X_\bullet^+)$) the category of abelian sheaves on $X_\bullet$ (resp. $X_\bullet^+$), and by

$$\mathrm{res} : Ab(X_\bullet) \longrightarrow Ab(X_\bullet^+) \tag{2.1.1}$$

the restriction functor. Note that this functor is exact.

**2.2.** For any abelian sheaf $F \in Ab(X_\bullet)$ and integer $q$, there is a natural map

$$H^q(X_\bullet, F) \to H^q(X_\bullet^+, \mathrm{res}(F)). \tag{2.2.1}$$

This map is obtained by observing that there is a natural isomorphism

$$H^0(X_\bullet, F) \simeq H^0(X_\bullet^+, \mathrm{res}(F)) \tag{2.2.2}$$

and that both $H^\bullet(X_\bullet, -)$ and $H^\bullet(X_\bullet^+, \mathrm{res}(-))$ are cohomological $\delta$–functors.

**Theorem 2.3.** *For any abelian sheaf $F \in Ab(X_\bullet)$ and integer $q$, the map* (2.2.1) *is an isomorphism.*

The proof is in several steps (2.4)–(2.9).

**2.4.** For $[n] \in \Delta$, let

$$r_n : Ab(X_\bullet) \to Ab(X_n), \quad r_n^+ : Ab(X_\bullet^+) \to Ab(X_n) \tag{2.4.1}$$

be the restriction functors $F_\bullet \mapsto F_n$. The functors $r_n$ and $r_n^+$ have right adjoints, denoted $e_n$ and $e_n^+$ respectively, given by the formulas

$$(e_n F)_k := \prod_{\rho \in \mathrm{Hom}_\Delta([k],[n])} \rho_* F, \quad (e_n^+ F)_k := \prod_{\rho \in \mathrm{Hom}_{\Delta^+}([k],[n])} \rho_* F \tag{2.4.2}$$

with the natural transition maps. Note that since $r_n$ and $r_n^+$ are exact, the functors $e_n$ and $e_n^+$ take injectives to injectives. The functors $r_n$ and $r_n^+$ also have left adjoints $l_n$ and $l_n^+$ respectively given by the formulas

$$(l_n G)_k = \bigoplus_{\rho \in \mathrm{Hom}_\Delta([n],[k])} \rho^* G, \quad (l_n^+ G)_k = \bigoplus_{\rho \in \mathrm{Hom}_{\Delta^+}([n],[k])} \rho^* G. \tag{2.4.3}$$

Since the functors $l_n$ and $l_n^+$ are exact (since $\rho^*$ for abelian sheaves is exact for each morphism of topoi $\rho : X_k \to X_n$), the functors $r_n$ and $r_n^+$ also take injectives to injectives.

**Lemma 2.5.** *Let $F \in Ab(X_n)$ be an abelian sheaf, and let $C^\bullet$ be the complex*

$$\Gamma(X_0, r_0 e_n^+ F) \xrightarrow{d_0 - d_1} \Gamma(X_1, r_1 e_n^+ F) \longrightarrow \cdots \xrightarrow{\sum(-1)^i d_i} \Gamma(X_p, r_p e_n^+ F) \longrightarrow \cdots . \tag{2.5.1}$$

*Then $H^q(C^\bullet) = 0$ for $q > 0$ and $H^0(C^\bullet) \simeq \Gamma(X_n, F)$.*

*Proof.* Let $T$ be the functor from $\Delta^+$ to the category of abelian groups

$$[p] \mapsto \prod_{\mathrm{Hom}_{\Delta^+}([p],[n])} \mathbb{Z} \tag{2.5.2}$$

and let $\widetilde{T}$ be the associated chain complex

$$\widetilde{T} : \cdots \longrightarrow T_p \xrightarrow{\sum(-1)^i d_i} T_{p+1} \longrightarrow \cdots . \tag{2.5.3}$$

Then $C^\bullet \simeq \widetilde{T} \otimes_{\mathbb{Z}} \Gamma(X_n, F)$, and since each term in $\widetilde{T}$ is a free $\mathbb{Z}$–module this is the same as $\widetilde{T} \otimes^{\mathbb{L}}_{\mathbb{Z}} \Gamma(X_n, F)$. Hence to prove the lemma it suffices to show that $H^q(\widetilde{T}) = 0$ for $q > 0$ and $H^0(\widetilde{T}) = \mathbb{Z}$. This follows from the observation that $\widetilde{T}$ is the complex computing the singular cohomology of the standard $n$–simplex (or a direct calculation).

□

**Corollary 2.6.** *Every object $F \in Ab(X_\bullet^+)$ admits an embedding $F \hookrightarrow I$ into an injective sheaf $I$ for which the sequence*

$$0 \longrightarrow \Gamma(X_\bullet^+, I) \longrightarrow \Gamma(X_0, I|_{X_0}) \longrightarrow \Gamma(X_1, I|_{X_1}) \longrightarrow \cdots \tag{2.6.1}$$

*is exact, and for which $I|_{X_n}$ is a flasque sheaf on $X_n$ for every $n$.*

*Proof.* For each $n$, choose an inclusion $F_n \hookrightarrow I_n$ with $I_n$ injective in $Ab(X_n)$. Then define $I := \prod_n e_n^+ I_n$ which has a natural inclusion $F \hookrightarrow I$ (to see injectivity, note that for every $n$ the map $F_n \to e_n^+ I_n$ is injective). Since a product of injectives is injective, the sheaf $I$ is injective and the exactness of the sequence (2.6.1) follows from (2.5). □

**Corollary 2.7.** *Let $F \in Ab(X_\bullet^+)$ be an abelian sheaf. Then there is a natural spectral sequence*

$$E_1^{pq} = H^q(X_p, F|_{X_p}) \implies H^{p+q}(X_\bullet^+, F). \tag{2.7.1}$$

*Proof.* By (2.6), there exists an injective resolution $F \to I^\bullet$ such that the natural map from $\Gamma(X_\bullet^+, I^\bullet)$ to the total complex of the double complex

$$\{\Gamma(X_p, I^q)\}_{p,q} \tag{2.7.2}$$

is a quasi–isomorphism. Furthermore, we can choose $I^\bullet$ so that the restriction $I^\bullet|_{X_p}$ is an injective resolution of $F|_{X_p}$. The spectral sequence (2.7.1) is that obtained from this double complex.

That it is independent of the choices follows from observing that if $J^\bullet$ is a second injective resolution of $F$, then there is a quasi–isomorphism of complexes $J^\bullet \to I^\bullet$ inducing the identity on $F$. This quasi–isomorphism induces a morphism of double complexes

$$\{\Gamma(X_p, J^q)\}_{p,q} \longrightarrow \{\Gamma(X_p, I^q)\}_{p,q} \tag{2.7.3}$$

which induces an isomorphism of spectral sequences which is independent of the choice of the map $J^\bullet \to I^\bullet$. □

**Remark 2.8.** If $F \to J^\bullet$ resolution by sheaves $J^s$ such that for every $n$ the restriction $J^s_n$ to $X_n$ is flasque, then the proof shows that for any $q$ the group $H^q(X^+_\bullet, F)$ is isomorphic to the $q$–th cohomology group of the total complex associated to the double complex

(2.8.1) $$\{\Gamma(X_p, J^s|_{X_p})\}_{p,s}.$$

**2.9.** Now let $F \in Ab(X_\bullet)$ be an abelian sheaf. By the same argument used in the proof of (2.6) replacing $e^+_n$ by $e_n$, there exists a resolution $F \to J^\bullet$ in $Ab(X_\bullet)$ such that the restriction of $J^s$ to $X_p$ is injective for every $p$ and $s$. Then the $q$–th cohomology of the total complex of the double complex

(2.9.1) $$\{\Gamma(X_p, J^s|_{X_p})\}_{p,s}$$

is by the same reasoning as above (see also ([De], 5.2.3)) the group $H^q(X_\bullet, F)$, and consideration of this double complex yields a natural spectral sequence

(2.9.2) $$E^{pq}_1 = H^q(X_p, F|_{X_p}) \implies H^{p+q}(X_\bullet, F).$$

It follows from the definition of the map $H^{p+q}(X_\bullet, F) \to H^{p+q}(X^+_\bullet, \mathrm{res}(F))$ that it extends to a morphism of spectral sequences between (2.7.1) and (2.9.2) with the identity maps on the $E^{pq}_1$ terms. In particular, the map (2.2.1) is an isomorphism. This completes the proof of (2.3). □

## 3. The lisse-étale topos and cartesian sheaves

**Definition 3.1** ([LM-B], 12.1)**.** Let $S$ be a scheme and let $\mathcal{X}/S$ be an algebraic stack. The *lisse–étale site* of $\mathcal{X}$, denoted Lis-Et($\mathcal{X}$), is the site with underlying category the full subcategory of $\mathcal{X}$–schemes whose objects are smooth $\mathcal{X}$–schemes and whose covering families $\{U_i \to U\}_{i\in I}$ are families of étale morphisms such that the amalgamation

(3.1.1) $$\coprod_{i\in I} U_i \longrightarrow U$$

is surjective. We denote by $\mathcal{X}_{\text{lis-et}}$ the associated topos. The topos $\mathcal{X}_{\text{lis-et}}$ is naturally ringed with structure sheaf $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$ which associates to any $U \in$ Lis-Et($\mathcal{X}$) the ring $\Gamma(U, \mathcal{O}_U)$.

**Remark 3.2.** A sheaf $F \in \mathcal{X}_{\text{lis-et}}$ defines for every object $U \in$ Lis-Et($\mathcal{X}$) a sheaf $F_U$ on $U_{\text{et}}$ by restriction. As proved in ([LM-B], 12.2.1) this gives an equivalence of categories between the category of sheaves $\mathcal{X}_{\text{lis-et}}$ and the category of systems $\{F_U, \theta_\varphi\}$ consisting of a sheaf $F_U \in U_{\text{et}}$ for every $U \in$ Lis-Et($\mathcal{X}$) and a morphism $\theta_\varphi : \varphi^{-1}F_V \to F_U$ for every morphism $\varphi : U \to V$ in Lis-Et($\mathcal{X}$) such that

(i) $\theta_\varphi$ is an isomorphism if $\varphi$ is étale.

(ii) For a composite

(3.2.1) $$U \xrightarrow{\ \varphi\ } V \xrightarrow{\ \psi\ } W$$

we have $\theta_\varphi \circ \varphi^*(\theta_\psi) = \theta_{\psi\circ\varphi}$.

**3.3.** An important observation in regards to the category Lis-Et($\mathcal{X}$) is that if $f, g : U \to V$ are two morphisms in Lis-Et($\mathcal{X}$), then the equalizer in the category of $\mathcal{X}$–spaces of $f$ and $g$ may not be an object of Lis-Et($\mathcal{X}$). For example, let $\mathcal{X}$ be the spectrum of a field $k$ and consider the two morphisms $f, g : \mathbb{A}^1_k \to \mathbb{A}^1_k$ in Lis-Et($\mathcal{X}$), where $f$ is induced by the map

$k[t] \to k[t]$ sending $t$ to 0 and $g$ is induced by the map sending $t$ to $t^2$. Then the equalizer of $f$ and $g$ is represented in the category of schemes by $\mathrm{Spec}(k[t]/t^2)$ which is not smooth over $k$.

This observation has the following consequence. If $f : \mathcal{Y} \to \mathcal{X}$ is a morphism of algebraic stacks, then there is an induced functor

$$\text{Lis-Et}(\mathcal{X}) \longrightarrow \text{Lis-Et}(\mathcal{Y}), \quad U \mapsto U \times_{\mathcal{X}} \mathcal{Y}. \tag{3.3.1}$$

This functor induces a pair of adjoint functors $(f^{-1}, f_*)$, where $f_*$ sends a sheaf $\mathcal{M}$ to the sheaf which to any $U \in \text{Lis-Et}(\mathcal{X})$ associates $\mathcal{M}(U \times_{\mathcal{X}} \mathcal{Y})$. The functor $f^{-1}$ sends a sheaf $\mathcal{N}$ to the sheaf associated to the presheaf which to any $V \in \text{Lis-Et}(\mathcal{Y})$ associates the limit

$$\varinjlim_{V \to U} \mathcal{N}(U). \tag{3.3.2}$$

Here the limit is taken over the category of morphisms over $f$ from $V$ to objects $U \in \text{Lis-Et}(\mathcal{X})$. Unfortunately, this functor $f^{-1}$ is not left exact. This is because the limit in (3.3.2) is not filtering (it is connected but equalizers do not always exist), and hence $f^{-1}$ is not left exact.

It follows that the functor $(f^{-1}, f_*)$ do not define a morphism of topoi $\mathcal{Y}_{\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}$ as asserted in ([LM-B], 12.5). It is this mistake that we aim to correct in this paper.

Note however that the functor $f^{-1}$ is left exact when $f$ is smooth (it is just the restriction functor), and hence in this case we do have a morphism of topoi $\mathcal{Y}_{\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}$.

**Example 3.4.** For an explicit example due to Behrend ([Be], 5.3.12), consider a field $k$ and $\mathcal{Y} = \mathrm{Spec}(k)$, $\mathcal{X} = \mathbb{A}^1_k = \mathrm{Spec}(k[t])$, and $f : \mathcal{Y} \to \mathcal{X}$ the inclusion of the origin. Consider the map of sheaves $g : \mathcal{O}_{\mathcal{X}_{\text{lis-et}}} \to \mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$ given by multiplication by $t$. This map has kernel equal to zero. On the other hand, since $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$ is representable by $\mathbb{A}^1_{\mathcal{X}}$, the pullback $f^{-1}\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$ is represented by $\mathbb{A}^1_k$. It follows that the pullback of $g$ to $\mathcal{Y}$ is the zero map $\mathcal{O}_{\mathrm{Spec}(k)_{\text{lis-et}}} \to \mathcal{O}_{\mathrm{Spec}(k)_{\text{lis-et}}}$. In particular $g$ has zero kernel whereas $f^{-1}(g)$ has nontrivial kernel so $f^{-1}$ is not exact.

**Lemma 3.5.** *Let $\mathfrak{X}$ be an algebraic stack and $f : V \to \mathfrak{X}$ a smooth representable morphism of algebraic stacks. Then for any abelian sheaf $F \in \mathfrak{X}_{\text{lis-et}}$ there is a natural isomorphism*

$$H^*(\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \simeq H^*(V_{\text{lis-et}}, f^{-1}F), \tag{3.5.1}$$

*where $\widetilde{V}$ denotes the sheaf represented by $V$ and $\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}$ denotes the topos of sheaves over $\widetilde{V}$.*

*Proof.* This is a special case of (A.6). However, there is also a simpler proof as follows.

To see the isomorphism in degree 0, note that the object $\mathrm{id} : \widetilde{V} \to \widetilde{V}$ is the initial object in the topos $\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}$. Therefore $H^0(\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}, F)$ is equal to $F(\widetilde{V})$. Since $\widetilde{V}$ is the initial object in $V_{\text{lis-et}}$ this gives an isomorphism

$$H^0(\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \simeq H^0(V_{\text{lis-et}}, F). \tag{3.5.2}$$

Since $\{H^*(\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}, F)\}$ is a universal cohomological $\delta$–functor there is a natural $\delta$–functorial map

$$H^*(\mathfrak{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \to H^*(V_{\text{lis-et}}, f^{-1}F). \tag{3.5.3}$$

We claim that this map is an isomorphism.

Consider first the case when $V$ is an algebraic space. In this case there is a morphism of topoi $\epsilon : \mathcal{X}_{\text{lis-et}}|_{\widetilde{V}} \to V_{\text{et}}$ induced by the inclusion $\text{Et}(V) \subset \text{Lis-Et}(\mathcal{X})|_V$ ([SGA4], IV.9.2). The functor $\epsilon_*$ is exact and so in particular we have

$$H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \simeq H^*(V_{\text{et}}, F). \tag{3.5.4}$$

Similarly we have

$$H^*(V_{\text{lis-et}}, F) \simeq H^*(V_{\text{et}}, F), \tag{3.5.5}$$

and by the universal property of the cohomological $\delta$–functor $H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{V}}, -)$ we conclude that the resulting isomorphism

$$H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \simeq H^*(V_{\text{lis-et}}, F) \tag{3.5.6}$$

agrees with the map (3.5.3). This proves the result in the case when $V$ is an algebraic space.

For the general case, let $U \to V$ be a smooth surjection with $U$ an algebraic space, and let $U_\bullet$ be the 0–coskeleton. Let $\mathcal{X}_{\text{lis-et}}|_{\widetilde{U}_\bullet}$ be the resulting simplicial topos. By ([SGA4], V.2.3.4 (1)) we have

$$H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{V}}, F) \simeq H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{U}_\bullet}, F). \tag{3.5.7}$$

Similarly we also have

$$H^*(V_{\text{lis-et}}, F) \simeq H^*(V_{\text{lis-et}}|_{\widetilde{U}_\bullet}, F) \tag{3.5.8}$$

which by the representable case already considered is isomorphic to

$$H^*(U_{\bullet,\text{lis-et}}, F). \tag{3.5.9}$$

To prove that our map is an isomorphism in general it therefore suffices to show that the natural map

$$H^*(\mathcal{X}_{\text{lis-et}}|_{\widetilde{U}_\bullet}, F) \to H^*(U_{\bullet,\text{lis-et}}, F) \tag{3.5.10}$$

is an isomorphism, which again follows from the representable case. □

**Remark 3.6.** The representability of the morphism $f : V \to \mathcal{X}$ is assumed in order that $\widetilde{V}$ is a sheaf. One can generalize (3.5) to non-representable smooth morphisms $f$, but this requires some foundational work to define $\mathcal{X}_{\text{lis-et}}|_{\widetilde{V}}$. Since we do not need this generalization we do not include it here.

**Definition 3.7.** Let $\mathcal{X}$ be an algebraic stack over a scheme $S$.

(i) A sheaf of rings $\mathcal{A}$ on $\mathcal{X}_{\text{lis-et}}$ is *flat* if for any smooth morphism $f : U \to V$ in Lis-Et($\mathcal{X}$), the natural map of sheaves on $U_{\text{et}}$

$$f^{-1}(\mathcal{A}_V) \to \mathcal{A}_U \tag{3.7.1}$$

is faithfully flat.

(ii) Let $\mathcal{A}$ be a flat sheaf of rings in $\mathcal{X}_{\text{lis-et}}$. An $\mathcal{A}$–module $\mathcal{M}$ is *cartesian* if for any morphism $f : U \to V$ in Lis-Et($\mathcal{X}$) the map of sheaves on $U_{\text{et}}$ induced by the map $\theta_f : f^{-1}\mathcal{M}_V \to \mathcal{M}_U$

$$\mathcal{A}_U \otimes_{f^{-1}(\mathcal{A}_V)} f^{-1}(\mathcal{M}_V) \to \mathcal{M}_U \tag{3.7.2}$$

is an isomorphism. We denote the category of cartesian $\mathcal{A}$–modules by $Mod_{\text{cart}}(\mathcal{A})$.

**Lemma 3.8.** *Let $S$ be a scheme, $\mathcal{X}$ an algebraic $S$–stack, and $\mathcal{A}$ a flat sheaf of rings in $\mathcal{X}_{\text{lis-et}}$. A sheaf of $\mathcal{A}$–modules $\mathcal{M}$ is cartesian if and only if for every smooth morphism $f : U \to V$ in* Lis-Et $(\mathcal{X})$ *the morphism* (3.7.2) *is an isomorphism.*

*Proof.* The "only if" direction is clear from the definition of cartesian sheaf. For the other direction assume that the maps $\mathcal{A}_{U'} \otimes_{f^{-1}\mathcal{A}_U} f^{-1}\mathcal{M}_U \to \mathcal{M}_{U'}$ are isomorphisms for smooth $f : U' \to U$, and let $g : V' \to V$ be any morphism in Lis-Et$(\mathcal{X})$. There exists a commutative diagram in Lis-Et$(\mathcal{X})$

$$\begin{array}{ccc} \widetilde{V}' & \xrightarrow{\tilde{g}} & \widetilde{V} \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ V' & \xrightarrow{g} & V, \end{array} \tag{3.8.1}$$

where $h$ and $h'$ are smooth and surjective, and furthermore there exists a smooth morphism $\widetilde{V} \to U$ in Lis-Et$(\mathcal{X})$ such that the composite $\widetilde{V}' \to \widetilde{V} \to U$ is smooth. Indeed we can take for $U$ any smooth presentation of $\mathcal{X}$ and define $\widetilde{V} := V \times_{\mathcal{X}} U$ and $\widetilde{V}' := V \times_{\mathcal{X}} U$. Since the map $h'^{-1}\mathcal{A}_{V'} \to \mathcal{A}_{\widetilde{V}'}$ is faithfully flat by assumption, to verify that $\mathcal{A}_{V'} \otimes_{g^{-1}\mathcal{A}_V} g^{-1}\mathcal{M}_V \to \mathcal{M}_{V'}$ is an isomorphism, it suffices to verify that the map

$$\mathcal{A}_{\widetilde{V}'} \otimes_{h'^{-1}\mathcal{A}_{V'}} h'^{-1}(\mathcal{A}_{V'} \otimes_{g^{-1}\mathcal{A}_V} g^{-1}\mathcal{M}_V) \to \mathcal{A}_{\widetilde{V}'} \otimes_{h'^{-1}\mathcal{A}_{V'}} h'^{-1}\mathcal{M}_{V'} \tag{3.8.2}$$

is an isomorphism. By our assumptions we have

$$\mathcal{A}_{\widetilde{V}'}\otimes_{h'^{-1}\mathcal{A}_{V'}}h'^{-1}(\mathcal{A}_{V'}\otimes_{g^{-1}\mathcal{A}_V}g^{-1}\mathcal{M}_V) \simeq \mathcal{A}_{\widetilde{V}'}\otimes_{\tilde{g}^{-1}\mathcal{A}_{\widetilde{V}}}\tilde{g}^{-1}(\mathcal{A}_{\widetilde{V}}\otimes_{h^{-1}\mathcal{A}_V}h^{-1}\mathcal{M}_V) \simeq \mathcal{A}_{\widetilde{V}'}\otimes_{\tilde{g}^{-1}\mathcal{A}_{\widetilde{V}}}\mathcal{M}_{\widetilde{V}}, \tag{3.8.3}$$

and

$$\mathcal{A}_{\widetilde{V}'} \otimes_{h'^{-1}\mathcal{A}_{V'}} h'^{-1}\mathcal{M}_{V'} \simeq \mathcal{M}_{\widetilde{V}'}, \tag{3.8.4}$$

and the arrow (3.8.2) is identified with the map $\mathcal{A}_{\widetilde{V}'} \otimes_{g^{-1}\mathcal{A}_{\widetilde{V}}} \mathcal{M}_{\widetilde{V}} \to \mathcal{M}_{\widetilde{V}'}$. On the other hand, the maps to $U$ identify this arrow with the pullback of the identity morphism $\mathcal{M}_U \to \mathcal{M}_U$ whence the Lemma. □

**Corollary 3.9.** *For an algebraic stack $\mathcal{X}$ with a flat sheaf of rings $\mathcal{A}$, the subcategory $Mod_{\text{cart}}(\mathcal{A}) \subset Mod(\mathcal{A})$ is closed under the formation of kernels, cokernels, and extensions.*

*Proof.* This follows from (3.8) since for a smooth morphism $f : U \to V$ in Lis-Et$(\mathcal{X})$ the functor

$$Mod(\mathcal{A}_V) \to Mod(\mathcal{A}_U), \quad \mathcal{M} \mapsto \mathcal{A}_U \otimes_{f^{-1}\mathcal{A}_V} f^{-1}\mathcal{M} \tag{3.9.1}$$

is exact. □

**3.10.** Let $\mathcal{X}$ be an algebraic stack and $\mathcal{A}$ a flat sheaf of rings on $\mathcal{X}_{\text{lis-et}}$. For $\alpha \in \{[a, b], +, -, b\}$, let $D^\alpha(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ denote the usual derived category of sheaves of $\mathcal{A}$–modules in $\mathcal{X}_{\text{lis-et}}$. The corollary implies that there is a well–defined triangulated subcategory $D^\alpha_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \subset D^\alpha(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ consisting of objects all of whose cohomology sheaves are cartesian.

**Definition 3.11** ([LM-B], 12.7.2)**.** Let $S$ be a scheme. A *ringed algebraic $S$–stack* is a pair $(\mathcal{X}, \mathcal{A})$, where $\mathcal{X}$ is an algebraic $S$–stack and $\mathcal{A}$ is a flat sheaf of rings in $\mathcal{X}_{\text{lis-et}}$. A *morphism of ringed algebraic $S$–stacks* $f : (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ is a morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$

together with a map of sheaves of rings $f^{-1}\mathcal{B} \to \mathcal{A}$ on $\mathcal{Y}_{\text{lis-et}}$ (note that this makes sense even though $f$ does not induce a morphism of topoi). For such a morphism $f$, we write

$$f^* : Mod(\mathcal{B}) \to Mod(\mathcal{A}) \tag{3.11.1}$$

for the functor

$$\mathcal{M} \mapsto \mathcal{A} \otimes_{f^{-1}\mathcal{B}} f^{-1}\mathcal{M}. \tag{3.11.2}$$

**Lemma 3.12.** *Let $f : (\mathcal{X}, \mathcal{A}) \to (\mathcal{Y}, \mathcal{B})$ be a morphism of ringed algebraic $S$–stacks. Then for any cartesian sheaf $\mathcal{M} \in Mod_{\text{cart}}(\mathcal{B})$ the pullback $f^*\mathcal{M}$ is a cartesian sheaf of $\mathcal{A}$–modules.*

*Proof.* The sheaf $f^*\mathcal{M}$ is equal to the sheaf associated to the presheaf which to any $U \in$ Lis-Et$(\mathcal{X})$ associates the module

$$(\varinjlim_{g:U\to W} \mathcal{M}(W)) \otimes_{\varinjlim_{g:U\to W} \mathcal{B}(W)} \mathcal{A}(U), \tag{3.12.1}$$

where the limit is taken over morphisms $g : U \to W$ over $f$ from $U$ to objects $W \in$ Lis-Et$(\mathcal{Y})$. By the universal property of $\varinjlim$ and $\otimes$, for any $\mathcal{A}(U)$–module $N$ we have

$$\begin{array}{rc} & \mathrm{Hom}_{\mathcal{A}(U)}((\varinjlim_g \mathcal{M}(W)) \otimes_{\varinjlim_g \mathcal{B}(W)} \mathcal{A}(U), N) \\ \simeq & \mathrm{Hom}_{\varinjlim_g \mathcal{B}(W)}(\varinjlim_g \mathcal{M}(W), N) \\ \simeq & \varprojlim_g \mathrm{Hom}_{\mathcal{B}(W)}(\mathcal{M}(W), N) \\ \simeq & \varprojlim_g \mathrm{Hom}_{\mathcal{A}(U)}(\mathcal{M}(W) \otimes_{\mathcal{B}(W)} \mathcal{A}(U), N) \\ \simeq & \mathrm{Hom}_{\mathcal{A}(U)}(\varinjlim_g (\mathcal{M}(W) \otimes_{\mathcal{B}(W)} \mathcal{A}(U)), N). \end{array} \tag{3.12.2}$$

By Yoneda's Lemma, it follows that $f^*\mathcal{M}$ is isomorphic to the sheaf associated to the presheaf which to any $U \in$ Lis-Et$(\mathcal{X})$ associates

$$\varinjlim_{g:U\to W} g^*\mathcal{M}_W(U), \tag{3.12.3}$$

where $\mathcal{M}_W$ denotes the restriction of $\mathcal{M}$ to $W_{\text{et}}$, and $g^*\mathcal{M}_W$ denotes the pullback of $\mathcal{M}_W$ via the morphism of ringed topoi $(U_{\text{et}}, \mathcal{A}|_{U_{\text{et}}}) \to (W_{\text{et}}, \mathcal{B}|_{W_{\text{et}}})$ induced by $g$.

We claim that for any fixed $g : U \to W$, the natural map

$$g^*\mathcal{M}_W(U) \to \varinjlim_{g:U\to W} g^*\mathcal{M}_W(U) \tag{3.12.4}$$

is an isomorphism, where $g^*\mathcal{M}_W$ denotes the pullback of $\mathcal{M}_W$ under the morphism of ringed topoi $(U_{\text{et}}, \mathcal{A}_U) \to (W_{\text{et}}, \mathcal{B}_W)$ induced by $g$. This certainly implies that $f^*\mathcal{M}$ is cartesian. For this note first that the category over which this limit is taken is connected, and by the definition of cartesian sheaf every transition morphism is an isomorphism. To prove that (3.12.4) is an isomorphism, it therefore suffices to show that for any two morphisms $p_1, p_2 : T' \to T$ in Lis-Et$(\mathcal{Y})$ and morphisms $g' : U \to T'$ and $g : U \to T$ such that $p_1 \circ g' = g = p_2 \circ g'$, the two induced maps

$$p_1^*, p_2^* : g^*\mathcal{M}_T \to g'^*\mathcal{M}_{T'} \tag{3.12.5}$$

are equal. For then the limit in (3.12.4) can be replaced by the limit over the partially ordered set whose elements are morphisms $g : U \to W$ as above, and for which $(g : U \to W) \geq (g' : U \to W')$ if there exists an $\mathcal{X}$–morphism $p : W \to W'$ such that $g' = p \circ g$.

Let $Z$ denote the fiber product of the diagram

$$\begin{array}{ccc} & & T' \\ & & \downarrow{\scriptstyle p_1\times p_2} \\ T & \xrightarrow{\Delta} & T\times_{\mathcal{X}} T. \end{array} \tag{3.12.6}$$

Then the map $g': U \rightarrow T'$ factors through $Z$. On the other hand, the map $p_i^*|_Z : p_i^*\mathcal{M}_T|_Z \rightarrow \mathcal{M}_{T'}|_Z$ is equal to the map

$$\Delta^* p_i^* \mathcal{M}_T|_Z \rightarrow \Delta^* \mathcal{M}_{T\times_{\mathcal{X}} T} \tag{3.12.7}$$

which by the definition of $\Delta$ is the same for $i = 1, 2$. □

## 4. Descriptions of $D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ via hypercovers

**4.1.** Let $\mathcal{X}$ be an algebraic stack over some scheme $S$, $\mathcal{A}$ a flat sheaf of rings in $\mathcal{X}_{\text{lis-et}}$, and let $P : X \rightarrow \mathcal{X}$ be a smooth cover by an algebraic space $X$. Denote by $X_\bullet \rightarrow \mathcal{X}$ the simplicial algebraic space obtained by taking the 0–coskeleton of $P$, and let $X_\bullet^+$ be associated strictly simplicial algebraic space. Observe that for every morphism $[n] \rightarrow [m]$ in $\Delta^+$ the corresponding morphism $X_m \rightarrow X_n$ is smooth.

**4.2.** Since each morphism in $X_\bullet^+$ is smooth, we can define the strictly simplicial topos $X^+_{\bullet,\text{lis-et}}$. The restriction of $\mathcal{A}$ defines a sheaf in this topos which we denote by $\mathcal{A}_{X_\bullet^+,\text{lis-et}}$. We also have the étale topos $X_{\bullet,\text{et}}$ with the restriction $\mathcal{A}_{X_{\bullet,\text{et}}}$ of $\mathcal{A}$, the strictly simplicial ringed topos $(X^+_{\bullet,\text{et}}, \mathcal{A}_{X^+_{\bullet,\text{et}}})$, and there is a flat morphism of ringed topoi

$$\epsilon : (X^+_{\bullet,\text{lis-et}}, \mathcal{A}_{X^+_{\bullet,\text{lis-et}}}) \longrightarrow (X^+_{\bullet,\text{et}}, \mathcal{A}_{X^+_{\bullet,\text{et}}}) \tag{4.2.1}$$

induced by the natural morphisms of topoi $\epsilon_n : X_{n,\text{lis-et}} \rightarrow X_{n,\text{et}}$. The functor $\epsilon_*$ is restriction, and for a sheaf $F_\bullet \in X_{\bullet,\text{et}}$ the sheaf $\epsilon^* F_\bullet$ can be described as follows. The restriction of $\epsilon^* F_\bullet$ to $X_{n,\text{lis-et}}$ is $\epsilon_n^* F_n$, and for a morphism $\delta : [n] \rightarrow [m]$ in $\Delta^+$ the morphism $\delta^*_{\text{lis-et}}(\epsilon_n^* F_n) \rightarrow \epsilon_m^* F_m$ is defined to be the composite

$$\delta^*_{\text{lis-et}}(\epsilon_n^* F_n) \simeq \epsilon_m^* \delta^*_{\text{et}} F_n \rightarrow \epsilon_m^* F_m, \tag{4.2.2}$$

where the first isomorphism is obtained from the commutative diagram of topoi

$$\begin{array}{ccc} X_{m,\text{lis-et}} & \xrightarrow{\epsilon_m} & X_{m,\text{et}} \\ {\scriptstyle \delta_{\text{lis-et}}}\downarrow & & \downarrow{\scriptstyle \delta_{\text{et}}} \\ X_{n,\text{lis-et}} & \xrightarrow{\epsilon_n} & X_{n,\text{et}} \end{array} \tag{4.2.3}$$

and the second morphism is induced by the simplicial structure on $F_\bullet$.

**Definition 4.3.** A sheaf $F \in Mod(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $F \in Mod(\mathcal{A}_{X^+_{\bullet,\text{et}}})$, $F \in Mod(\mathcal{A}_{X_{\bullet,\text{et}}})$) is *cartesian* if for each morphism $\rho : [m] \rightarrow [n]$ in $\Delta^+$ the map $\rho^* F_m \rightarrow F_n$ is an isomorphism, where $F_n$ denotes the restriction of $F$ to $X_n$, and in the case of $X^+_{\bullet,\text{lis-et}}$ each $F_n$ is a cartesian sheaf on $X_{n,\text{lis-et}}$. We denote by $Mod_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $Mod_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{et}}})$, $Mod_{\text{cart}}(\mathcal{A}_{X_{\bullet,\text{et}}})$) the category of cartesian sheaves in $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$).

**Proposition 4.4.** *The categories $Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$, $Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}})$, and $Mod_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}})$ are all naturally equivalent and are also equivalent to the category $Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ of cartesian $\mathcal{A}$–modules on the lisse-étale site of $\mathcal{X}$.*

*Proof.* We have natural functors

$$Mod_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}}) \xrightarrow{\mathrm{res}} Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}) \xrightarrow{\epsilon^*} Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}}). \tag{4.4.1}$$

The functor $\epsilon^*$ has a right adjoint $\epsilon_*$ (which is just restriction) and the composite

$$\epsilon_* \circ \epsilon^* : Mod(\mathcal{A}_{X_{\bullet,et}}) \rightarrow Mod(\mathcal{A}_{X_{\bullet,et}}) \tag{4.4.2}$$

is the identity. By the definition of cartesian sheaf, if $F \in Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}})$ is a cartesian sheaf then the natural map $\epsilon^*\epsilon_*(F) \rightarrow F$ is an isomorphism. From this it follows that $\epsilon^*$ is an equivalence.

To see that res is fully faithful, observe that to give a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ in $Mod_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}})$ is equivalent to giving a morphism $\varphi_0 : \mathcal{F}_0 \rightarrow \mathcal{G}_0$ such that the diagram

$$\begin{array}{ccccc} \mathrm{pr}_1^*\mathcal{F}_0 & \xrightarrow{\mathrm{can}} & \mathcal{F}_1 & \xleftarrow{\mathrm{can}} & \mathrm{pr}_2^*\mathcal{F}_0 \\ \downarrow{\scriptstyle \mathrm{pr}_1^*(\varphi_0)} & & & & \downarrow{\scriptstyle \mathrm{pr}_2^*(\varphi_0)} \\ \mathrm{pr}_1^*\mathcal{G}_0 & \xrightarrow{\mathrm{can}} & \mathcal{G}_1 & \xleftarrow{\mathrm{can}} & \mathrm{pr}_2^*\mathcal{G}_0 \end{array} \tag{4.4.3}$$

commutes, where "can" denotes the morphisms provided by the simplicial structure on $\mathcal{F}$ and $\mathcal{G}$ and $\mathrm{pr}_i : X_1 \rightarrow X_0$ $(i = 1, 2)$ denote the two projections.. Indeed a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ induces such a morphism $\varphi_0$ by restricting to $X_0$. On the other hand, given $\varphi_0$ define $\varphi|_{X_n}$ by choosing a morphism $\rho : [0] \rightarrow [n]$ and letting $\varphi|_{X_n} : \mathcal{F}_n \rightarrow \mathcal{G}_n$ be the unique morphism such that the diagram

$$\begin{array}{ccc} \rho^*\mathcal{F}_0 & \xrightarrow{\rho^*(\varphi_0)} & \rho^*\mathcal{G}_0 \\ \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathcal{F}_n & \xrightarrow{\varphi|_{X_n}} & \mathcal{G}_n. \end{array} \tag{4.4.4}$$

The commutativity of (4.4.3) then implies that $\varphi|_{X_n}$ so defined is independent of the choice of the map $\rho$.

To show that res is an equivalence and to establish the equivalence with $Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$, let $Des(X/\mathcal{X})$ denote the category of pairs $(\mathcal{G}, \iota)$, where $\mathcal{G}$ is a sheaf of $\mathcal{A}_X$–modules on $X_{\mathrm{et}}$ and $\iota : \mathrm{pr}_1^*\mathcal{G} \rightarrow \mathrm{pr}_2^*\mathcal{G}$ is an isomorphism on the étale site of $X_1 := X \times_{\mathcal{X}} X$ such that the two isomorphisms

$$\mathrm{pr}_{13}^*(\iota), \mathrm{pr}_{23}^* \circ \mathrm{pr}_{12}^* : \mathrm{pr}_1^*\mathcal{G} \longrightarrow \mathrm{pr}_3^*\mathcal{G} \tag{4.4.5}$$

on $X_2 := X \times_{\mathcal{X}} X \times_{\mathcal{X}} X$ are equal. Note that any sheaf $\mathcal{M} \in Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ defines an object of $Des(X/\mathcal{X})$ by setting $\mathcal{G} := \mathcal{M}_X$ and defining $\iota$ to be the isomorphism

$$\mathrm{pr}_1^*\mathcal{M}_X \xrightarrow{\mathrm{pr}_1^*} \mathcal{M}_{X_1} \xrightarrow{\mathrm{pr}_2^{*-1}} \mathrm{pr}_2^*\mathcal{M}_X. \tag{4.4.6}$$

**Lemma 4.5.** *The induced functor $A : Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \rightarrow Des(X/\mathcal{X})$ is an equivalence of categories.*

*Proof.* For $(\mathcal{G}, \iota) \in Des(X/\mathcal{X})$, define a sheaf $\mathcal{G}^a$ as follows. For an object $U \in$ Lis-Et$(\mathcal{X})$ and $n \in \{0, 1, 2\}$, let $X_{n,U}$ denote the fiber product $X_n \times_{\mathcal{X}} U$ , and let $\delta_n : X_{n,U} \to X_n$ and $\pi^n : X_{n,U} \to U$ be the projections. Define $\mathcal{G}^a$ to be the sheaf associated to the presheaf

$$U \mapsto \mathrm{Ker}(\pi^0_* \delta_0^* \mathcal{G} \rightrightarrows \pi^1_* \delta_1^* \mathrm{pr}_1^* \mathcal{G}). \tag{4.5.1}$$

We claim that $\mathcal{G}^a$ is in $Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ and that the functor $(\mathcal{G}, \iota) \mapsto \mathcal{G}^a$ defines a quasi–inverse to the functor in the Lemma.

In the case when the morphism $U \to \mathcal{X}$ admits a lifting $s : U \to X$, the equalizer (4.5.1) can be described as follows. If $\mathcal{G}_U$ denotes $s^*\mathcal{G}$, then $\mathcal{G}^a|_{U_{\mathrm{et}}}$ is equal to

$$\mathrm{Ker}(\pi^0_* \pi^{0*} \mathcal{G}_U \rightrightarrows \pi^1_* \pi^{1*} \mathcal{G}_U). \tag{4.5.2}$$

Since the map $X_{0,U} \to U$ admits a section, this kernel is equal to $\mathcal{G}_U$ (([SGA4], $\mathrm{V}^{\mathrm{bis}}$.3.3.1 (a))). From this it also follows that $A(\mathcal{G}^a) \simeq (\mathcal{G}, \iota)$.

Since any morphism $U \to \mathcal{X}$ étale locally lifts to a morphism to $X$, it follows that $\mathcal{G}^a_U$ is étale locally on $U$ isomorphic to the sheaf $s^*\mathcal{G}$ obtained by choosing a section $s : U \to X$. In particular, the sheaf $\mathcal{G}^a$ is cartesian. Furthermore, it follows from the definitions that the the functor $\mathcal{G} \mapsto \mathcal{G}^a$ is left adjoint to the functor $A$, and from the preceding discussion it follows that for any $\mathcal{M} \in Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ the adjunction map $\mathcal{M} \to A(\mathcal{M})^a$ is an isomorphism. □

A sheaf $\mathcal{F} \in Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}})$ defines a pair $(\mathcal{G}, \iota) \in Des(X/\mathcal{X})$ by setting $\mathcal{G} := \mathcal{F}_0$ and $\iota$ the composite

$$\mathrm{pr}_1^* \mathcal{F}_0 \xrightarrow{\mathrm{can}} \mathcal{F}_1 \xrightarrow{\mathrm{can}^{-1}} \mathrm{pr}_2^* \mathcal{F}_0. \tag{4.5.3}$$

The cocycle condition on $\iota$ follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} \mathrm{pr}_1^*\mathcal{F}_0 & \xrightarrow{\mathrm{pr}_{12}^*(\mathrm{can})} & \mathrm{pr}_{12}^*\mathcal{F}_1 & \xleftarrow{\mathrm{pr}_{12}^*(\mathrm{can})} & \mathrm{pr}_2^*\mathcal{F}_0 \\ \downarrow{\scriptstyle \mathrm{pr}_{13}^*(\mathrm{can})} & & \downarrow{\scriptstyle \mathrm{can}} & & \downarrow{\scriptstyle \mathrm{pr}_{23}^*(\mathrm{can})} \\ \mathrm{pr}_{13}^*\mathcal{F}_1 & \xrightarrow{\mathrm{can}} & \mathcal{F}_2 & \xleftarrow{\mathrm{can}} & \mathrm{pr}_{23}^*\mathcal{F}_1 \\ \uparrow{\scriptstyle \mathrm{pr}_{13}^*(\mathrm{can})} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{pr}_{23}^*(\mathrm{can})} \\ \mathrm{pr}_3^*\mathcal{F}_0 & \xrightarrow{\mathrm{can}} & \mathcal{F}_2 & \xleftarrow{\mathrm{can}} & \mathrm{pr}_3^*\mathcal{F}_0. \end{array} \tag{4.5.4}$$

Here $\mathrm{pr}_i : X_2 \to X_0$ $(i = 1, 2, 3)$ denotes the map induced by the inclusion $[0] \to [2]$ sending $0$ to $i$, and $\mathrm{pr}_{ij} : X_2 \to X_1$ $(1 \leq i < j \leq 2)$ denote the map obtained from the unique map $[1] \to [2]$ with image $\{i, j\}$.

This defines a functor $Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}) \to Des(X/\mathcal{X})$, and hence also a functor $Mod_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}}) \to Des(X/\mathcal{X})$. By the argument showing that res is an equivalence, both these functors are fully faithful. It follows that they are equivalences with quasi–inverses given by the equivalence

$$Des(X/\mathcal{X}) \simeq Mod_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \tag{4.5.5}$$

composed with the restriction functor.

Finally we leave to the reader the verification that the composite

$$Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}) \longrightarrow Des(X/\mathcal{X}) \simeq Mod_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}}) \xrightarrow{\mathrm{res}} Mod_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}) \tag{4.5.6}$$

is isomorphic to the identity functor, thereby completing the proof that res is an equivalence. □

**4.6.** Denote by $D^+_{\text{cart}}(\mathcal{A}_{X_\bullet,\text{et}})$ (resp. $D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{et}}})$, $D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$, $D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$) the full subcategory of the derived category $D^+(Mod(\mathcal{A}_{X_\bullet,\text{et}}))$ (resp. $D^+(Mod(\mathcal{A}_{X^+_{\bullet,\text{et}}}))$, $D^+(Mod(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}}))$, $D^+(Mod(\mathcal{A}))$) consisting of complexes with cartesian cohomology sheaves.

Denote by

$$\pi : X^+_{\bullet,\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}, \quad \epsilon : X^+_{\bullet,\text{lis-et}} \to X^+_{\bullet,\text{et}} \tag{4.6.1}$$

the natural flat morphisms of ringed topoi. Here $\pi^*$ is the functor which sends a sheaf $\mathcal{F}$ on $\mathcal{X}_{\text{lis-et}}$ to the family of sheaves $\{\mathcal{F}|_{X_n}\}_n$ on the $X_n$ with the natural transition maps, and $\epsilon_*$ is the restriction functor.

**Theorem 4.7.** *The functors*

$$\pi^* : D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \longrightarrow D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}}) \tag{4.7.1}$$

*and*

$$\epsilon^* : D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{et}}}) \longrightarrow D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}}), \tag{4.7.2}$$

*are both equivalences of triangulated categories with quasi–inverses given by $R\pi_*$ and $R\epsilon_*$ respectively. For any $F \in D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$, the maps*

$$H^*(\mathcal{X}_{\text{lis-et}}, F) \to H^*(X^+_{\bullet,\text{lis-et}}, \pi^*F) \to H^*(X^+_{\bullet,\text{et}}, R\epsilon_*\pi^*F) \tag{4.7.3}$$

*are isomorphisms, and these isomorphisms are functorial with respect to smooth base change $\mathcal{X}' \to \mathcal{X}$.*

*Proof.* That (4.7.1) and (4.7.2) are equivalences follows from the following (4.8) and (4.9).

**Lemma 4.8.** *For any $F \in D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{et}}})$ (resp. $G \in D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$) the natural map $F \to R\epsilon_*\epsilon^*F$ (resp. $G \to R\pi_*\pi^*G$) is an isomorphism.*

*Proof.* That $F \to R\epsilon_*\epsilon^*F$ is an isomorphism is clear because $\epsilon^*$ and $\epsilon_*$ are both exact functors and $\epsilon_* \circ \epsilon^*$ is the identity functor.

To prove that $G \to R\pi_*\pi^*G$ is an isomorphism, note first that for any fixed $q$ we can choose an integer $n$ (for example $n = q$) so that the maps

$$\mathcal{H}^q(\tau_{\leq n}G) \to \mathcal{H}^q(G), \quad \mathcal{H}^q(R\pi_*\pi^*\tau_{\leq n}G) \to \mathcal{H}^q(R\pi_*\pi^*G) \tag{4.8.1}$$

are isomorphisms. From this we see that it suffices to consider the case when $G = \tau_{\leq n}G$. Moreover, induction and consideration of the distinguished triangle

$$\tau_{\leq n-1}G \to \tau_{\leq n}G \to \mathcal{H}^n(G)[-n] \to \tau_{\leq n-1}G[1] \tag{4.8.2}$$

shows that it suffices to consider the case when $G \in Mod_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$.

By ([SGA4], V.5.1 (1)) and (3.5), the sheaf $R^q\pi_*\pi^*G$ is the sheaf associated to the presheaf which to any smooth morphism $V \to \mathcal{X}$ from a scheme $V$ associates the group $H^q(X^+_{V,\bullet,\text{lis-et}}, \pi^*G|_{X^+_{V,\bullet}})$, where $X_{V,\bullet}$ denotes the simplicial algebraic space obtained from $X_\bullet$ by base change to $V$. Since $\epsilon_*$ is exact and takes injectives to injectives (since $\epsilon^*$ is also exact), this group is isomorphic to $H^q(X^+_{V,\bullet,\text{et}}, \epsilon^*G|_{X^+_{V,\bullet}})$. Let $G_V$ be the restriction of $G$ to $V_{\text{et}}$ and let $\lambda : X^+_{V,\bullet,\text{et}} \to V_{\text{et}}$ be the projection. Then $\pi^*G|_{X^+_{V,\bullet,\text{et}}}$ is isomorphic to $\lambda^*G_V$ (since $G$ is cartesian). By (2.3) it follows that there is a natural isomorphism

$$H^q(X^+_{V,\bullet,\text{lis-et}}, \pi^*G|_{X^+_{V,\bullet}}) \simeq H^q(X_{V,\bullet,\text{et}}, \lambda^*G_V). \tag{4.8.3}$$

Since $X \to \mathcal{X}$ is smooth, étale locally on $V$ the map $V \to \mathcal{X}$ admits a lifting to a map $V \to X$. In this case the map $X_{V,\bullet} \to V$ admits a section, and by ([SGA4], $\text{V}^{\text{bis}}$.3.3.1 (a)) the natural map

$$H^q(V_{\text{et}}, G_V) \to H^q(X_{V,\bullet,\text{et}}, \lambda^*G_V) \tag{4.8.4}$$

is an isomorphism. Passing to the associated sheaves we find that $\mathcal{G} \to R^0\pi_*\pi^*\mathcal{G}$ is an isomorphism, and that $R^q\pi_*\pi^*\mathcal{G} = 0$ for $q > 0$. □

**Lemma 4.9.** *For any $F \in D^+_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$ the natural maps $\pi^*R\pi_*F \to F$ and $\epsilon^*R\epsilon_*F \to F$ are isomorphisms.*

*Proof.* As in the proof of (4.8), consideration of the truncations $\tau_{\leq n}$ shows that it suffices to consider the case when $F \in Mod_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{lis-et}}})$. In this case (4.4) shows that there exists $G \in Mod_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ and $H \in Mod_{\text{cart}}(\mathcal{A}_{X^+_{\bullet,\text{et}}})$ such that $F \simeq \pi^*G$ and $F \simeq \epsilon^*H$. From this and (4.8) the result follows. □

To see that the maps in (4.7.3) are isomorphisms, note first that since $\pi^*$ is exact, the functor $\pi_*$ takes injectives to injectives. It follows that

$$H^*(X^+_{\bullet,\text{lis-et}}, \pi^*F) \simeq H^*(\mathcal{X}_{\text{lis-et}}, R\pi_*\pi^*F) \tag{4.9.1}$$

which by (4.8) is isomorphic to $H^*(\mathcal{X}_{\text{lis-et}}, F)$. That the second map is an isomorphism follows from the fact that $\epsilon_*$ takes injectives to injectives since $\epsilon^*$ is exact.

That these isomorphisms are compatible with smooth base change can be seen as follows. Let $\mathcal{X}' \to \mathcal{X}$ be a smooth morphism and $X'_\bullet$ the base change of $X_\bullet$ to $\mathcal{X}'$. Then there are natural commutative diagrams of topoi

$$\begin{array}{ccc} X^+_{\bullet,\text{lis-et}} & \xleftarrow{a} & X'^+_{\bullet,\text{lis-et}} \\ \pi \downarrow & & \downarrow \pi' \\ \mathcal{X}_{\text{lis-et}} & \xleftarrow{b} & \mathcal{X}'_{\text{lis-et}}, \end{array} \tag{4.9.2}$$

$$\begin{array}{ccc} X^+_{\bullet,\text{lis-et}} & \xleftarrow{c} & X'^+_{\bullet,\text{lis-et}} \\ \epsilon \downarrow & & \downarrow \epsilon' \\ X^+_{\bullet,\text{et}} & \xleftarrow{d} & X'^+_{\bullet,\text{et}}. \end{array} \tag{4.9.3}$$

The compatibility with smooth base change then reduces to the statement that for any $F \in D^+_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$ (resp. $G \in D^+_{\mathrm{cart}}(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}})$), the following diagrams commute:

$$\begin{array}{ccc} F & \longrightarrow & R\pi_*\pi^*F \\ \downarrow & & \downarrow \\ Rb_*b^*F & & R\pi_*Ra_*a^*\pi^*F \\ \mathrm{id}\downarrow & & \downarrow w \\ Rb_*b^*F & \longrightarrow & Rb_*R\pi'_*\pi'^*b^*F, \end{array} \tag{4.9.4}$$

$$\begin{array}{ccc} G & \longrightarrow & R\epsilon_*\epsilon^*G \\ \downarrow & & \downarrow \\ Rd_*d^*G & & R\epsilon_*Rc_*c^*\epsilon^*G \\ \mathrm{id}\downarrow & & \downarrow u \\ Rd_*d^*G & \longrightarrow & Rd_*R\epsilon'_*\epsilon'^*b^*G. \end{array} \tag{4.9.5}$$

Here $w$ and $u$ are obtained from the natural isomorphisms of functors

$$R\pi_*Ra_* \simeq R(\pi \circ a)_* \simeq R(b \circ \pi')_* \simeq Rb_*R\pi'_*, \tag{4.9.6}$$

$$R\epsilon_*Rc_* \simeq R(\epsilon \circ c)_* \simeq R(d \circ \epsilon')_* \simeq Rd_*R\epsilon'_*. \tag{4.9.7}$$

That these diagrams commute then follows from the universal properties of the adjunction maps. This completes the proof of (4.7). □

## 5. Ext–groups

Let $\mathcal{X}$ be an algebraic stack, $P : X \to \mathcal{X}$ a smooth cover with $X$ a scheme, and let $\mathcal{A}$ be a flat sheaf of rings on $\mathcal{X}_{\text{lis-et}}$. Let $X_\bullet$ denote the 0–coskeleton of $P$, and let $\mathcal{A}_{X^+_{\bullet,\text{lis-et}}}$ (resp. $\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}$, $\mathcal{A}_{X_{\bullet,\mathrm{et}}}$) be the restriction of $\mathcal{A}$ to $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\mathrm{et}}$, $X_{\bullet,\mathrm{et}}$).

**5.1.** The restriction fuctor $\mathrm{res} : X_{\bullet,\mathrm{et}} \to X^+_{\bullet,\mathrm{et}}$ is exact, and hence induces a functor

$$\mathrm{res} : D(\mathcal{A}_{X_{\bullet,\mathrm{et}}}) \to D(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}) \tag{5.1.1}$$

of triangulated categories. By ([SGA4], $\mathrm{V}^{\mathrm{bis}}$.1.2.10) the functor res also has a right adjoint $\gamma$. This functor $\gamma$ can be described as follows. Let $[n]/\Delta^+$ denote the category whose objects are arrows $\rho : [n] \to [k]$ in $\Delta$ and whose morphisms $(\rho : [n] \to [k]) \to (\rho' : [n] \to [k'])$ are maps $j : [k] \to [k']$ in $\Delta^+$ such that $\rho' = j \circ \rho$. Then for $G_\bullet \in X^+_{\bullet,\mathrm{et}}$ we have

$$\gamma(G)_n = \varprojlim_{(\rho:[n]\to[k])\in[n]/\Delta^+} \rho_*G_k. \tag{5.1.2}$$

Let $\mathcal{F}$ be a sheaf of $\mathcal{A}_{X_{\bullet,\mathrm{et}}}$–modules. Since res is an exact functor the functor $\gamma$ takes injectives to injectives. The isomorphism of functors on the category of $\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}$–modules

$$\mathrm{Hom}_{\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}}(\mathrm{res}(\mathcal{F}), -) \simeq \mathrm{Hom}_{\mathcal{A}_{X_{\bullet,\mathrm{et}}}}(\mathcal{F}, \gamma(-)) \tag{5.1.3}$$

therefore induces for any sheaf $G$ of $\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}$–modules a morphism

$$RHom(\mathcal{F}, \gamma(G)) \to RHom(\mathrm{res}(\mathcal{F}), G). \tag{5.1.4}$$

In particular, if $\mathcal{G}$ is a sheaf of $\mathcal{A}_{X_{\bullet,\mathrm{et}}}$–modules, then the adjunction map $\mathcal{G} \to \gamma(\mathrm{res}(\mathcal{G}))$ induces a canonical map

$$RHom(\mathcal{F}, \mathcal{G}) \to RHom(\mathrm{res}(\mathcal{F}), \mathrm{res}(\mathcal{G})). \tag{5.1.5}$$

**Theorem 5.2.** *If $\mathcal{F}$ and $\mathcal{G}$ are cartesian $\mathcal{A}_{X_{\bullet,\mathrm{et}}}$–modules, then the map* (5.1.5) *is an isomorphism.*

*Proof.* It suffices to show that for every integer $q$ the map

$$\mathrm{Ext}^q_{\mathcal{A}_{X_{\bullet,\mathrm{et}}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^q_{\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}}(\mathrm{res}(\mathcal{F}), \mathrm{res}(\mathcal{G})) \tag{5.2.1}$$

induced by (5.1.5) is an isomorphism. Note that by construction this map is $\delta$–functorial in both $\mathcal{F}$ and $\mathcal{G}$.

**Lemma 5.3.** *Every sheaf $\mathcal{G} \in Mod(\mathcal{A}_{X_\bullet,\mathrm{et}})$ admits an embedding $G \hookrightarrow J$ such that the restriction $J_n$ of $J$ to each $X_n$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{A}_{X_n}}(\mathcal{F}_n, -)$. Similarly every sheaf $\mathcal{G} \in Mod(\mathcal{A}_{X^+_\bullet,\mathrm{et}})$ admits an embedding $G \hookrightarrow J$ such that the restriction $J_n$ of $J$ to each $X_n$ is acyclic for the functor $\mathrm{Hom}_{\mathcal{A}_{X_n}}(\mathcal{F}_n, -)$*

*Proof.* This follows from ([SGA4], $\mathrm{V_{bis}}$.1.3.10), but for the convenience of the reader we prove the first statement (for the second replace $e_n$ with $e_n^+$). Consider the embedding $\mathcal{F} \hookrightarrow \prod_m e_m I_m$ obtained from the choice of embeddings $\mathcal{F}_m \hookrightarrow I_m$ of each $\mathcal{F}_n$ into an injective sheaf $I_m \in Mod(\mathcal{A}_{X_m})$. For any $q > 0$ the group

$$\mathrm{Ext}^q_{\mathcal{A}_{X_n}}(\mathcal{F}_n, e_m I_m) \tag{5.3.1}$$

is zero. To see this, note that by definition

$$e_m I_m = \prod_{\rho:[n]\to[m]} \rho_* I_m, \tag{5.3.2}$$

and since $\mathrm{Ext}^q_{\mathcal{A}_{X_n}}(\mathcal{F}_n, -)$ commutes with products in the second variable, it suffices to show that for each $\rho : [n] \to [m]$ the group

$$\mathrm{Ext}^q_{\mathcal{A}_{X_n}}(\mathcal{F}_n, \rho_* I_m) \tag{5.3.3}$$

is zero. Since $I_m$ is injective this group is canonically isomorphic to

$$\mathrm{Ext}^q_{\mathcal{A}_{X_n}}(\mathcal{F}_n, R\rho_* I_m) \simeq \mathrm{Ext}^q_{\mathcal{A}_{X_m}}(L\rho^*\mathcal{F}_n, I_m). \tag{5.3.4}$$

Since $\mathcal{F}$ is cartesian, there exists a sheaf $\mathcal{F}_0$ on $X_0$ so that $\mathcal{F}_n$ is pulled back from $\mathcal{F}_0$. Since the corresponding map $X_m \to X_0$ is smooth it follows that $L\rho^*\mathcal{F}_n \simeq \mathcal{F}_m$. From this and the fact that $I_m$ is injective the assertion follows. □

**5.4.** For a sheaf $\mathcal{G} \in Mod(\mathcal{A}_{X^+_{\bullet,\mathrm{et}}})$ there is an injective resolution $\mathcal{G} \to I^\bullet$ such that for each $s$ and $n$ the sheaf $I^s_n \in Mod(\mathcal{A}_{X_{n,\mathrm{et}}})$ is injective and the sequence
(5.4.1)

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}}(\mathrm{res}(\mathcal{F}), I^s) \longrightarrow \mathrm{Hom}_{\mathcal{A}_{X_{0,\mathrm{et}}}}(\mathcal{F}_0, I^s_0) \xrightarrow{d_0 - d_1} \mathrm{Hom}_{\mathcal{A}_{X_{1,\mathrm{et}}}}(\mathcal{F}_1, I^s_1) \longrightarrow \cdots$$

is exact. To see that this last sequence is exact, it suffices to consider the case of $I^s = e_n^+ J$ for some injective sheaf $J$ on $X_n$. In this case,
(5.4.2)
$$\mathrm{Hom}_{\mathcal{A}_{X_{i,\mathrm{et}}}}(\mathcal{F}_i, I^s) \simeq \mathrm{Hom}_{\mathcal{A}_{X_{i,\mathrm{et}}}}(\mathcal{F}_i, \prod_{\rho\in \mathrm{Hom}_{\Delta^+}([i],[n])} \rho_* J) \simeq \prod_{\rho\in \mathrm{Hom}_{\Delta^+}([i],[n])} \mathrm{Hom}_{\mathcal{A}_{X_{n,\mathrm{et}}}}(\rho^*\mathcal{F}_i, J),$$
which since $\mathcal{F}$ is cartesian is canonically isomorphic to
$$\prod_{\rho\in \mathrm{Hom}_{\Delta^+}([i],[n])} \mathrm{Hom}_{\mathcal{A}_{X_{n,\mathrm{et}}}}(\mathcal{F}_n, J). \tag{5.4.3}$$
This implies that the sequence
$$\mathrm{Hom}_{\mathcal{A}_{X_{0,\mathrm{et}}}}(\mathcal{F}_0, I_0^s) \xrightarrow{d_0-d_1} \mathrm{Hom}_{\mathcal{A}_{X_{1,\mathrm{et}}}}(\mathcal{F}_1, I_1^s) \longrightarrow \cdots \tag{5.4.4}$$
is isomorphic to $\mathrm{Hom}_{\mathcal{A}_{X_{n,\mathrm{et}}}}(\mathcal{F}_n, J)$ tensored with the complex denoted $\widetilde{T}$ in the proof of (2.6). Hence the proof of (2.6) implies the exactness of (5.4.1).

It follows that $\mathrm{Ext}^q_{\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}}(\mathrm{res}(\mathcal{F}), \mathcal{G})$ can be computed by choosing a resolution $\mathcal{G} \to J^\bullet$ such that the restriction of $J^s$ to each $X_n$ is acyclic for $\mathrm{Hom}_{\mathcal{A}_{X_n}}(\mathcal{F}_n, -)$, and then taking the $q$–th cohomology of the total complex of the double complex
$$\{\mathrm{Hom}_{\mathcal{A}_{X_p}}(\mathcal{F}_p, J_p^s)\}_{p,s}. \tag{5.4.5}$$
Indeed by the above discussion there exists an injective resolution $\mathcal{G} \to I^\bullet$ for which this holds. Choosing any quasi–isomorphism $J^\bullet \to I^\bullet$ we obtain a morphism of double complexes
$$\{\mathrm{Hom}_{\mathcal{A}_{X_p}}(\mathcal{F}_p, J_p^s)\}_{p,s} \to \{\mathrm{Hom}_{\mathcal{A}_{X_p}}(\mathcal{F}_p, I_p^s)\}_{p,s}. \tag{5.4.6}$$
Since the $J_p^s$ are acyclic for $\mathrm{Hom}_{\mathcal{A}_{X_n}}(\mathcal{F}_n, -)$, for any fixed $p$, the map
$$\mathrm{Hom}_{\mathcal{A}_{X_p}}(\mathcal{F}_p, J_p^\bullet) \to \mathrm{Hom}_{\mathcal{A}_{X_p}}(\mathcal{F}_p, I_p^\bullet) \tag{5.4.7}$$
is a quasi–isomorphism, and hence the map of total complexes induced by (5.4.6) is a quasi–isomorphism as well.

By the same reasoning the groups $\mathrm{Ext}^q_{\mathcal{A}_{X_{\bullet,et}}}(\mathcal{F}, \mathcal{G})$ can be computed by such a complex. From this and (5.3) we obtain Theorem (5.2) (we leave to the reader the verification that these computations are compatible with the map in (5.2)).

□

**Corollary 5.5.** *For any* $\mathcal{F} \in D^-_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}})$ *and* $\mathcal{G} \in D^+_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}})$ *there is for every* $q \in \mathbb{Z}$ *a natural isomorphism*
$$\mathrm{Ext}^q_{\mathcal{A}_{X_{\bullet,\mathrm{et}}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^q_{\mathcal{A}_{X^+_{\bullet,\mathrm{et}}}}(\mathrm{res}(\mathcal{F}), \mathrm{res}(\mathcal{G})). \tag{5.5.1}$$

*Proof.* The same argument used in (5.1) shows that there is a natural map
$$R\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to R\mathrm{Hom}(\mathrm{res}(\mathcal{F}), \mathrm{res}(\mathcal{G})). \tag{5.5.2}$$
This map induces the morphism (5.5.1).

Note first that to prove the corollary for any fixed $q$ we may replace $\mathcal{F}$ by $\tau_{\geq n}\mathcal{F}$ for some $n$ sufficiently negative. We may therefore assume $\mathcal{F} \in D^b_{\mathrm{cart}}(\mathcal{A}_{X_{\bullet,\mathrm{et}}})$. Consideration of the distinguished triangles associated to the truncations of $\mathcal{F}$ then reduces the problem to the case when $\mathcal{F}$ is in $Mod_{\text{cart}}(\mathcal{A}_{X_\bullet,\text{et}})$.

We can also reduce to the case when $\mathcal{G}$ is in $Mod_{\text{cart}}(\mathcal{A}_{X_\bullet,\text{et}})$ by again first considering the truncation $\tau_{\leq n}\mathcal{G}$ for $n$ sufficiently big, and then considering the distinguished triangles associated to the truncations of $\mathcal{G}$. We are thus reduced to (5.2). □

**Corollary 5.6.** *The restriction functor $D^b_{\text{cart}}(\mathcal{A}_{X_\bullet,\text{et}}) \to D^b_{\text{cart}}(\mathcal{A}_{X^+_\bullet,\text{et}})$ is an equivalence of triangulated categories and these categories are also naturally equivalent to $D^b_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A})$.*

*Proof.* The restriction functor is fully faithful by (5.5) (taking $q = 0$) and essentially surjective since the equivalence $D^b_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{A}) \to D^b_{\text{cart}}(\mathcal{A}_{X^+_\bullet,\text{et}})$ provided by (4.7) factors through $D^b_{\text{cart}}(\mathcal{A}_{X_\bullet,\text{et}})$. □

## 6. Quasi–coherent sheaves

Let $S$ be a scheme and $\mathcal{X}/S$ an algebraic stack.

**Definition 6.1** ([LM-B], 13.2.2 and 15.1)**.** (i) A sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$–modules is called *quasi–coherent* if for every $U \in \text{Lis-Et}(\mathcal{X})$ the restriction $\mathcal{M}_U$ of $\mathcal{M}$ to the étale site of $U$ is a quasi–coherent sheaf, and if for every morphism $f : U' \to U$ in Lis-Et$(\mathcal{X})$ the map $f^*\mathcal{M}_U \to \mathcal{M}_{U'}$ is an isomorphism.

(ii) If $\mathcal{X}$ is locally noetherian, a quasi–coherent sheaf $\mathcal{M}$ is called *coherent* if for every $U \in$ Lis-Et$(\mathcal{X})$ the restriction $\mathcal{M}_U$ of $\mathcal{M}$ to $U$ is a coherent sheaf.

We write $Qcoh(\mathcal{X})$ (resp. $Coh(\mathcal{X})$) for the category of quasi–coherent (resp. coherent) sheaves on $\mathcal{X}$.

**Lemma 6.2.** *Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}$–modules such that for every $U \in$ Lis-Et$(\mathcal{X})$ the restriction $\mathcal{M}_U$ is quasi–coherent. Then $\mathcal{M}$ is quasi–coherent if and only if the map $f^*\mathcal{M}_U \to \mathcal{M}_{U'}$ is an isomorphism for smooth morphisms $f : U' \to U$.*

*Proof.* This follows from (3.8). □

**6.3.** For $* \in \{b, +, -, [a, b]\}$ we write $D^*_{\text{qcoh}}(\mathcal{X}) \subset D^*_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}}, \mathcal{O}_{\mathcal{X}_{\text{lis-et}}})$ for the full subcategory of objects whose cohomology sheaves are quasi–coherent sheaves. If $\mathcal{X}$ is locally noetherian, we also define $D^*_{\text{coh}}(\mathcal{X}) \subset D^*_{\text{qcoh}}(\mathcal{X})$ to be the full subcategory of objects with coherent cohomology sheaves.

The following proposition follows immediately from the definitions and the analogous results for schemes (combine ([EGA], III.6.5.13) and ([EGA] $0_{\text{III}}$.12.3.3) with the "way out Lemma" ([Ha], I.7.3)).

**Proposition 6.4** ([LM-B], (13.2.6 (i)) and (15.6 (i) and (ii)))**.** *Let $\mathcal{X}/S$ be an algebraic stack.*

(i) *If $\mathcal{M}$ and $\mathcal{N}$ are quasi–coherent sheaves on $\mathcal{X}$, then $\mathcal{M} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{N}$ is also a quasi–coherent sheaf. More generally, the functor $(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-)$ induces a functor*

$$(-) \otimes^{\mathbb{L}}_{\mathcal{O}_\mathcal{X}} (-) : D^-_{\text{qcoh}}(\mathcal{X}) \times D^-_{\text{qcoh}}(\mathcal{X}) \to D^-_{\text{qcoh}}(\mathcal{X}). \tag{6.4.1}$$

(ii) *If $\mathcal{X}$ is locally noetherian, then $(-)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{X}}}(-)$ induces a functor*

$$(-)\otimes^{\mathbb{L}}_{\mathcal{O}_{\mathcal{X}}}(-) : D^-_{\text{coh}}(\mathcal{X})\times D^-_{\text{coh}}(\mathcal{X})\to D^-_{\text{coh}}(\mathcal{X}). \tag{6.4.2}$$

(iii) *If $\mathcal{X}$ is locally noetherian, then the functor $R\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(-,-)$ induces functors*

$$R\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(-,-) : D^-_{\text{coh}}(\mathcal{X})\times D^+_{\text{qcoh}}(\mathcal{X})\to D^+_{\text{qcoh}}(\mathcal{X}), \tag{6.4.3}$$

$$R\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(-,-) : D^-_{\text{coh}}(\mathcal{X})\times D^+_{\text{coh}}(\mathcal{X})\to D^+_{\text{coh}}(\mathcal{X}), \tag{6.4.4}$$

**Lemma 6.5.** *Let $f:\mathcal{X}\to\mathcal{Y}$ be a quasi–compact morphism of algebraic stacks, and let $f_*$ be the functor defined in* (3.3)*. Denote by $f^*$ the functor $f^{-1}(-)\otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}}\mathcal{O}_{\mathcal{X}}$.*

(i) *For any quasi–coherent sheaf $\mathcal{M}\in Qcoh(\mathcal{X})$ the sheaf $f_*\mathcal{M}$ is a quasi–coherent sheaf on $\mathcal{Y}$.*

(ii) *For any quasi–coherent sheaf $\mathcal{N}$ on $\mathcal{Y}$ the sheaf $f^*\mathcal{N}$ is a quasi–coherent sheaf on $\mathcal{X}$. The induced functor $f^*: Qcoh(\mathcal{Y})\to Qcoh(\mathcal{X})$ is left adjoint to the functor $f_*: Qcoh(\mathcal{X})\to Qcoh(\mathcal{Y})$.*

(iii) *If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian, then for any coherent sheaf $\mathcal{N}$ on $\mathcal{Y}$ the sheaf $f^*\mathcal{N}$ is a coherent sheaf on $\mathcal{X}$.*

*Proof.* To see (i), note first that if $X\to\mathcal{X}$ is a smooth cover and $X'$ denotes $X\times_{\mathcal{X}}X$, then for any quasi–coherent sheaf $\mathcal{M}$ on $\mathcal{X}$ we have

$$f_*\mathcal{M}=\text{Ker}(f_{X*}\mathcal{M}|_X \xrightarrow{p_1^*-p_2^*} f_{X'*}\mathcal{M}|_{X'}), \tag{6.5.1}$$

where $f_X$ (resp. $f_{X'}$) denotes the induced map $X\to\mathcal{Y}$ (resp. $X'\to\mathcal{Y}$). It therefore suffices to consider the case when $\mathcal{X}$ is an algebraic space.

In this case, for any smooth morphism $U\to\mathcal{Y}$ we have

$$f_*\mathcal{M}(U)=\mathcal{M}(U\times_{\mathcal{Y}}\mathcal{X}). \tag{6.5.2}$$

Thus to prove (i) it suffices to show that if $f:V\to U$ is a quasi–compact morphism of algebraic spaces then the formation of $f_*\mathcal{M}$ is compatible with smooth base change. Repeating the above reduction with a cover of $V$ by a scheme, it follows that it suffices to consider the case when $f:V\to U$ is a quasi–compact and quasi–separated morphism of schemes which follows from ([EGA], IV.2.3.1).

To see (ii), note first that $f^*\mathcal{N}$ is cartesian by (3.12). In fact the proof of (3.12) shows the following. If $W\in\text{Lis-Et}(\mathcal{X})$ admits a morphism $\tilde{f}:W\to Z$ over $f$ to some object $Z\in\text{Lis-Et}(\mathcal{Y})$, then $(f^*\mathcal{N})_W$ is isomorphic to the pullback of the quasi–coherent sheaf $\mathcal{N}_Z$ via the morphism $W_{\text{et}}\to Z_{\text{et}}$ induced by $\tilde{f}$. Since any $W\in\text{Lis-Et}(\mathcal{X})$ étale locally admits such a morphism $\tilde{f}$, the sheaf $(f^*\mathcal{N})_W$ is quasi–coherent on $W_{\text{et}}$. This implies (ii) and also (iii). □

The following corollary whose proof is immediate from the proof of (i) will be superceded by (10.13) below, but we need it for the subsequent "devissage Lemma" which will be used in what follows.

**Corollary 6.6.** *Let $\mathcal{X}$ be a locally noetherian algebraic stack and $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ a closed substack. Then the functor $I_*$ is exact and takes coherent sheaves on $\mathcal{Y}$ to coherent sheaves on $\mathcal{X}$. In particular, $I_*$ induces a functor*

$$I_* : D^+_{\mathrm{coh}}(\mathcal{Y}) \to D^+_{\mathrm{coh}}(\mathcal{X}). \tag{6.6.1}$$

**Lemma 6.7** ([LM-B], (15.7))**.** *Let $\mathcal{X}$ be a noetherian algebraic stack and let $\mathcal{C}$ be a full subcategory of $D^+_{\mathrm{coh}}(\mathcal{X})$ such that the following conditions hold:*

(i) *$0 \in \mathcal{C}$ and for every distinguished triangle*

$$M' \to M \to M'' \to M'[1] \tag{6.7.1}$$

*in $D^+_{\mathrm{coh}}(\mathcal{X})$, if any two of $M'$, $M$, and $M''$ are in $\mathcal{C}$ then so is the third;*

(ii) *If $M \in D^+_{\mathrm{coh}}(\mathcal{X})$ and if $\tau_{\leq n} M \in \mathcal{C}$ for every integer $n$ then $M \in \mathcal{C}$;*

(iii) *For every integral closed substack $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ and every coherent $\mathcal{O}_{\mathcal{Y}}$–module $\mathcal{N}$ with support all of $\mathcal{Y}$, there exists $N \in D^+_{\mathrm{coh}}(\mathcal{Y})$ and a morphism $\alpha : \mathcal{N} \to N$ in $D^+_{coh}(\mathcal{Y})$ such that $I_* N \in \mathcal{C}$ and such that $\alpha$ is an isomorphism over some dense open substack of $\mathcal{Y}$.*

*Then in fact $\mathcal{C} = D^+_{\mathrm{coh}}(\mathcal{X})$.*

*Proof.* The proof of ([LM-B], 15.7) holds verbatim. □

**6.8.** Using the results of section 4, the categories $D^+_{\mathrm{qcoh}}(\mathcal{X})$ and $D^+_{\mathrm{coh}}(\mathcal{X})$ can also be described using hypercovers.

Assume that $\mathcal{X}$ is quasi–compact, and let $P : X \to \mathcal{X}$ be a smooth cover by a quasi–compact algebraic space $X$. Denote by $X_\bullet \to \mathcal{X}$ the simplicial algebraic space obtained by taking the 0–coskeleton of $P$, and let $X^+_\bullet$ be associated strictly simplicial algebraic space.

As in (4.2), we then have the strictly simplicial topos $X^+_{\bullet,\text{lis-et}}$ which is naturally a ringed topos with structure sheaf $\mathcal{O}_{X_\bullet,\text{lis-et}}$. We also have the étale topos $X_{\bullet,\text{et}}$ and there is a flat morphism of ringed topoi

$$\epsilon : (X^+_{\bullet,\text{lis-et}}, \mathcal{O}_{X^+_{\bullet,\text{lis-et}}}) \longrightarrow (X^+_{\bullet,\text{et}}, \mathcal{O}_{X_{\bullet,\text{et}}}) \tag{6.8.1}$$

induced by the natural morphisms of topoi $\epsilon_n : X_{n,\text{lis-et}} \to X_{n,\text{et}}$.

**Definition 6.9.** A sheaf $F \in Mod(\mathcal{O}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $F \in Mod(\mathcal{O}_{X^+_{\bullet,\text{et}}})$, $F \in Mod(\mathcal{O}_{X_{\bullet,\text{et}}})$) is *quasi–coherent* if for each $n$ the restriction $F_n \in Mod(\mathcal{O}_{X_{n,\text{lis-et}}})$ (resp. $F_n \in Mod(\mathcal{O}_{X_{n,\text{et}}})$) is quasi–coherent and for each morphism $\rho : [m] \to [n]$ in $\Delta^+$ the map $\rho^* F_m \to F_n$ is an isomorphism. We denote by $Qcoh(X^+_{\bullet,\text{lis-et}})$ (resp. $Qcoh(X^+_{\bullet,\text{et}})$, $Qcoh(X_{\bullet,\text{et}})$) the category of quasi–coherent sheaves on $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,et}$).

If $\mathcal{X}$ is locally noetherian, a quasi–coherent sheaf $F$ in $Mod(\mathcal{O}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $Mod(\mathcal{O}_{X^+_{\bullet,\text{et}}})$, $Mod(\mathcal{O}_{X_{\bullet,\text{et}}})$) is called *coherent* if its restriction to each $X_n$ is a coherent sheaf. We write $Coh(X^+_{\bullet,\text{lis-et}})$ (resp. $Coh(X^+_{\bullet,\text{et}})$, $Coh(X_{\bullet,\text{et}})$) for the resulting categories of coherent sheaves.

**Remark 6.10.** It follows from the condition that the map $\rho^* F_m \to F_n$ is an isomorphism for every $\rho : [m] \to [n]$ that a sheaf $F$ in $Mod_{\text{cart}}(\mathcal{O}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $Mod_{\text{cart}}(\mathcal{O}_{X^+_{\bullet,\text{et}}})$, $Mod_{\text{cart}}(\mathcal{O}_{X_{\bullet,\text{et}}})$) is quasi–coherent if and only if $F_0$ is quasi–coherent on $X_{0,\text{et}}$.

Similarly, if $\mathcal{X}$ is locally noetherian then a sheaf $F$ in $Mod_{\text{cart}}(\mathcal{O}_{X^+_{\bullet,\text{lis-et}}})$ (resp. $Mod_{\text{cart}}(\mathcal{O}_{X^+_{\bullet,\text{et}}})$, $Mod_{\text{cart}}(\mathcal{O}_{X_{\bullet,\text{et}}})$) is coherent if and only if $F_0$ is coherent on $X_{0,\text{et}}$.

**Lemma 6.11.** *A cartesian sheaf $\mathcal{M} \in Mod_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{O}_{\mathcal{X}_{\text{lis-et}}})$ is quasi–coherent if and only if the restriction of $\mathcal{M}$ to $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) is quasi–coherent in the sense of* (6.9).

*Similarly, if $\mathcal{X}$ is locally noetherian then a cartesian sheaf $\mathcal{M} \in Mod_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \mathcal{O}_{\mathcal{X}_{\text{lis-et}}})$ is coherent if and only if the restriction of $\mathcal{M}$ to $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) is coherent.*

*Proof.* Since any object $W \in \text{Lis-Et}(\mathcal{X})$ étale locally admits an $\mathcal{X}$–morphism to $X$, the sheaf $\mathcal{M}$ is quasi–coherent if and only if the sheaf $\mathcal{M}_X$ is a quasi–coherent sheaf on $X_{\text{et}}$. From this and (6.10) the lemma follows. □

**Proposition 6.12.** *The categories $Qcoh(X_{\bullet,\text{et}})$, $Qcoh(X^+_{\bullet,\text{et}})$, and $Qcoh(X^+_{\bullet,\text{lis-et}})$ are all naturally equivalent and are also equivalent to the category $Qcoh(\mathcal{X})$ of quasi–coherent sheaves on the lisse-étale site of $\mathcal{X}$.*

*If $\mathcal{X}$ is locally noetherian, then the categories $Coh(X^+_{\bullet,\text{lis-et}})$,$Coh(X^+_{\bullet,\text{et}})$, and $Coh(X_{\bullet,\text{et}})$ are also naturally equivalent, and are also equivalent to the category $Coh(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$.*

*Proof.* This follows from (4.4) and (6.11). □

**6.13.** Denote by $D^+_{\text{qcoh}}(X_{\bullet,\text{et}})$ (resp. $D^+_{\text{qcoh}}(X^+_{\bullet,\text{et}})$, $D^+_{\text{qcoh}}(X^+_{\bullet,\text{lis-et}})$, $D^+_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}})$) the full subcategory of the derived category $D^+(Mod(\mathcal{O}_{X_{\bullet,\text{et}}}))$ (resp. $D^+(Mod(\mathcal{O}_{X^+_{\bullet,\text{et}}}))$, $D^+(Mod(\mathcal{O}_{X^+_{\bullet,\text{lis-et}}}))$, $D^+(Mod(\mathcal{O}_{\mathcal{X}_{\text{lis-et}}}))$) consisting of complexes with quasi–coherent cohomology sheaves.

Denote by

$$\pi : X^+_{\bullet,\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}, \quad \epsilon : X^+_{\bullet,\text{lis-et}} \to X^+_{\bullet,\text{et}} \tag{6.13.1}$$

the natural flat morphisms of ringed topoi. Here $\pi^*$ is the functor which sends a sheaf $\mathcal{F}$ on $\mathcal{X}_{\text{lis-et}}$ to the family of sheaves $\{\mathcal{F}|_{X_n}\}_n$ on the $X_n$ with the natural transition maps, and $\epsilon_*$ is the restriction functor.

**Theorem 6.14.** *The functors*

$$\pi^* : D^+_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}}) \longrightarrow D^+_{\text{qcoh}}(X^+_{\bullet,\text{lis-et}}) \tag{6.14.1}$$

*and*

$$\epsilon^* : D^+_{\text{qcoh}}(X^+_{\bullet,\text{et}}) \longrightarrow D^+_{\text{qcoh}}(X^+_{\bullet,\text{lis-et}}), \tag{6.14.2}$$

*are both equivalences of triangulated categories with quasi–inverses given by $R\pi_*$ and $R\epsilon_*$ respectively. For any $F \in D^+_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}})$, the maps*

$$H^*(\mathcal{X}_{\text{lis-et}}, F) \to H^*(X^+_{\bullet,\text{lis-et}}, \pi^*F) \to H^*(X^+_{\bullet,\text{et}}, R\epsilon_*\pi^*F) \tag{6.14.3}$$

*are isomorphisms, and these isomorphisms are functorial with respect to smooth base change $\mathcal{X}' \to \mathcal{X}$.*

*Proof.* This follows from (4.7) and (6.11). □

**Corollary 6.15.** *If $\mathcal{X}$ is locally noetherian, then the functors*

$$\pi^* : D^+_{\mathrm{coh}}(\mathcal{X}_{\text{lis-et}}) \longrightarrow D^+_{\mathrm{coh}}(X^+_{\bullet,\text{lis-et}}) \tag{6.15.1}$$

*and*

$$\epsilon^* : D^+_{\mathrm{coh}}(X^+_{\bullet,\mathrm{et}}) \longrightarrow D^+_{\mathrm{coh}}(X^+_{\bullet,\text{lis-et}}) \tag{6.15.2}$$

*are equivalences.*

We also note the following Proposition which will be used in what follows:

**Proposition 6.16.** *Let $X \to \mathcal{X}$ and $X' \to \mathcal{X}$ be smooth quasi–compact and quasi–separated presentations, and let $j : X \to X'$ be a morphism over $\mathcal{X}$. Then for any $F \in D^b_{\mathrm{qcoh}}(X'^+_{\bullet,\mathrm{et}})$, the pullback $Lj^*F$ lies in $D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$ and the resulting functor*

$$Lj^* : D^b_{\mathrm{qcoh}}(X'^+_{\bullet,\mathrm{et}}) \longrightarrow D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}}) \tag{6.16.1}$$

*is an equivalence of triangulated categories.*

*Proof.* Let

$$\mathrm{res}' : D^b_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}}) \to D^b_{\mathrm{qcoh}}(X'^+_{\bullet,\mathrm{et}}), \quad \mathrm{res} : D^b_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}}) \to D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}}) \tag{6.16.2}$$

be the restriction functors, which are equivalences by (4.7). For any $F \in D^b_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$, there is a natural map

$$Lj^*\mathrm{res}'(F) \longrightarrow \mathrm{res}(F). \tag{6.16.3}$$

To prove the Proposition, it suffices to show that this map is an isomorphism for all $F$.

For this, consideration of the distinguished triangles associated to the truncations of $F$ reduces the question to the case when $F$ is a quasi–coherent sheaf on $\mathcal{X}_{\text{lis-et}}$. Choose a smooth cover $U \to \mathcal{X}$ by a scheme and let $X_{U\bullet}$ and $X'_{U\bullet}$ denote the base changes of $X_\bullet$ and $X'_\bullet$ to $U$ so that there is a cartesian square

$$\begin{array}{ccc} X_{U\bullet} & \xrightarrow{\tilde{j}} & X'_{U\bullet} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X_\bullet & \xrightarrow{j} & X'_\bullet. \end{array} \tag{6.16.4}$$

Since $g$ induces a smooth surjective map $X_{U,n} \to X_n$ for all $n$, to prove that (6.16.3) is an isomorphism it suffices to show that the induced map

$$L\tilde{j}^*g'^*\mathrm{res}'(F) \simeq Lg^*Lj^*\mathrm{res}'(F) \longrightarrow Lg^*\mathrm{res}(F) \simeq g^*\mathrm{res}(F) \tag{6.16.5}$$

is an isomorphism. Since $F$ is quasi–coherent this reduces the problem to the case when $\mathcal{X} = U$ in which case the result is clear since $\mathrm{res}(F)$ and $\mathrm{res}'(F)$ are obtained by pulling from a quasi–coherent sheaf $G$ on $\mathcal{X}_{\mathrm{et}}$. □

**Remark 6.17.** The fact that $Lj^*$ has image in $D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$ can also be seen by observing that $j$ is a local complete intersection morphism since both $X$ and $X'$ are smooth over $\mathcal{X}$.

**Theorem 6.18.** *For any $\mathcal{F} \in D^-_{\mathrm{qcoh}}(X_{\bullet,\mathrm{et}})$ and $\mathcal{G} \in D^+_{\mathrm{qcoh}}(X_{\bullet,\mathrm{et}})$ there is for every $q \in \mathbb{Z}$ a natural isomorphism*

$$\mathrm{Ext}^q_{X_{\bullet,\mathrm{et}}}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^q_{X^+_{\bullet,\mathrm{et}}}(\mathrm{res}(\mathcal{F}), \mathrm{res}(\mathcal{G})). \tag{6.18.1}$$

*Proof.* This follows from (5.5). □

**Corollary 6.19.** *The restriction functor $D^b_{\mathrm{qcoh}}(X_{\bullet,\mathrm{et}}) \to D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$ is an equivalence of triangulated categories and these categories are also naturally equivalent to $D^b_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$.*

*Proof.* The restriction functor is fully faithful by (6.18) (taking $q = 0$) and essentially surjective since the equivalence $D^b_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}}) \to D^b_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$ provided by (6.14) factors through $D^b_{\mathrm{qcoh}}(X_{\bullet,\mathrm{et}})$. □

We can also use the above techniques to compute derived functors. Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi–compact morphism of algebraic stacks.

**Lemma 6.20.** *For any $F \in D^+_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$, the image $Rf_*F \in D^+(\mathcal{Y}_{\text{lis-et}})$ lies in $D^+_{\mathrm{qcoh}}(\mathcal{Y}_{\text{lis-et}})$.*

*Proof.* The assertion is local on $\mathcal{Y}$, and hence we can assume that $\mathcal{X}$ is quasi–compact and admits a quasi–separated presentation $X \to \mathcal{X}$. Furthermore, as in the proof of (4.8), it suffices to consider the case when $F$ is in $Qcoh(\mathcal{X})$. In this case for any integer $q$ the sheaf $R^qf_*F$ can be described as follows. The sheaf $R^qf_*F$ is the sheaf associated to the presheaf which to any smooth $V \to \mathcal{Y}$, with $V$ an affine scheme, associates $H^q(\mathcal{X}_V, F|_{\mathcal{X}_V})$. By (4.7), this is the same as the sheaf associated to the presheaf which to $V$ as above associates $H^q(X^+_{V,\bullet,\mathrm{et}}, F)$, and this in turn is by (2.3) equal to the sheaf associated to the presheaf $V \mapsto H^q(X_{V,\bullet,\mathrm{et}}, F)$. The spectral sequence (see for example ([De], 5.2.3.1))

$$E_1^{pq} = H^q((X_p \times_{\mathcal{Y}} V)_{\mathrm{et}}, F|_{X_p \times_{\mathcal{Y}} V}) \implies H^q(X_{V,\bullet,\mathrm{et}}, F) \tag{6.20.1}$$

then reduces the proof to showing that if $f_p : X_p \times_{\mathcal{Y}} V \to V$ denotes the projection and $f_p^{\mathrm{et}} : (X_p \times_{\mathcal{Y}} V)_{\mathrm{et}} \to V_{\mathrm{et}}$ the associated morphism of topoi, then the sheaf $R^qf^{\mathrm{et}}_{p*}F|_{X_p\times_{\mathcal{X}}V}$ on $V_{\mathrm{et}}$ is quasi–coherent and its formation commutes with smooth base change $V' \to V$. Since the morphism $f_p$ is quasi–compact and quasi–separated this follows from ([EGA], III.1.4.10) (note that though only stated there for separated morphisms, the same argument also proves the result for quasi–separated morphisms) and ([SGA6], IV.3.1.0). □

## 7. The derived pullback functor

**7.1.** If $\mathcal{A}$ is an abelian category, we denote by $D^+(\mathcal{A})$ (resp. $D^-(\mathcal{A})$, $D^b(\mathcal{A})$) the derived category of complexes bounded below (resp. bounded above, bounded), and for $n \in \mathbb{Z}$ we denote by $\tau_{\geq n}$ the "canonical truncation in degree $\geq n$" functor ([SGA4], VII.1.1.13). Let $D'(\mathcal{A})$ denote the category of projective systems

$$K = (\cdots \longrightarrow K_{\geq -n-1} \longrightarrow K_{\geq -n} \longrightarrow \cdots \longrightarrow K_{\geq 0}),$$

where each $K_{\geq -n} \in D^+(\mathcal{A})$ and the maps

$$K_{\geq -n} \longrightarrow \tau_{\geq -n}K_{\geq -n}, \quad \tau_{\geq -n}K_{\geq -n-1} \longrightarrow \tau_{\geq -n}K_{\geq -n}$$

are all isomorphisms. We denote by $D'^b(\mathcal{A})$ the full subcategory of $D'(\mathcal{A})$ consisting of objects $K$ for which $K_{\geq -n} \in D^b(\mathcal{A})$ for all $n$.

**Example 7.2.** The main example we will consider is when $\mathcal{A}$ is the category of $\mathcal{O}_{\mathcal{X}}$–modules on the lisse-étale site of an algebraic stack $\mathcal{X}$ in which case we write $D'(\mathcal{X})$ for $D'(\mathcal{A})$. In this setting we will also consider the subcategory $D'_{\mathrm{qcoh}}(\mathcal{X}) \subset D'(\mathcal{X})$ consisting of systems $K$

for which $K_{\geq -n}$ is in $D^+_{\mathrm{qcoh}}(\mathcal{X})$ for all $n$. If $\mathcal{X}$ is locally noetherian we also have the category $D'_{\mathrm{coh}}(\mathcal{X})$.

**7.3.** The shift functor $(-)[1] : D(\mathcal{A}) \to D(\mathcal{A})$ is extended to $D'(\mathcal{A})$ by defining

$$K[1] := (\cdots \to K_{\geq -n-1}[1] \to K_{\geq -n}[1] \to \cdots \to K_0[1] \to (\tau_{\geq 1}K_0)[1])$$

for $K \in D'(\mathcal{A})$ ([LM-B], 17.4 (3)). We say that a triangle of $D'(\mathcal{A})$

$$K_1 \xrightarrow{u} K_2 \xrightarrow{v} K_3 \xrightarrow{w} K_1[1] \tag{7.3.1}$$

is *distinguished* if for every $n \geq 0$, there exists a commutative diagram

$$\begin{array}{ccccccc} K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v'} & L & \xrightarrow{w'} & K_{1,\geq -n}[1] \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v} & K_{3,\geq -n} & \xrightarrow{w} & K_{1,\geq -n+1}[1] \end{array} \tag{7.3.2}$$

where the top row is a distinguished triangle in $D(\mathcal{A})$, $\gamma$ is the map obtained from the given map $K_{1,\geq -n} \to K_{1,\geq -n+1}$, and the map

$$\tau_{\geq -n}L \longrightarrow K_{3,\geq -n} \tag{7.3.3}$$

induced by $\beta$ is an isomorphism. For example, if

$$L_1 \longrightarrow L_2 \longrightarrow L_3 \longrightarrow L_1[1]$$

is a distinguished triangle in $D(\mathcal{A})$, then

$$(\tau_{\geq -n}L_1) \longrightarrow (\tau_{\geq -n}L_2) \longrightarrow (\tau_{\geq -n}L_3) \longrightarrow (\tau_{\geq -n}L_1)[1]$$

is distinguished in $D'(\mathcal{A})$, where $(\tau_{\geq -n}L_i)$ denotes the system with

$$(\tau_{\geq -n}L_i)_{\geq -n} := \tau_{\geq -n}L_i.$$

**Remark 7.4.** In the context of (7.2), the notion of distinguished triangle on $D'(\mathcal{X})$ restricts to give a notion of distinguished triangle in $D'_{\mathrm{qcoh}}(\mathcal{X})$ and $D'_{\mathrm{coh}}(\mathcal{X})$. Observe that in (7.3.2) above, if $K_{1,\geq -n}$ and $K_{2,\geq -n}$ are in $D^+_{\mathrm{qcoh}}(\mathcal{X})$ (resp. $D^+_{\mathrm{coh}}(\mathcal{X})$) and the top triangle is distinguished in $D^+(\mathcal{X})$, then $L$ automatically is in the subcategory $D^+_{\mathrm{qcoh}}(\mathcal{X})$ (resp. $D^+_{\mathrm{coh}}(\mathcal{X})$).

**Remark 7.5.** The above definition of a distinguished triangle in $D'(\mathcal{A})$ differs from that in ([LM-B], 17.4 (3)). In (loc. cit.), a triangle (7.3.1) is said to be distinguished if for all $n \geq 0$ and for all commutative diagrams

$$\begin{array}{ccccccc} K_1 & \xrightarrow{u} & K_2 & \xrightarrow{v} & K_3 & \xrightarrow{w} & K_1[1] \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \alpha_1[1]} \\ K_{1,\geq -n} & \xrightarrow{u} & K_{2,\geq -n} & \xrightarrow{v'} & L & \xrightarrow{w'} & K_{1,\geq -n}[1], \end{array} \tag{7.5.1}$$

where the bottom row is a distinguished triangle in $D(\mathcal{A})$ and $\alpha_1$ and $\alpha_2$ denote the natural maps, the map $K_{3,\geq -n} \to \tau_{\geq -n}L$ is an isomorphism. This definition is not suitable for this paper. For example, suppose $\mathcal{A}$ is the category of $R$-modules for some ring $R$, and let $M \in \mathcal{A}$ be an object. Defining $K_1 = 0$, $K_2 = K_3 = M$ (placed in degree 0), and $v$ to be the zero map, we obtain a triangle in $D'(\mathcal{A})$

$$0 \longrightarrow M \xrightarrow{0} M \longrightarrow 0[1] \tag{7.5.2}$$

for which there does not exist a diagram as in (7.5.1) for any $n$ (exercise). Hence the triangle (7.5.2) is distinguished in $D'(\mathcal{A})$ according to ([LM-B], 17.4 (3)) even though it is not distinguished in $D(\mathcal{A})$.

**7.6.** The cohomology functors

$$\mathcal{H}^i : D^+(\mathcal{A}) \longrightarrow \mathcal{A}$$

extend naturally to $D'(\mathcal{A})$: for $K \in D'(\mathcal{A})$, the pro-object "$\varprojlim$"$\mathcal{H}^i(K_{\geq -n})$ is essentially constant, and we define $\mathcal{H}^i(K)$ to be "the" corresponding object of $\mathcal{A}$ under the natural equivalence between $\mathcal{A}$ and the category of essentially constant pro-objects in $\mathcal{A}$. It follow from the definition of a distinguished triangle in $D'(\mathcal{A})$ (7.3) that a distinguished triangle (7.3.1) induces a long exact sequence of cohomology groups

$$\cdots \longrightarrow \mathcal{H}^i(K_1) \longrightarrow \mathcal{H}^i(K_2) \longrightarrow \mathcal{H}^i(K_3) \longrightarrow \mathcal{H}^{i+1}(K_1) \longrightarrow \cdots .$$

**7.7.** Suppose now that $\mathcal{A}$ has enough injectives and let $M$ be an object of $\mathcal{A}$. Then the functor

$$\mathrm{RHom}_{D^b(\mathcal{A})}(-, M) : D^b(\mathcal{A}) \longrightarrow D^+(\mathrm{Ab}),$$

where Ab denotes the category of abelian groups, induces a natural functor

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(-, M) : D'^b(\mathcal{A}) \longrightarrow D'^b(\mathrm{Ab}^{op})$$

defined as follows. For $K \in D'^b(\mathcal{A})$, define

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} := \tau_{\leq n}\mathrm{RHom}_{D^b(\mathcal{A})}(K_{\geq -n}, M).$$

There is a natural map

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} \to \mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n+1}$$

which induces an isomorphism

$$\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n} \simeq \tau_{\leq n}\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n+1}.$$

We write $\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)$ for the ind–object $\{\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n}\}$ in $D(\mathrm{Ab})$. In the case when $K = (\tau_{\geq -n}L)$ for some $L \in D^b(\mathcal{A})$, the object $\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)$ is simply the inductive system $\{\tau_{\leq n}\mathrm{RHom}_{D(\mathcal{A})}(L, M)\}$.

For any integer $i$, the ind–group $H^i(\mathrm{RHom}_{D'^b(\mathcal{A})}(K, M)_{\leq n})$ is essentially constant and we define $\mathrm{Ext}^i(K, M) \in \mathrm{Ab}$ to be the limit. For any $n \geq i$ the natural map

$$\mathrm{Ext}^i_{D^b(\mathcal{A})}(K_{\geq -n}, M) \to \mathrm{Ext}^i(K, M)$$

is an isomorphism.

It follow from the definition of a distinguished triangle in $D'(\mathcal{A})$ (7.3) that a distinguished triangle (7.3.1) induces a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^i(K_3, M) \longrightarrow \mathrm{Ext}^i(K_2, M) \longrightarrow \mathrm{Ext}^i(K_1, M) \longrightarrow \mathrm{Ext}^{i+1}(K_3, M) \longrightarrow \cdots .$$

**7.8.** Let $\mathcal{X}$ be an algebraic stack and $P : X \to \mathcal{X}$ a smooth cover with associated 0-coskeleton $X_\bullet \to \mathcal{X}$. Let $D'_{\mathrm{qcoh}}(\mathcal{X})$ (resp. $D'_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$, etc.) denote the category of projective systems

$$K = (\cdots \to K_{\geq -n-1} \to K_{\geq -n} \to \cdots \to K_{\geq 0}) \tag{7.8.1}$$

where each $K_{-n}\in D^{[-n,\infty[}_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$ (resp. $K_{-n}\in D_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$, etc.) and for every $n$ the natural map

$$\tau_{\geq -n}K_{\geq -n-1}\longrightarrow K_{\geq -n} \tag{7.8.2}$$

is an isomorphism. We view these categories as categories with triangles using the definition of triangles given in (7.3) above.

**Lemma 7.9.** *Let $f:\mathcal{X}\to\mathcal{Y}$ be a quasi–compact morphism of algebraic stacks. For any integer $a\in\mathbb{Z}$, the functor $Rf_*:D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})\to D^{[a,\infty[}_{\mathrm{qcoh}}(\mathcal{Y}_{\text{lis-et}})$ has a left adjoint denoted (abusively) $\tau_{\geq a}Lf^*$.*

*Proof.* Choose a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\tilde f} & Y \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array} \tag{7.9.1}$$

where $P$ and $Q$ are smooth, surjective, quasi–compact and quasi–separated, and the map $X\to Y\times_{\mathcal{Y}}\mathcal{X}$ induced by $\tilde f$ is smooth. Applying the 0–coskeleton functor we obtain a commutative diagram

$$\begin{array}{ccccc} X_\bullet & \xrightarrow{g} & Y_{\mathcal{X}\bullet} & \xrightarrow{h} & Y_\bullet \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X_\bullet & \xrightarrow{\beta} & \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array} \tag{7.9.2}$$

where $X_\bullet$ denotes the 0–coskeleton of $X\to\mathcal{X}$, $Y_\bullet$ denotes the 0–coskeleton of $Y\to\mathcal{Y}$, and $Y_{\mathcal{X}\bullet}$ denotes $Y_\bullet\times_{\mathcal{Y}}\mathcal{X}$. Note that since $X\to Y\times_{\mathcal{Y}}\mathcal{X}$ is smooth, the map $g$ induces a smooth map $X_n\to Y_{\mathcal{X},n}$ for every $n$. In particular, $g$ induces a morphism of topoi $X^+_{\bullet,\text{lis-et}}\to Y^+_{\mathcal{X}\bullet,\text{lis-et}}$. The diagram (7.9.2) induces a 2–commutative diagram of triangulated categories

$$\begin{array}{ccccc} D^+_{\mathrm{qcoh}}(X^+_{\bullet,\text{lis-et}}) & \xrightarrow{Rg_*} & D^+_{\mathrm{qcoh}}(Y^+_{\mathcal{X}\bullet,\text{lis-et}}) & & \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle R\alpha_*}\downarrow & & \\ D^+_{\mathrm{qcoh}}(X^+_{\bullet,\text{lis-et}}) & \xrightarrow{R\beta_*} & D^+_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}}) & \xrightarrow{Rf_*} & D^+_{\mathrm{qcoh}}(\mathcal{Y}_{\text{lis-et}}) \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle \beta^*} \\ D^+_{\mathrm{qcoh}}(X^+_{\bullet,\text{lis-et}}) & \xrightarrow{Rg_*} & D^+_{\mathrm{qcoh}}(Y^+_{\mathcal{X}\bullet,\text{lis-et}}) & \xrightarrow{Rh_*} & D^+_{\mathrm{qcoh}}(Y^+_{\bullet,\text{lis-et}}), \end{array} \tag{7.9.3}$$

where the isomorphism $Rg_*\simeq\alpha^*\circ R\alpha_*\circ Rg_*$ is obtained from the natural map $\alpha^*R\alpha_*\to\mathrm{id}$ which the same argument used in the proof of (4.8) shows is an isomorphism. It follows that via the equivalences in (4.7) the functor $Rf_*$ is identified with the functor

$$Rh_*\circ Rg_*:D^+_{\mathrm{qcoh}}(X^+_{\bullet,\text{lis-et}})\longrightarrow D^+_{\mathrm{qcoh}}(Y^+_{\bullet,\text{lis-et}}). \tag{7.9.4}$$

Since $g_*$ is obtained from a morphism of topoi, the functor $g_*$ takes injectives to injectives. It follows that there is a natural isomorphism of functors $R\tilde f_*\simeq Rh_*\circ Rg_*$. On the other hand it follows from the definitions that the diagram

$$
\begin{array}{ccc}
D^+_{\text{qcoh}}(X^+_{\bullet,\text{lis-et}}) & \xrightarrow{R\tilde{f}_*} & D^+_{\text{qcoh}}(Y^+_{\bullet,\text{lis-et}}) \\
\text{res}\downarrow & & \downarrow\text{res} \\
D^+_{\text{qcoh}}(X^+_{\bullet,\text{et}}) & \xrightarrow{R\tilde{f}^{\text{et}}_*} & D^+_{\text{qcoh}}(Y^+_{\bullet,\text{et}})
\end{array}
\tag{7.9.5}
$$

commutes. Hence it suffices to show that the functor $R\tilde{f}^{\text{et}}_* : D^{[a,\infty[}_{\text{qcoh}}(Y^+_{\bullet,\text{et}}) \to D^{[a,\infty[}_{\text{qcoh}}(Y^+_{\bullet,\text{et}})$ has a left adjoint. This follows from Spaltenstein's theory ([Sp]) which shows that one can define $Lf^{\text{et}*}$ on the unbounded derived category $D_{\text{qcoh}}(Y^+_{\bullet,\text{et}})$ (see ([Sp], Theorem A). Though (loc. cit.) is stated only for topological spaces the definition of $Lf^*$ for unbounded complexes works equally well for simplicial schemes with the étale topology. For a more detailed discussion of this see also ([KS], 2.4.12, 1.7.8, Exercise I.23, and the discussion on page 110). The functor $Rf^{\text{et}}_*$ therefore has a left adjoint $\tau_{\geq a}Lf^{\text{et}*}$. □

**7.10.** It follows from the construction that for any integer $a' \geq a$ there is a natural isomorphism of functors

$$\tau_{\geq a'}Lf^* \simeq \tau_{\geq a'}Lf^*\tau_{\geq a'}. \tag{7.10.1}$$

In particular, given an object

$$K = (\cdots \to K_{\geq -n-1} \to K_{\geq -n} \to \cdots \to K_{\geq 0}) \tag{7.10.2}$$

in $D'_{\text{qcoh}}(\mathcal{Y}_{\text{lis-et}})$, we obtain an object $Lf^*K \in D'_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}})$ by defining

$$(Lf^*K)_{\geq -n} := \tau_{\geq -n}Lf^*K_{\geq -n}. \tag{7.10.3}$$

We thus obtain a functor

$$Lf^* : D'_{\text{qcoh}}(\mathcal{Y}_{\text{lis-et}}) \longrightarrow D'_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}}). \tag{7.10.4}$$

It follows from the construction that this functor takes distinguished triangles to distinguished triangles.

**7.11.** If $\mathcal{F}$ is a quasi–coherent sheaf on $\mathcal{Y}$, then $\tau_{\geq 0}Lf^*\mathcal{F}$ is isomorphic to the pullback $f^*\mathcal{F}$ defined in (7.3). In particular, for any $a \geq 0$ there is a natural morphism $\tau_{\geq a}Lf^*\mathcal{F} \to f^*\mathcal{F}$ in $D^b_{\text{qcoh}}(\mathcal{X})$.

**Corollary 7.12.** *If the morphism $f : \mathcal{X} \to \mathcal{Y}$ is flat, then the map $\tau_{\geq a}Lf^*\mathcal{F} \to f^*\mathcal{F}$ is an isomorphism for every $a \geq 0$.*

*Proof.* This follows from the proof of (7.9.2) by observing that in (7.9.2) we can choose $g$ and $h$ to be flat. □

**Corollary 7.13.** *If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian, then for any $\mathcal{F} \in D^{[a,b]}_{\text{coh}}(\mathcal{Y})$ the derived pullback $\tau_{\geq a}Lf^*\mathcal{F}$ is in $D^{[a,b]}_{\text{coh}}(\mathcal{X})$.*

*Proof.* This follows from the construction of $\tau_{\geq a}Lf^*\mathcal{F}$ and the corresponding result for schemes ([EGA], III.6.5.13). □

## 8. The cotangent complex

We explain how with the above notion of derived pullback the theory of cotangent complexes for stacks developed in ([LM-B], Chapitre 17) can be carried through.

**Theorem 8.1** ([LM-B], 17.3)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi–compact and quasi–separated morphism of algebraic stacks. Then to $f$ one can associate an object $L_{\mathcal{X}/\mathcal{Y}} \in D'_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}})$ called the cotangent complex of $f$ such that the following hold:*

(i) *If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces, then $L_{\mathcal{X}/\mathcal{Y}}$ is canonically isomorphic to the object $\{\tau_{\geq -n}\pi^* L_{\mathcal{X}/\mathcal{Y},\text{et}}\}_n$, where $\pi : \mathcal{X}_{\text{lis-et}} \to \mathcal{X}_{\text{et}}$ denotes the projection to the étale topos and $L_{\mathcal{X}/\mathcal{Y},\text{et}}$ denotes the usual cotangent complex of the morphism of ringed topoi $\mathcal{X}_{\text{et}} \to \mathcal{Y}_{\text{et}}$.*

(ii) *For any 2–commutative square of algebraic stacks*

(8.1.1)
$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{A} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{B} & \mathcal{Y} \end{array}$$

*there is a natural functoriality morphism*

(8.1.2)
$$LA^* L_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}'/\mathcal{Y}'}.$$

*If the square is cartesian and either $f$ or $B$ is flat then this functoriality morphism is an isomorphism, and the sum map*

(8.1.3)
$$LA^* L_{\mathcal{X}/\mathcal{Y}} \oplus Lf'^* L_{\mathcal{Y}'/\mathcal{Y}} \to L_{\mathcal{X}'/\mathcal{Y}}$$

*is an isomorphism.*

(iii) *If $g : \mathcal{Y} \to \mathcal{Z}$ is another morphism of algebraic stacks, then there is a natural map*

(8.1.4)
$$L_{\mathcal{X}/\mathcal{Y}} \longrightarrow Lf^*(L_{\mathcal{Y}/\mathcal{Z}})[1]$$

*such that*

(8.1.5)
$$Lf^* L_{\mathcal{Y}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}} \to Lf^* L_{\mathcal{Y}/\mathcal{Z}}[1]$$

*is a distinguished triangle in $D'_{\text{qcoh}}(\mathcal{X}_{\text{lis-et}})$.*

The proof is in several steps (8.2)–(8.10).

**8.2.** To define $L_{\mathcal{X}/\mathcal{Y}}$, let $C_{\mathcal{X}/\mathcal{Y}}$ be the category of 2–commutative diagrams

(8.2.1)
$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{Y}, \end{array}$$

where $X$ and $Y$ are algebraic spaces, $Y \to \mathcal{Y}$ is a smooth cover, and the map $X \to Y \times_{\mathcal{Y}} \mathcal{X}$ is smooth and surjective. We usually denote such an object of $C_{\mathcal{X}/\mathcal{Y}}$ simply by $X/Y$. Let $X_\bullet$ (resp. $Y_\bullet$) be the 0–coskeleton of the morphism $X \to \mathcal{X}$ (resp. $Y \to \mathcal{Y}$), and let $\mathcal{X}'_\bullet$ denote the fiber product $Y_\bullet \times_{\mathcal{Y}} \mathcal{X}$. There is then a natural sequence

(8.2.2)
$$X_\bullet \to \mathcal{X}'_\bullet \to Y_\bullet,$$

where each $X_n \to \mathcal{X}'_n$ is smooth. Define $L^{X/Y}_{\mathcal{X}/\mathcal{Y}}$ to be the complex on $X_{\bullet,\mathrm{et}}$ whose restriction to $X_n$ is the complex

(8.2.3) $$L_{X_n/Y_n,\mathrm{et}} \xrightarrow{\partial} \Omega^1_{X_n/\mathcal{X}'_n}.$$

Here the map $\partial$ is obtained from the map of differentials

(8.2.4) $$H_0(L_{X_n/Y_n,\mathrm{et}}) \simeq \Omega^1_{X_n/Y_n} \to \Omega^1_{X_n/\mathcal{X}'_n},$$

and $\Omega^1_{X_n/\mathcal{X}'_n}$ is placed in degree 1.

**Lemma 8.3.** *The complex $L^{X/Y}_{\mathcal{X}/\mathcal{Y}}$ is in $D^{\leq 1}_{\mathrm{qcoh}}(X_{\bullet,\mathrm{et}})$.*

*Proof.* For any $Y_n \to Y_{n-1}$ corresponding to an injective morphism $[n-1] \to [n]$ we have commutative diagrams

(8.3.1) $$\begin{array}{ccccc} X_n & \longrightarrow & Y_n \times_{Y_{n-1}} X_{n-1} & \longrightarrow & X_{n-1} \\ & & \downarrow & & \downarrow \\ & & Y_n & \longrightarrow & Y_{n-1}, \end{array}$$

(8.3.2) $$\begin{array}{ccccc} X_n & \longrightarrow & X_{n-1} \times_{\mathcal{X}'_{n-1}} \mathcal{X}'_n & \longrightarrow & \mathcal{X}'_n \\ & & \downarrow & & \downarrow \\ & & X_{n-1} & \longrightarrow & \mathcal{X}'_{n-1}, \end{array}$$

(8.3.3) $$\begin{array}{ccc} X_{n-1} \times_{\mathcal{X}'_{n-1}} \mathcal{X}'_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & Y_{n-1}, \end{array}$$

where the squares are cartesian.

If $\delta : X_n \to X_{n-1}$ is the morphism induced by $[n-1] \to [n]$, then chasing through these diagrams we see that there is a natural morphism of distinguished triangle on $X_{n,\mathrm{et}}$

(8.3.4) $$\begin{array}{ccccc} \delta^* L_{X_{n-1}/Y_{n-1},\mathrm{et}} & \longrightarrow & L_{X_n/Y_n,\mathrm{et}} & \longrightarrow & \Omega^1_{X_n/X_{n-1}\times_{Y_{n-1}} Y_n} \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \delta^*\Omega^1_{X_{n-1}/\mathcal{X}'_{n-1}} & \longrightarrow & \Omega^1_{X_n/\mathcal{X}'_n} & \longrightarrow & \Omega^1_{X_n/X_{n-1}\times_{Y_{n-1}} Y_n}. \end{array}$$

This and ([Ill], II.2.3.7) implies the lemma. □

We therefore obtain an object $L^{X/Y,\prime}_{\mathcal{X}/\mathcal{Y}} \in D'_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$ from the system $\{\mathrm{res}(\tau_{\geq -n} L^{X/Y}_{\mathcal{X}/Y})\} \in D'_{\mathrm{qcoh}}(X^+_{\bullet,\mathrm{et}})$ and the equivalences in (4.7). Moreover, for any commutative square

(8.3.5) $$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ Y' & \longrightarrow & Y & \longrightarrow & \mathcal{Y}, \end{array}$$

defining a morphism in $C_{\mathcal{X}/\mathcal{Y}}$ there is a natural morphism

$$L^{X/Y,\prime}_{\mathcal{X}/\mathcal{Y}} \to L^{X'/Y',\prime}_{\mathcal{X}/\mathcal{Y}}. \tag{8.3.6}$$

We therefore obtain a functor

$$L^{-,\prime}_{\mathcal{X}/\mathcal{Y}} : C^{op}_{\mathcal{X}/\mathcal{Y}} \longrightarrow D'_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}}). \tag{8.3.7}$$

**Lemma 8.4.** *The map* (8.3.6) *is an isomorphism and is independent of the choice of the morphism $X'/Y' \to X/Y$ in $C_{\mathcal{X}/\mathcal{Y}}$.*

*Proof.* Let

$$\begin{array}{ccc} X' & \xrightarrow{A} & X \\ \downarrow & & \downarrow f \\ Y' & \xrightarrow{B} & Y \end{array} \tag{8.4.1}$$

be the commutative square defining the morphism $(X'/Y') \to (X/Y)$ in $C_{\mathcal{X}/\mathcal{Y}}$. It then suffices to consider the following two cases: (i) When $B$ is the identity; (ii) The square (8.4.1) is cartesian.

For case (i), note that there is a natural morphism

$$\begin{array}{ccc} A^*L_{X_\bullet/Y_\bullet,\mathrm{et}} & \longrightarrow & L_{X'_\bullet/Y_\bullet,\mathrm{et}} \\ \downarrow & & \downarrow \\ A^*\Omega^1_{X_\bullet/\mathcal{X}'_\bullet} & \longrightarrow & \Omega^1_{X'_\bullet/\mathcal{X}'_\bullet} \end{array} \tag{8.4.2}$$

which we claim is an equivalence. To verify this it suffices to verify that it becomes an equivalence after base change by a smooth surjective morphism $U \to \mathcal{X}$. If we denote by $X_{U\bullet}$ and $X_{U\bullet'}$ the base changes of $X'_\bullet$ and $X_\bullet$ to $U$, then we have a natural morphism of distinguished triangles

$$\begin{array}{ccccc} L_{X_\bullet/Y_\bullet,\mathrm{et}}|_{X_{U\bullet}} & \longrightarrow & L_{X_{U_\bullet}/Y_\bullet,\mathrm{et}} & \longrightarrow & \Omega^1_{U/\mathcal{X}}|_{X_{U\bullet}} \\ \downarrow & & \downarrow & & \\ \Omega^1_{X_\bullet/\mathcal{X}'_\bullet}|_{X_{U\bullet}} & \xrightarrow{\simeq} & \Omega^1_{X_{U_\bullet}/U\times_{\mathcal{Y}}Y_\bullet}, & & \end{array} \tag{8.4.3}$$

and a similarly a morphism of triangles on $X'_{U\bullet}$

$$\begin{array}{ccccc} L_{X'_\bullet/Y_\bullet,\mathrm{et}}|_{X'_{U\bullet}} & \longrightarrow & L_{X'_{U_\bullet}/Y_\bullet,\mathrm{et}} & \longrightarrow & \Omega^1_{U/\mathcal{X}}|_{X'_{U\bullet}} \\ \downarrow & & \downarrow & & \\ \Omega^1_{X'_\bullet/\mathcal{X}'_\bullet}|_{X'_{U\bullet}} & \xrightarrow{\simeq} & \Omega^1_{X'_{U_\bullet}/U\times_{\mathcal{Y}}Y_\bullet}. & & \end{array} \tag{8.4.4}$$

This reduces the problem to the case when $\mathcal{X} = U$ in which case the result follows from consideration of the morphism of distinguished triangles

$$\begin{array}{ccccc} A^*L_{X_\bullet/Y_\bullet,\mathrm{et}} & \longrightarrow & L_{X'_\bullet/Y_\bullet,et} & \longrightarrow & L_{X'_\bullet/X_\bullet,\mathrm{et}} \\ \downarrow & & \downarrow & & \mathrm{id}\downarrow \\ A^*\Omega^1_{X_\bullet/\mathcal{X}'_\bullet} & \longrightarrow & \Omega^1_{X'_\bullet/\mathcal{X}'_\bullet} & \longrightarrow & L_{X'_\bullet/X_\bullet,\mathrm{et}}. \end{array} \tag{8.4.5}$$

For case (ii) it suffices to show that the natural map $A^*L_{X_\bullet/Y_\bullet,\mathrm{et}} \to L_{X'_\bullet/Y'_\bullet,\mathrm{et}}$ is a quasi–isomorphism. This can be verified locally in the flat topology on $X_\bullet$. In particular, if $U \to \mathcal{Y}$ is a smooth cover, it suffices to consider the base change to $U$. We may therefore assume that $\mathcal{Y}$ is equal to an algebraic space. Furthermore, if $\widetilde{\mathcal{X}} \to \mathcal{X}$ is a smooth morphism, and $\widetilde{X} \to \widetilde{\mathcal{X}}$ the base change of $X \to \mathcal{X}$, then there is a natural morphism of distinguished triangle

$$\begin{array}{ccccc} \pi^*A^*L_{X_\bullet/Y_\bullet,\mathrm{et}} & \longrightarrow & \widetilde{A}^*L_{\widetilde{X}_\bullet/Y_\bullet,\mathrm{et}} & \longrightarrow & A^*\Omega^1_{\widetilde{X}_\bullet/X_\bullet} \\ \downarrow & & \downarrow & & \downarrow \simeq \\ \pi^*L_{X'_\bullet/Y'_\bullet,\mathrm{et}} & \longrightarrow & L_{\widetilde{X}'_\bullet/Y'_\bullet,\mathrm{et}} & \longrightarrow & \Omega^1_{\widetilde{X}'_\bullet/X'_\bullet}, \end{array} \tag{8.4.6}$$

where $\widetilde{X}_\bullet$ and $\widetilde{X}'_\bullet$ denote the base changes of $X_\bullet$ and $X'_\bullet$ to $\widetilde{\mathcal{X}}$ and $\pi : \widetilde{X}'_\bullet \to X'_\bullet$ denotes the projection. From this it follows that it suffices to consider the case when both $\mathcal{X}$ and $\mathcal{Y}$ are algebraic spaces. Assuming this, let $Z_\bullet$ (resp. $Z'_\bullet$) denote $Y_\bullet \times_{\mathcal{Y}} \mathcal{X}$ (resp. $Y'_\bullet \times_{\mathcal{Y}} \mathcal{X}$). Then there is a natural morphism of distinguished triangles

$$\begin{array}{ccccc} L_{Z_\bullet/Y_\bullet,\mathrm{et}}|_{X'_\bullet} & \longrightarrow & A^*L_{X_\bullet/Y_\bullet,\mathrm{et}} & \longrightarrow & A^*\Omega^1_{X_\bullet/Z_\bullet} \\ \downarrow & & \downarrow & & \downarrow \simeq \\ L_{Z'_\bullet/Y'_\bullet,\mathrm{et}} & \longrightarrow & L_{X'_\bullet/Y'_\bullet,\mathrm{et}} & \longrightarrow & \Omega^1_{X'_\bullet/Z'_\bullet}, \end{array} \tag{8.4.7}$$

which shows that we may further assume that $X_\bullet = Z_\bullet$ and $X'_\bullet = Z'_\bullet$. In this case the result is clear for the morphisms $Y'_\bullet \to \mathcal{Y}$ and $Y_\bullet \to \mathcal{Y}$ are both flat, and hence by ([Ill], II.2.2.3) the complexes $L_{X_\bullet/Y_\bullet,et}$ and $L_{X'_\bullet/Y'_\bullet,\mathrm{et}}$ are both quasi–isomorphic to the pullback of $L_{\mathcal{X}/\mathcal{Y},\mathrm{et}}$.

To prove that the map is independent of the choice of the morphism $X'/Y' \to X/Y$, note that if $f, g : (X'/Y') \to (X/Y)$ are two morphisms in $C_{\mathcal{X}/\mathcal{Y}}$, then there exists a morphism

$$\Gamma : (X'/Y') \longrightarrow (X \times_{\mathcal{X}} X/Y \times_{\mathcal{Y}} Y) \tag{8.4.8}$$

such that $\mathrm{pr}_1 \circ \Gamma = f$ and $\mathrm{pr}_2 \circ \Gamma = g$. Hence to prove that $f$ and $g$ induce the same map it suffices to consider $\mathrm{pr}_1$ and $\mathrm{pr}_2$. In this case we also have the diagonal map

$$\Delta : (X/Y) \longrightarrow (X \times_{\mathcal{X}} X/Y \times_{\mathcal{Y}} Y). \tag{8.4.9}$$

We thus obtain a diagram of isomorphisms in $D'_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$

$$L^{X/Y}_{\mathcal{X}/\mathcal{Y}} \rightrightarrows L^{X\times_{\mathcal{X}}X/Y\times_{\mathcal{Y}}Y}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\Delta^*} L^{X/Y}_{\mathcal{X}/\mathcal{Y}} \tag{8.4.10}$$

such that the two composites $L^{X/Y}_{\mathcal{X}/\mathcal{Y}} \to L^{X/Y}_{\mathcal{X}/\mathcal{Y}}$ are the identity. Since $\Delta^*$ is an isomorphism this implies that the maps induced by $\mathrm{pr}_1$ and $\mathrm{pr}_2$ are equal.

□

**8.5.** The opposite category $C^{\mathrm{op}}_{\mathcal{X}/\mathcal{Y}}$ is not filtering (as asserted in ([LM-B], 17.6)) as equalizers do not exist. It is however connected. Define a new category $\overline{C}_{\mathcal{X}/\mathcal{Y}}$ whose objects are the same as those of $C_{\mathcal{X}/\mathcal{Y}}$ and for which $\mathrm{Hom}(a, b) = \{*\}$ if there exists a morphism $a \to b$ in $C_{\mathcal{X}/\mathcal{Y}}$ and the emptyset otherwise. The category $\overline{C}^{\mathrm{op}}_{\mathcal{X}/\mathcal{Y}}$ is filtering as $C^{\mathrm{op}}_{\mathcal{X}/\mathcal{Y}}$ is connected.

There is a natural functor $C_{\mathcal{X}/\mathcal{Y}} \to \overline{C}_{\mathcal{X}/\mathcal{Y}}$, and (8.4) implies that the functor $L^{-,\prime}_{\mathcal{X}/\mathcal{Y}}$ factors through $\overline{C}_{\mathcal{X}/\mathcal{Y}}$. It follows that

$$L_{\mathcal{X}/\mathcal{Y}} := \varinjlim_{C^{\mathrm{op}}_{\mathcal{X}/\mathcal{Y}}} L^{-,\prime}_{\mathcal{X}/\mathcal{Y}} \tag{8.5.1}$$

exists in $D'_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$ and that for any object $X/Y \in C_{\mathcal{X}/\mathcal{Y}}$ the map

$$L^{X/Y,\prime}_{\mathcal{X}/\mathcal{Y}} \longrightarrow L_{\mathcal{X}/\mathcal{Y}} \tag{8.5.2}$$

is an isomorphism.

**8.6.** Observe that property (8.1 (i)) follows immediately from the construction as we can take $X/Y = \mathcal{X}/\mathcal{Y}$ in the above.

**8.7.** (Functoriality morphism) Consider a 2–commutative square as in (8.1, (ii)), and for any object $X/Y \in C_{\mathcal{X}/\mathcal{Y}}$ let $D_{X/Y}$ denote the category of commutative squares

$$\begin{array}{ccc} X' & \xrightarrow{s} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{t} & Y \end{array} \tag{8.7.1}$$

over the square (8.1.1) with $X'/Y' \in C_{\mathcal{X}'/\mathcal{Y}'}$. The category $D_{X/Y}$ is nonempty and connected. Let $X'\bullet$ (resp. $X_\bullet$, $Y'_\bullet$, $Y_\bullet$) denote the 0–coskeleton of $X' \to \mathcal{X}'$ (resp. $X \to \mathcal{X}$, $Y' \to \mathcal{Y}'$, $Y \to \mathcal{Y}$), and let $Z'_\bullet$ (resp. $Z_\bullet$) denote $Y'_\bullet \times_{\mathcal{Y}'} \mathcal{X}'$ (resp. $Y_\bullet \times_{\mathcal{Y}} \mathcal{X}$). Then there is a commutative diagram

$$\begin{array}{ccccc} X'_\bullet & \longrightarrow & Z'_\bullet & \longrightarrow & Y'_\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X_\bullet & \longrightarrow & Z_\bullet & \longrightarrow & Y_\bullet \end{array} \tag{8.7.2}$$

which induces a morphism

$$(L_{X_\bullet/Y_\bullet,\mathrm{et}} \to \Omega^1_{X_\bullet/Z_\bullet})|_{X'_\bullet} \to (L_{X'_\bullet/Y'_\bullet,\mathrm{et}} \to \Omega^1_{X'_\bullet/Z'_\bullet}). \tag{8.7.3}$$

The following lemma shows that this map is independent of the choices, and hence these maps induce a canonical map $LA^*L_{\mathcal{X}/\mathcal{Y}} \to L_{\mathcal{X}'/\mathcal{Y}'}$.

**Lemma 8.8.** *The induced morphism $LA^*L^{X/Y}_{\mathcal{X}/\mathcal{Y}} \to L^{X'/Y'}_{\mathcal{X}'/\mathcal{Y}'}$ is independent of the choices of the morphisms $s$ and $t$.*

*Proof.* Given a second pair of morphisms $(s', t')$ defining a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{s'} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{t'} & Y \end{array} \tag{8.8.1}$$

over (8.1.1), there exists a commutative square

(8.8.2)
$$\begin{array}{ccc} X' & \xrightarrow{g} & X\times_{\mathcal{X}}X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{h} & Y\times_{\mathcal{Y}}Y \end{array}$$

over (8.1.1) such that $(s,t)$ (resp. $(s',t')$) equals $(\mathrm{pr}_1\circ g, \mathrm{pr}_1\circ h)$ (resp. $(\mathrm{pr}_2\circ g, \mathrm{pr}_2\circ h)$). The result therefore follows from (8.4). □

**8.9.** (Proof of additional statements in (8.1 (ii)))

To verify that the base change morphism (8.1.2) is an isomorphism when the square (8.1.1) is cartesian and either $f$ or $B$ is flat, note first that it follows from the construction of the cotangent complex above that it holds in the case when $\mathcal{Y}'$ is representable by an algebraic space.

This special case implies the general case, for if $Y'\to\mathcal{Y}'$ is a smooth cover with $Y'$ an algebraic space, then we have

(8.9.1)
$$L_{\mathcal{X}/\mathcal{Y}}|_{\mathcal{X}\times_{\mathcal{Y}}Y'}\simeq L_{\mathcal{X}\times_{\mathcal{Y}}Y'/Y'}\simeq L_{\mathcal{X}'\times_{\mathcal{Y}'}Y'/Y'}\simeq L_{\mathcal{X}'/\mathcal{Y}'}|_{\mathcal{X}'\times_{\mathcal{Y}'}Y'}.$$

To prove that the sum map (8.1.3) is an isomorphism note first that by the base change property just shown, it suffices to prove the result after base changing by a smooth morphism $Y\to\mathcal{Y}$. We may therefore assume that $\mathcal{Y}$ is an algebraic space. Note in addition that if $X\to\mathcal{X}$ is a smooth cover, then there is a natural morphism of distinguished triangles

(8.9.2)
$$\begin{array}{ccccc} L_{\mathcal{X}/\mathcal{Y}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'}\oplus L_{\mathcal{Y}'/\mathcal{Y}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'} & \longrightarrow & L_{X/\mathcal{Y}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'}\oplus L_{\mathcal{Y}'/\mathcal{Y}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'} & \longrightarrow & \Omega^1_{X/\mathcal{X}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'} \\ \downarrow & & \downarrow & & \downarrow \\ L_{\mathcal{X}'/\mathcal{Y}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'} & \longrightarrow & L_{X\times_{\mathcal{Y}}\mathcal{Y}'/\mathcal{Y}} & \longrightarrow & \Omega^1_{X/\mathcal{X}}|_{X\times_{\mathcal{Y}}\mathcal{Y}'}. \end{array}$$

This problem reduces the question to the case when $\mathcal{X}$ is an algebraic space, and the symmetric argument shows that we can also assume that $\mathcal{Y}'$ is an algebraic space. It thus suffices to consider the case when $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Y}'$ are all algebraic spaces in which case the result follows from ([Ill], II.2.2.3).

**8.10.** (Distinguished triangle) Consider a diagram

(8.10.1)
$$\mathcal{X}\xrightarrow{f}\mathcal{Y}\xrightarrow{g}\mathcal{Z}$$

of algebraic stacks. We construct a morphism $L_{\mathcal{X}/\mathcal{Y}}\to Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ in $D'_{\mathrm{qcoh}}(\mathcal{X}_{\text{lis-et}})$ such that

(8.10.2)
$$Lf^*L_{\mathcal{Y}/\mathcal{Z}}\to L_{\mathcal{X}/\mathcal{Z}}\to L_{\mathcal{X}/\mathcal{Y}}\to Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$$

is a distinguished triangle. For this choose first a commutative diagram

(8.10.3)
$$\begin{array}{ccc} \mathcal{X} & \longleftarrow & X \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longleftarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{Z} & \longleftarrow & Z, \end{array}$$

where $X/Y \in C_{\mathcal{X}/\mathcal{Y}}$ and $Y/Z \in C_{\mathcal{Y}/\mathcal{Z}}$. Note that this implies that $X/Z \in C_{\mathcal{X}/\mathcal{Z}}$. Denote by $X_\bullet$, $Y_\bullet$, and $Z_\bullet$ the associated simplicial algebraic spaces. Consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{X} & \longleftarrow & \mathcal{X}\times_{\mathcal{Z}} Z_\bullet & \longleftarrow & \mathcal{X}\times_{\mathcal{Y}} Y_\bullet & \longleftarrow & X_\bullet \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{Y} & \longleftarrow & \mathcal{Y}\times_{\mathcal{Z}} Z_\bullet & \longleftarrow & Y_\bullet & & \\ \downarrow & & \downarrow & & & & \\ \mathcal{Z} & \longleftarrow & Z_\bullet. & & & & \end{array} \tag{8.10.4}$$

From this diagram and the construction of ([Ill], II.2.1.1) it follows that there is a natural commutative diagram of simplicial modules (here it is necessary to view the cotangent complex as a simplicial module as in ([Ill], II.1.2.3) rather than applying the normalization functor)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{Y_\bullet/Z_\bullet}|_{X_\bullet} & \longrightarrow & L_{X_\bullet/Z_\bullet} & \longrightarrow & L_{X_\bullet/Y_\bullet} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^1_{Y_\bullet/\mathcal{Y}\times_{\mathcal{Z}}Z_\bullet}|_{X_\bullet} & \longrightarrow & \Omega^1_{X_\bullet/\mathcal{X}\times_{\mathcal{Z}}Z_\bullet} & \longrightarrow & \Omega^1_{X_\bullet/\mathcal{X}\times_{\mathcal{Y}}Y_\bullet} & \longrightarrow & 0. \end{array} \tag{8.10.5}$$

Normalizing, taking total complexes, and using the equivalences in (4.7) we then obtain the desired distinguished triangle (8.10.2). That the resulting map $L_{\mathcal{X}/\mathcal{Y}} \to Lf^*L_{\mathcal{Y}/\mathcal{Z}}[1]$ is independent of the choices follows from an argument using the diagonal as in the proofs of (8.4) and (8.8).

## 9. Constructible sheaves

Let $S$ be a scheme and $\mathcal{X}/S$ an algebraic stack. Throughout this section $\Lambda$ denotes a torsion noetherian ring annihilated by an integer invertible in $S$. For the definitions of locally constant sheaves and constructible sheaves on a scheme see ([SGA4], IX.2.3).

**Lemma 9.1.** *Let $\mathcal{F}$ be a cartesian sheaf of sets on $\mathcal{X}_{\text{lis-et}}$. Then the following are equivalent.*

(i) *For every $U \in \text{Lis-Et}(\mathcal{X})$ the sheaf $\mathcal{F}_U$ is a locally constant (resp. constructible) sheaf of sets on $U_{\text{et}}$.*

(ii) *There exists a presentation $P : X \to \mathcal{X}$ such that the sheaf $\mathcal{F}_X$ on $X_{\text{et}}$ is locally constant (resp. constructible).*

*Moreover, the analogous statement for locally constant (resp. constructible) sheaves of $\Lambda$–modules also holds.*

*Proof.* It is clear that (i) implies (ii).

For (ii) implies (i), note that for $U \in \text{Lis-Et}(\mathcal{X})$ the sheaf $\mathcal{F}_U$ is locally constant (resp. constructible) if and only if its restriction to an étale cover $U' \to U$ is locally constant (resp. constructible). It follows that it suffices to verify (i) for smooth morphisms $U \to \mathcal{X}$ which factor through $X$. Choosing any morphism $\rho : U \to X$ over $\mathcal{X}$ we obtain an isomorphism $\mathcal{F}_U \simeq \rho^*\mathcal{F}_X$, and since $\mathcal{F}_X$ is locally constant (resp. constructible) it follows that $\mathcal{F}_U$ is also locally constant (resp. constructible).

The same argument gives the statement about Λ–modules. □

**Definition 9.2.** A sheaf of sets (resp. Λ–modules) on $\mathcal{X}_{\text{lis-et}}$ is *constructible* if it is cartesian and if the equivalent conditions in (9.1) hold.

We denote the category of constructible sheaves of Λ–modules on $\mathcal{X}_{\text{lis-et}}$ by $Mod_c(\mathcal{X}_{\text{lis-et}}, \Lambda)$.

Let $P : X \to \mathcal{X}$ be a smooth surjection with $X$ a scheme, and let $X_\bullet$ denote the 0–coskeleton of $P$.

**Definition 9.3.** A *constructible sheaf* of Λ–modules on $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) is a cartesian sheaf of Λ–modules $F$ in the sense of (4.3), such that for every $[n] \in \Delta$ the restriction $F_n$ of $F$ to $X_{n,\text{lis-et}}$ (resp. $X_{n,\text{et}}$) is a constructible sheaf.

We denote the category of constructible sheaves of Λ–modules on $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) by $Mod_c(X^+_{\bullet,\text{lis-et}}, \Lambda)$ (resp. $Mod_c(X^+_{\bullet,\text{et}}, \Lambda)$, $Mod_c(X_{\bullet,\text{et}}, \Lambda)$).

**Remark 9.4.** As in (6.10), a cartesian sheaf of Λ–modules $F$ on $X^+_{\bullet,\text{lis-et}}$ (resp. $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) is constructible if and only if the sheaf $F_0$ on $X_0$ is a constructible sheaf.

**Lemma 9.5.** *The natural restriction functors*

$$Mod_c(\mathcal{X}_{\text{lis-et}}, \Lambda) \to Mod_c(X^+_{\bullet,\text{lis-et}}, \Lambda) \to Mod_c(X^+_{\bullet,\text{et}}, \Lambda), \tag{9.5.1}$$

*and*

$$Mod_c(\mathcal{X}_{\text{lis-et}}, \Lambda) \to Mod_c(X_{\bullet,\text{et}}, \Lambda) \to Mod_c(X^+_{\bullet,\text{et}}, \Lambda) \tag{9.5.2}$$

*are all equivalences.*

*Proof.* This follows form (4.4) and (9.4). □

**9.6.** It follows from (4.8) that the category of constructible sheaves of Λ–modules on $\mathcal{X}_{\text{lis-et}}$ (resp. $X^+_{\bullet,\text{lis-et}}$, $X^+_{\bullet,\text{et}}$, $X_{\bullet,\text{et}}$) is closed under the formation of kernels, cokernels, and cokernels. For $* \in \{b, +, -, [a, b]\}$ let $D^*_c(\mathcal{X}_{\text{lis-et}}, \Lambda)$ (resp. $D^*_c(X^+_{\bullet,\text{lis-et}}, \Lambda)$, $D^*_c(X^+_{\bullet,\text{et}}, \Lambda)$, $D^*_c(X_{\bullet,\text{et}}, \Lambda)$) denote the triangulated subcategory of the derived category $D^*(\mathcal{X}_{\text{lis-et}}, \Lambda)$ (resp. $D^*(X^+_{\bullet,\text{lis-et}}, \Lambda)$, $D^*(X^+_{\bullet,\text{et}}, \Lambda)$, $D^*(X_{\bullet,\text{et}}, \Lambda)$) consisting of complexes with constructible cohomology sheaves.

As in (4.6) let

$$\pi : X^+_{\bullet,\text{lis-et}} \to \mathcal{X}_{\text{lis-et}}, \quad \epsilon : X^+_{\bullet,\text{lis-et}} \to X^+_{\bullet,\text{et}} \tag{9.6.1}$$

be the natural morphisms of topoi.

**Theorem 9.7.** *The functors*

$$\pi^* : D^+_c(\mathcal{X}_{\text{lis-et}}, \Lambda) \longrightarrow D^+_c(X^+_{\bullet,\text{lis-et}}, \Lambda) \tag{9.7.1}$$

*and*

$$\epsilon^* : D^+_c(X^+_{\bullet,\text{et}}, \Lambda) \longrightarrow D^+_c(X^+_{\bullet,\text{lis-et}}, \Lambda), \tag{9.7.2}$$

*are both equivalences of triangulated categories with quasi–inverses given by $R\pi_*$ and $R\epsilon_*$ respectively. For any $F \in D^+_c(\mathcal{X}_{\text{lis-et}}, \Lambda)$, the maps*

$$H^*(\mathcal{X}_{\text{lis-et}}, F) \to H^*(X^+_{\bullet,\text{lis-et}}, \pi^*F) \to H^*(X^+_{\bullet,\text{et}}, R\epsilon_*\pi^*F) \tag{9.7.3}$$

*are isomorphisms, and these isomorphisms are functorial with respect to smooth base change $\mathcal{X}' \to \mathcal{X}$.*

*Proof.* This follows from (4.7). □

**Proposition 9.8.** (i) *Let $f : \mathcal{Y} \to \mathcal{X}$ be a representable morphism of algebraic stacks and $F$ a sheaf of $\Lambda$–modules on $\mathcal{Y}_{\text{lis-et}}$. Then for any object $U \in \text{Lis-Et}(\mathcal{X})$ and integer $i \geq 0$ the restriction $(R^if_*F)_U$ of $R^if_*F$ to $U_{\text{et}}$ is canonically isomorphic to $R^if_{U_{\text{et}},*}F_{\mathcal{Y}\times_{\mathcal{X}}U}$, where $f_{U,\text{et}} : (\mathcal{Y}\times_{\mathcal{X}} U)_{\text{et}} \to U_{\text{et}}$ denotes the morphism of topoi induced by base change.*

(ii) *Let $f : \mathcal{Y} \to \mathcal{X}$ be an arbitrary morphism of algebraic stacks, and let $Y \to \mathcal{Y}$ be a smooth surjection with $Y$ a scheme. Let $U \in \text{Lis-Et}(\mathcal{X})$ be an object, let $Y_\bullet$ denote the 0–coskeleton of $Y \to \mathcal{Y}$, and let $f_{U,n} : Y_n \times_{\mathcal{X}} U \to U$ be the morphism induced by $f$. Then for any cartesian sheaf of $\Lambda$–modules $F$ on $\mathcal{Y}_{\text{lis-et}}$ there is a canonical spectral sequence*

$$E_1^{pq} = R^qf_{U,n*}F_{Y_p\times_{\mathcal{X}}U} \implies (R^{p+q}f_*F)_U. \tag{9.8.1}$$

*Proof.* For (i), note that $R^if_*F$ is by ([SGA4], V.5.1) equal to the sheaf associated to the presheaf which to any $U \in \text{Lis-Et}(\mathcal{X})$ associates $H^i(\mathcal{Y}|_{\widetilde{\mathcal{Y}\times_{\mathcal{X}}U}}, F)$ in the notation of (3.5). It therefore suffices to show that there is a natural isomorphism

$$H^i(\mathcal{Y}|_{\widetilde{\mathcal{Y}\times_{\mathcal{X}}U}}, F) \simeq H^i((\mathcal{Y}\times_{\mathcal{X}} U)_{\text{et}}, F_{\mathcal{Y}\times_{\mathcal{X}}U}). \tag{9.8.2}$$

For this note that by (3.5) there is a canonical isomorphism

$$H^i(\mathcal{Y}|_{\widetilde{\mathcal{Y}\times_{\mathcal{X}}U}}, F) \simeq H^i((\mathcal{Y}\times_{\mathcal{X}} U)_{\text{lis-et}}, F). \tag{9.8.3}$$

On the other hand, if $q : (\mathcal{Y}\times_{\mathcal{X}}U)_{\text{lis-et}} \to (\mathcal{Y}\times_{\mathcal{X}}U)_{\text{et}}$ denotes the morphism of topoi induced by the inclusion $\text{Et}(\mathcal{Y}\times_{\mathcal{X}}U) \subset \text{Lis-Et}(\mathcal{Y}\times_{\mathcal{X}}U)$, then $q_*$ is exact so there are natural isomorphisms

$$H^i((\mathcal{Y}\times_{\mathcal{X}} U)_{\text{lis-et}}, F) \simeq H^i((\mathcal{Y}\times_{\mathcal{X}} U)_{\text{et}}, q_*F) = H^i((\mathcal{Y}\times_{\mathcal{X}} U)_{\text{et}}, F_{\mathcal{Y}\times_{\mathcal{X}}U}). \tag{9.8.4}$$

For (ii), note that as in the proof of (i) the sheaf $R^if_*F$ is isomorphic to the sheaf associated to the presheaf which to any $U \in \text{Lis-Et}(\mathcal{X})$ associates $H^i(\mathcal{Y}|_{\widetilde{\mathcal{Y}\times_{\mathcal{X}}U}}, F)$. By (4.7), this sheaf is isomorphic to the sheaf associated to the presheaf which to any $U \in \text{Lis-Et}(\mathcal{X})$ associates $H^i((Y_\bullet^+ \times_{\mathcal{X}} U)_{\text{et}}, F_{Y_\bullet^+\times_{\mathcal{X}}U})$. The spectral sequence (9.8.1) is then induced by the spectral sequences (2.7)

$$E_1^{pq} = H^q(Y_p \times_{\mathcal{X}} U, F_{Y_p\times_{\mathcal{X}}U}) \implies H^{p+q}((Y_\bullet^+ \times_{\mathcal{X}} U)_{\text{et}}, F_{Y_\bullet^+\times_{\mathcal{X}}U}). \tag{9.8.5}$$

□

**Proposition 9.9.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi–compact morphism of algebraic stacks. Then for every integer $i \geq 0$ and cartesian sheaf of $\Lambda$–modules $F$ in $\mathcal{Y}_{\text{lis-et}}$, the sheaf $R^if_*F$ is a cartesian sheaf in $\mathcal{X}_{\text{lis-et}}$.*

*Proof.* By (3.8) and (3.5), it suffices to verify the result after replacing $\mathcal{X}$ by a smooth cover. We may therefore assume that $\mathcal{X}$ is a quasi–compact scheme. Choose a smooth surjection $Y \to \mathcal{Y}$ with $Y$ a quasi–compact scheme, and let $Y_\bullet \to \mathcal{Y}$ be the 0–coskeleton. Consideration of the spectral sequence (9.8.1) then shows that it suffices to prove the result for each of the morphisms $Y_n \to \mathcal{X}$. This reduces the proof to the case when $\mathcal{X}$ is a scheme and $\mathcal{Y}$ is an algebraic space. Repeating this reduction argument with an étale cover of $\mathcal{Y}$ further reduces the proof to the case when $\mathcal{Y}$ is also a scheme. In this case the result follows from ([SGA4], XVI.1.2). □

**Theorem 9.10.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic stacks of finite type over $S$, and let $F$ be a constructible sheaf of $\Lambda$-modules on $\mathcal{Y}$. Then for every $i \geq 0$ there exists a dense open subscheme $S^o \subset S$ such that the following hold:*

(i) *The restriction of $R^i f_* F$ to $S^o \times_S \mathcal{X}$ is constructible.*

(ii) *The formation of $R^i f_* F$ is compatible with arbitrary base change $S' \to S^o$.*

*Proof.* Choose a smooth surjection $X \to \mathcal{X}$ with $X$ a quasi–compact scheme. Then as in the proof of (9.9) it suffices to prove the Theorem after replacing $f$ by $\mathcal{Y} \times_{\mathcal{X}} X \to X$. We may therefore assume that $\mathcal{X}$ is a quasi–compact scheme. Also choose a smooth surjection $Y \to \mathcal{Y}$ with $Y$ a quasi–compact scheme, and let $Y_\bullet$ be the 0–coskeleton. Consideration of the spectral sequence (9.8.1) then shows that it suffices to prove the result for each of the morphisms $Y_n \to \mathcal{X}$. Repeating the argument with an étale cover of $Y_n$ we are then reduced to the case when $\mathcal{Y}$ and $\mathcal{X}$ are both schemes. In this case the result is ([SGA4.5], [Th. Finitude], 1.9). □

For an algebraic stack $\mathcal{X}/S$ we can also consider the big étale topos $\mathcal{X}_{\rm Et}$ defined in (10.1).

**Definition 9.11.** A sheaf of $\Lambda$–modules $F \in \mathcal{X}_{\rm Et}$ is constructible if for every $\mathcal{X}$–scheme $U$ the restriction $F_U$ of $F$ to $U_{\rm et}$ is a constructible sheaf of $\Lambda$–modules, and if for every morphism $\rho : V \to U$ over $\mathcal{X}$ the natural map $\rho^* F_U \to F_V$ is an isomorphism.

We denote the category of constructible $\Lambda$–modules in $\mathcal{X}_{\rm Et}$ by $Mod_c(\mathcal{X}_{\rm Et}, \Lambda)$.

**Lemma 9.12.** *The restriction functor*

(9.12.1) $$\mathrm{res} : Mod_c(\mathcal{X}_{\rm Et}, \Lambda) \to Mod_c(\mathcal{X}_{\text{lis-et}}, \Lambda)$$

*is an equivalence of categories.*

*Proof.* The same argument used in the proof of (10.5) reduces the proof to the case when $\mathcal{X}$ is a scheme. In this case the result follows from observing that both categories are equivalent to the category of constructible sheaves on the small étale site of $\mathcal{X}$. □

**Proposition 9.13.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of algebraic $S$–stacks, and let $F$ be a constructible sheaf of $\Lambda$–modules in $\mathcal{Y}_{\rm Et}$. Then for every $i \geq 0$ there is a natural isomorphism*

(9.13.1) $$\mathrm{res}_{\mathcal{X}}(R^i f^{\rm Et}_* F) \to R^i f_*(\mathrm{res}_{\mathcal{Y}}(F)),$$

*where $f^{\rm Et} : \mathcal{Y}_{\rm Et} \to \mathcal{X}_{\rm Et}$ denotes the morphism of topoi induced by $f$ and $\mathrm{res}_{\mathcal{X}} : \mathcal{X}_{\rm Et} \to \mathcal{X}_{\text{lis-et}}$ and $\mathrm{res}_{\mathcal{Y}} : \mathcal{Y}_{\rm Et} \to \mathcal{Y}_{\text{lis-et}}$ are the restriction functors.*

*Proof.* The arrow (9.13.1) is defined as in (10.6) by observing that there is a natural isomorphism of functors

(9.13.2) $$\mathrm{res}_{\mathcal{X}} \circ f^{\rm Et}_* \simeq f_* \circ \mathrm{res}_{\mathcal{Y}},$$

and that $\mathrm{res}_{\mathcal{X}}(R^i f^{\rm Et}_*(-))$ is a universal cohomological $\delta$–functor.

The same reduction argument used in the proof of (10.7) shows that to prove the Proposition it suffices to prove that if $X$ is an $S$–scheme and $F$ is a constructible sheaf on $X_{\rm Et}$, then the natural map

(9.13.3) $$H^*(X_{\rm Et}, F) \to H^*(X_{\rm et}, F|_{X_{\rm et}})$$

is an isomorphism. For this let $\pi : X_{\mathrm{Et}} \to X_{\mathrm{et}}$ be the morphism of topoi induced by the inclusion of the small étale site of $X$ into the big étale site. The functor $\pi_*$ is exact, so we have

$$H^*(X_{\mathrm{Et}}, F) \simeq H^*(X_{\mathrm{et}}, \pi_* F) = H^*(X_{\mathrm{et}}, F|_{X_{\mathrm{et}}}). \tag{9.13.4}$$

□

**Theorem 9.14.** *Let $f : \mathcal{Y} \to \mathcal{X}$ be a proper morphism of algebraic $S$–stacks, and let $F$ be a constructible sheaf of $\Lambda$–modules on $\mathcal{Y}_{\text{lis-et}}$. Then for every $i \geq 0$ the sheaf $R^i f_* F$ is a constructible sheaf on $\mathcal{X}_{\text{lis-et}}$ whose formation is compatible with arbitrary base change $\mathcal{X}' \to \mathcal{X}$.*

*Proof.* We abusively write also $F$ for the constructible sheaf on the big étale site $\mathcal{Y}_{\mathrm{Et}}$ obtained from $F$ using the equivalence (9.12). For a morphism $W \to \mathcal{X}$, let $f_W : \mathcal{Y}_W := \mathcal{Y} \times_{\mathcal{X}} W \to W$ be the morphism obtained by base change.

First we claim that it suffices to prove the Theorem after replacing $\mathcal{X}$ by a smooth cover $X \to \mathcal{X}$ by a scheme.

For this note that by (9.9) the sheaf $R^i f_* F$ is a cartesian sheaf on $\mathcal{X}_{\text{lis-et}}$. Hence to verify that $R^i f_* F$ is constructible it suffices to show that $R^i f_{X*} F|_{\mathcal{Y}_X}$ is constructible. Furthermore, since any morphism $g : W \to \mathcal{X}$ from a scheme $W$ étale locally factors through $X$, to prove that the natural map

$$g^* R^i f_* F \to R^i f_{W*}(F|_{\mathcal{Y}_W}) \tag{9.14.1}$$

is an isomorphism, it suffices to consider morphisms $g : W \to \mathcal{X}$ which admit a factorization $\tilde{g} : W \to X$. But in this case the arrow (9.14.1) is identified with the morphism

$$\tilde{g}^* R^i f_{X*}(F|_{\mathcal{Y}_X}) \to R^i f_{W*}(F|_{\mathcal{Y}_W}) \tag{9.14.2}$$

which is an isomorphism if the Theorem holds for $\mathcal{Y}_X \to X$.

We may therefore assume that $\mathcal{X}$ is an affine scheme in which case there exists by ([Ol], 1.1) a proper surjection $Y \to \mathcal{Y}$ with $Y$ a scheme proper over $X$. Let $Y_\bullet$ be the 0–coskeleton of this surjection, and let $f_n : Y_n \to \mathcal{X}$ be the map induced by $f$. Let $q : Y_{\bullet,\mathrm{Et}} \to \mathcal{Y}_{\mathrm{Et}}$ be the morphism of big étale topoi induced by the map $Y_\bullet \to \mathcal{Y}$.

**Lemma 9.15.** *For any constructible sheaf $\mathcal{G}$ on $\mathcal{Y}_{\mathrm{Et}}$, the adjunction map $\mathcal{G} \to Rq_* q^* \mathcal{G}$ is an isomorphism.*

*Proof.* It suffices to show that for every morphism $U \to \mathcal{Y}$ with $U$ a scheme and integer $i$ the natural map

$$H^i(U_{\mathrm{Et}}, \mathcal{G}) \to H^i((Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{Et}}, q^* G) \tag{9.15.1}$$

is an isomorphism. Let $\epsilon : U_{\mathrm{Et}} \to U_{\mathrm{et}}$ and $\epsilon_\bullet : (Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{Et}} \to (Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}}$ be the projections to the small étale sites, and let $\bar{q} : (Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}} \to U_{\mathrm{et}}$ be the natural morphism of topoi. There is then a commutative diagram of topoi

$$\begin{array}{ccc} (Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{Et}} & \xrightarrow{\epsilon_\bullet} & (Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \bar{q}} \\ U_{\mathrm{Et}} & \xrightarrow{\epsilon} & U_{\mathrm{et}}. \end{array} \tag{9.15.2}$$

Since the functors $\epsilon_*$ and $\epsilon_{\bullet *}$ are exact, there are isomorphisms

$$(9.15.3)\quad H^i(U_{\mathrm{Et}}, \mathcal{G}) \simeq H^i(U_{\mathrm{et}}, \mathcal{G}|_{U_{\mathrm{et}}}),\ \ H^i((Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{Et}}, q^*G) \simeq H^i((Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}}, q^*G|_{(Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}}}).$$

Since $G$ is cartesian, $q^*G|_{(Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}}} \simeq \overline{q}^*(G|_{U_{\mathrm{et}}})$ and via these isomorphisms the arrow (9.15.1) is identified with the natural map

$$(9.15.4)\qquad H^i(U_{\mathrm{et}}, \mathcal{G}|_{U_{\mathrm{et}}}) \to H^i((Y_\bullet \times_{\mathcal{Y}} U)_{\mathrm{et}}, \overline{q}^*\mathcal{G}|_{U_{\mathrm{et}}})$$

which is an isomorphism by ([SGA4], $\mathrm{V}^{\mathrm{bis}}$.4.3.2). □

From this it follows that $Rf_*^{\mathrm{Et}}F \in D^+(\mathcal{X}_{\mathrm{Et}}, \Lambda)$ is isomorphic to $R(f^{\mathrm{Et}} \circ q)_*F$. By (2.9), there is therefore a spectral sequence

$$(9.15.5)\qquad E_1^{pq} = R^q f_{n*}^{\mathrm{Et}}(F|_{Y_{n,\mathrm{Et}}}) \implies R^{p+q} f_*^{\mathrm{Et}}(F).$$

Restricting this spectral sequence to $\mathcal{X}_{\text{lis-et}}$ and using (9.13), we obtain a spectral sequence

$$(9.15.6)\qquad E_1^{pq} = R^q f_{n*}(F|_{Y_{n,\text{lis-et}}}) \implies R^{p+q} f_*(F).$$

Furthermore, by construction this spectral sequence is functorial with respect to base change $\mathcal{X}' \to \mathcal{X}$.

From this it follows that it suffices to prove the Theorem for each of the morphisms $f_n : Y_n \to \mathcal{X}$. This reduces the proof to the case when $\mathcal{X}$ is an affine scheme and $\mathcal{Y}$ is an algebraic space. Repeating the above argument with a proper surjection to $\mathcal{Y}$ from a scheme further reduces the proof to the case when both $\mathcal{X}$ and $\mathcal{Y}$ are schemes. In this case the result follows from ([SGA4], XII.5.1 and XIV.1.1) and (9.8 (i)). □

**9.16.** Let $S$ be a scheme and $\mathcal{X}$ an algebraic $S$–stack. For $* \in \{b, +, [a, b]\}$ let $D^*_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$ be the full subcategory of the triangulated category of $\Lambda$–modules in $\mathcal{X}_{\text{lis-et}}$ consisting of complexes whose cohomology sheaves are cartesian in the sense of (3.7). Let $p : X \to \mathcal{X}$ be a smooth surjection with $X$ a scheme, and let $X_\bullet$ be the 0–coskeleton of $p$. Let $D^*_{\mathrm{cart}}(X^+_{\bullet,\mathrm{et}})$ denote the subcategory of the derived category of $\Lambda$–modules on $X^+_{\bullet,\mathrm{et}}$ whose cohomology sheaves are cartesian in the sense of (4.3). Then by (4.7) restriction induces an equivalence of triangulated categories

$$(9.16.1)\qquad D^*_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \simeq D^*_{\mathrm{cart}}(X^+_{\bullet,\mathrm{et}}, \Lambda).$$

Let $f : \mathcal{Y} \to \mathcal{X}$ be a quasi–compact morphism of algebraic $S$-stacks. The equivalence (9.16.1) enables us to define a functor

$$(9.16.2)\qquad f^{-1} : D^*_{\mathrm{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \to D^*_{\mathrm{cart}}(\mathcal{Y}_{\text{lis-et}}, \Lambda)$$

for $* \in \{b, +, [a, b]\}$ as follows. Choose a commutative diagram

$$(9.16.3)\qquad \begin{array}{ccc} Y & \xrightarrow{\tilde{f}} & X \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle P} \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X}, \end{array}$$

where $X$ and $Y$ are schemes, $P$ and $Q$ are smooth and surjective, and $\tilde{f}$ is quasi–compact, and let $\tilde{f}_\bullet : Y_\bullet \to X_\bullet$ be the morphism of simplicial spaces obtained by applying the 0–coskeleton functor. Define $f^{-1}$ to be the composite

(9.16.4)

$$D^*_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \xrightarrow{(9.16.1)} D^*_{\text{cart}}(X^+_{\bullet,\text{et}}, \Lambda) \xrightarrow{\tilde{f}^{-1}_{\bullet}} D^*_{\text{cart}}(Y^+_{\bullet,\text{et}}, \Lambda) \xrightarrow{(9.16.1)} D^*_{\text{cart}}(\mathcal{Y}_{\text{lis-et}}, \Lambda),$$

where $\tilde{f}^{-1}_{\bullet}$ denotes the usual pullback functor obtained from the morphism of topoi $\tilde{f}_{\bullet} : Y^+_{\bullet,\text{et}} \to X^+_{\bullet,\text{et}}$.

**Lemma 9.17.** *The functor $f^{-1} : D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda) \to D^+_{\text{cart}}(\mathcal{Y}_{\text{lis-et}}, \Lambda)$ is left adjoint to the functor $Rf_*$ (note that by* (9.9) *this functors takes values in $D^+_{\text{cart}}(\mathcal{X}_{\text{lis-et}}, \Lambda)$). In particular the above definition of $f^{-1}$ is independent of the choice of the diagram* (9.16.3).

*Proof.* This follows from the construction and the fact that for a diagram (9.16.3), the functor

(9.17.1) $$\tilde{f}^{-1} : D^+_{\text{cart}}(X^+_{\bullet,\text{et}}, \Lambda) \to D^+_{\text{cart}}(Y^+_{\bullet,\text{et}}, \Lambda)$$

is left adjoint to the functor $R\tilde{f}_*(-)$. □

**Remark 9.18.** With the above results in hand, the remaining parts of ([LM-B], Chapter 18) apply essentially verbatim. In particular, the discussion of the functor $RF_!$ and Bernstein–Lunts stacks in (loc. cit.) can now be carried through as written.

## 10. Comparison with big étale and flat topologies, and finiteness of coherent cohomology

**10.1.** Let $\mathcal{X}$ be an algebraic stack, and define ringed topoi $(\mathcal{X}_{\text{fpqc}}, \mathcal{O}_{\mathcal{X}_{\text{fpqc}}})$ and $(\mathcal{X}_{\text{Et}}, \mathcal{O}_{\mathcal{X}_{\text{Et}}})$ as follows. The ringed topos $(\mathcal{X}_{\text{fpqc}}, \mathcal{O}_{\mathcal{X}_{\text{fpqc}}})$ is obtained from the site whose objects are all quasi–compact and quasi–separated $\mathcal{X}$–schemes and whose topology is generated by finite families $\{U_i \to U\}$ of flat morphisms such that the induced map $\coprod_i U_i \to U$ is surjective (note that since we restrict to quasi–compact $\mathcal{X}$–schemes the morphisms $U_i \to U$ are automatically quasi–compact). The ringed topos $(\mathcal{X}_{\text{Et}}, \mathcal{O}_{\mathcal{X}_{\text{Et}}})$ is obtained from the site whose objects are all quasi–compact $\mathcal{X}$–schemes and whose topology is generated by finite families $\{U_i \to U\}$ of étale morphisms such that the induced map $\coprod_i U_i \to U$ is surjective.

**Definition 10.2.** (i) A sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}_{\text{fpqc}}}$–modules (resp. $\mathcal{O}_{\mathcal{X}_{\text{Et}}}$–modules) on $\mathcal{X}_{\text{fpqc}}$ (resp. $\mathcal{X}_{\text{Et}}$) is *quasi–coherent* if for every quasi–compact $\mathcal{X}$–scheme $U$ the restriction $\mathcal{M}_U$ of $\mathcal{M}$ to $U_{\text{et}}$ is a quasi–coherent and if for every $\mathcal{X}$–morphism $f : U' \to U$ the natural map $f^*\mathcal{M}_U \to \mathcal{M}_{U'}$ is an isomorphism.

(ii) If $\mathcal{X}$ is locally noetherian, a sheaf $\mathcal{M}$ of $\mathcal{O}_{\mathcal{X}_{\text{fpqc}}}$–modules (resp. $\mathcal{O}_{\mathcal{X}_{\text{Et}}}$–modules) on $\mathcal{X}_{\text{fpqc}}$ (resp. $\mathcal{X}_{\text{Et}}$) is called *coherent* if it is quasi–coherent and if for every locally noetherian $\mathcal{X}$–scheme $U$ the restriction $\mathcal{M}_U$ is a coherent sheaf on $U_{\text{et}}$.

We denote by $Qcoh^{\text{fpqc}}(\mathcal{X})$ (resp. $Coh^{\text{fpqc}}(\mathcal{X})$) the category of quasi–coherent (resp. coherent) sheaves of $\mathcal{O}_{\mathcal{X}_{\text{fpqc}}}$–modules, and by $Qcoh^{\text{Et}}(\mathcal{X})$ (resp. $Coh^{\text{Et}}(\mathcal{X})$) the category of quasi–coherent (resp. coherent) sheaves of $\mathcal{O}_{\mathcal{X}_{\text{Et}}}$–modules.

**Remark 10.3.** Any sheaf $\mathcal{M} \in Qcoh^{\text{Et}}(\mathcal{X})$ is by descent theory for quasi–coherent sheaves automatically a sheaf for the fpqc topology. Hence there are natural equivalences of categories

(10.3.1) $$Qcoh^{\text{fpqc}}(\mathcal{X}) \simeq Qcoh^{\text{Et}}(\mathcal{X}), \quad Coh^{\text{fpqc}}(\mathcal{X}) \simeq Coh^{\text{Et}}(\mathcal{X}).$$

**10.4.** The functor which restricts a sheaf on $\mathcal{X}_{\rm Et}$ to $\mathcal{X}_{\text{lis-et}}$ induces a functor

$$r_{\rm qcoh} : Qcoh^{\rm Et}(\mathcal{X}) \to Qcoh(\mathcal{X}), \tag{10.4.1}$$

and in the locally noetherian case also a functor

$$r_{\rm coh} : Coh^{\rm Et}(\mathcal{X}) \to Coh(\mathcal{X}). \tag{10.4.2}$$

**Lemma 10.5.** *The functor $r_{\rm qcoh}$ is an equivalence of categories and in the locally noetherian case the functor $r_{\rm coh}$ is also an equivalence of categories.*

*Proof.* The statement that $r_{\rm coh}$ is an equivalence follows from the statement that $r_{\rm qcoh}$ is an equivalence and the observation that a sheaf $\mathcal{M} \in Qcoh^{\rm Et}(\mathcal{X})$ is coherent if and only if the restriction $r_{\rm qcoh}(\mathcal{M})$ is coherent.

Thus to prove the Lemma it suffices to prove that $r_{\rm qcoh}$ is an equivalence. Fix a smooth cover $X \to \mathcal{X}$, let $X'$ denote $X \times_{\mathcal{X}} X$, and write $p_1, p_2 : X' \to X$ for the two projections. For any quasi–compact $\mathcal{X}$–scheme $T$, there exists étale locally on $T$ a factorization $s : T \to X$ of $T \to \mathcal{X}$ through $X$, and for any two such factorizations $s_1$ and $s_2$ there exists a map $\rho : T \to X'$ such that $s_i = p_i \circ \rho$. From this we conclude that the category $Qcoh^{\rm Et}(\mathcal{X})$ is equivalent to the category $Des^{\rm Et}(X/\mathcal{X})$ of pairs $(\mathcal{M}, \iota)$, where $\mathcal{M}$ is in $Qcoh^{\rm Et}(X)$ and $\iota : p_1^*\mathcal{M} \to p_2^*\mathcal{M}$ is an isomorphism satisfying the usual cocycle condition (as in the proof of (4.4)) on $X \times_{\mathcal{X}} X \times_{\mathcal{X}} X$. If $Des(X/\mathcal{X})$ denotes the category defined in the proof of (4.4), then restriction also defines a functor

$$r_{Des} : Des^{\rm Et}(X/\mathcal{X}) \longrightarrow Des(X/\mathcal{X}) \tag{10.5.1}$$

such that the following diagram commutes

$$\begin{array}{ccc} Des^{\rm Et}(X/\mathcal{X}) & \xrightarrow{r_{Des}} & Des(X/\mathcal{X}) \\ \simeq\downarrow & & \downarrow\simeq \\ Qcoh^{\rm Et}(X) & \xrightarrow{r_{\rm qcoh}} & Qcoh(X). \end{array} \tag{10.5.2}$$

Thus it suffices to prove that $r_{Des}$ is an equivalence which reduces the proof of the Lemma to the case when $\mathcal{X}$ is an algebraic space. Applying the same argument to an étale cover of $\mathcal{X}$ then further reduces the proof to the case when $\mathcal{X}$ is a scheme. In this case the result is immediate. $\square$

**10.6.** If $f : \mathcal{X} \to \mathcal{Y}$ is a quasi–compact morphism of algebraic stacks, then there is a natural isomorphism of functors

$$\text{res}_{\mathcal{Y}} \circ f_*^{\rm Et} \to f_* \circ \text{res}_{\mathcal{X}}, \tag{10.6.1}$$

where $f^{\rm Et} : \mathcal{X}_{\rm Et} \to \mathcal{Y}_{\rm Et}$ denotes the morphism of topoi induced by $f$ and $\text{res}_{\mathcal{X}}$ (resp. $\text{res}_{\mathcal{Y}}$) denotes the functor which restrict a big étale sheaf to the lisse-étale site. Since the functor $\text{res}_{\mathcal{Y}}$ is exact, the cohomological $\delta$–functor $\{\text{res}_{\mathcal{Y}} R^i f_*^{\rm Et}(-)\}$ is isomorphic to the cohomological $\delta$–functor $\{R^i(\text{res}_{\mathcal{Y}} \circ f_*^{\rm Et})(-)\}$. In particular, the cohomological $\delta$–functor $\{\text{res}_{\mathcal{Y}} R^i f_*^{\rm Et}(-)\}$ is universal. Since $\{R^i f_*(\text{res}_{\mathcal{X}}(-))\}$ also forms a cohomological $\delta$–functor, it follows that there exists a unique morphism of cohomological $\delta$-functors

$$\{\text{res}_{\mathcal{Y}} R^i f_*^{\rm Et}(-)\} \to \{R^i f_*(\text{res}_{\mathcal{X}}(-))\} \tag{10.6.2}$$

which agrees with the map (10.6.1) for $i = 0$.

**Theorem 10.7.** *For any quasi–coherent sheaf $\mathcal{M}$ on $\mathcal{X}_{\mathrm{Et}}$ and $i \geq 0$, the map*

$$\mathrm{res}_{\mathcal{Y}} R^i f_*^{\mathrm{Et}}(\mathcal{M}) \to R^i f_*(\mathrm{res}_{\mathcal{X}}(\mathcal{M})) \tag{10.7.1}$$

*induced by* (10.6.2) *is an isomorphism.*

*Proof.* It suffices to prove that for every smooth $\mathcal{Y}$–scheme $U$ the map

$$H^i((\mathcal{X} \times_{\mathcal{Y}} U)_{\mathrm{Et}}, \mathcal{M}) \to H^i((\mathcal{X} \times_{\mathcal{Y}} U)_{\text{lis-et}}, \mathcal{M}|_{(\mathcal{X} \times_{\mathcal{Y}} U)_{\text{lis-et}}}) \tag{10.7.2}$$

defined as in (10.6.1) is an isomorphism. Thus by replacing $\mathcal{X}$ by $\mathcal{X} \times_{\mathcal{Y}} U$ it suffices to show that for every $\mathcal{M} \in Qcoh^{\mathrm{Et}}(\mathcal{X})$ the natural map

$$H^i(\mathcal{X}_{\mathrm{Et}}, \mathcal{M}) \to H^i(\mathcal{X}_{\text{lis-et}}, r_{\mathrm{qcoh}}(\mathcal{M})) \tag{10.7.3}$$

is an isomorphism. Let $P : X \to \mathcal{X}$ be a smooth presentation with $X$ a scheme, and let $P_\bullet : X_\bullet \to \mathcal{X}$ be the 0–coskeleton of $P$. We can then furthermore compose the restriction functor $r_{\mathrm{qcoh}}$ with the restriction functor from sheaves on $\mathcal{X}_{\text{lis-et}}$ to sheaves on $X_{\bullet,\mathrm{et}}$. We then obtain a commutative diagram

$$\begin{array}{ccc} H^i(\mathcal{X}_{\mathrm{Et}}, \mathcal{M}) & \xrightarrow{\simeq} & H^i(X_{\bullet,\mathrm{Et}}, \mathcal{M}|_{X_{\bullet,\mathrm{Et}}}) \\ \downarrow & & \downarrow \beta \\ H^i(\mathcal{X}_{\text{lis-et}}, r_{\mathrm{qcoh}}(\mathcal{M})) & \xrightarrow{\simeq} & H^i(X_{\bullet,\mathrm{et}}, \mathcal{M}|_{X_{\bullet,\mathrm{et}}}) \end{array} \tag{10.7.4}$$

where the top horizontal map is an isomorphism since $P : X \to \mathcal{X}$ defines a covering of the final object in $\mathcal{X}_{\mathrm{Et}}$ ([SGA4], V.2.3.4) and the bottom horizontal map is an isomorphism by (2.3) and (4.7). Thus to prove the Theorem it suffices to prove that the arrow labelled $\beta$ is an isomorphism. It follows from the construction of the map $\beta$ and the argument of (2.9) that $\beta$ extends to a map of spectral sequences

$$\begin{array}{ccc} E_1^{st} = H^t(X_{s,\mathrm{Et}}, \mathcal{M}|_{X_s}) & \Longrightarrow & H^{s+t}(X_{\bullet,\mathrm{Et}}, \mathcal{M}) \\ & & \downarrow \\ E_1^{st} = H^t(X_{s,\mathrm{et}}, \mathcal{M}|_{X_{s,\mathrm{et}}}) & \Longrightarrow & H^{s+t}(X_{\bullet,\mathrm{et}}, \mathcal{M}|_{X_{\bullet,\mathrm{et}}}). \end{array} \tag{10.7.5}$$

This reduces the proof of the Theorem to showing that if $\mathcal{X}$ is an algebraic space and $\mathcal{M}$ a quasi–coherent sheaf on $\mathcal{X}_{\mathrm{Et}}$ then the natural map

$$H^i(\mathcal{X}_{\mathrm{Et}}, \mathcal{M}) \to H^i(\mathcal{X}_{\mathrm{et}}, \mathcal{M}|_{\mathcal{X}_{\mathrm{et}}}) \tag{10.7.6}$$

is an isomorphism. This is clear because by the construction this map is obtained by first choosing an injective resolution $\mathcal{M} \to I^\bullet$ in the category of abelian sheaves on $\mathcal{X}_{\mathrm{Et}}$, and then observing that the restriction $\mathcal{M}|_{\mathcal{X}_{\mathrm{et}}} \to I^\bullet|_{\mathcal{X}_{\mathrm{et}}}$ is again a resolution. Then the map (10.7.6) can be described as

$$H^i(\Gamma(\mathcal{X}_{\mathrm{et}}, I^\bullet|_{\mathcal{X}_{\mathrm{et}}})) \to H^i(\Gamma(\mathcal{X}_{\mathrm{et}}, \mathrm{Tot}(J^{\bullet\bullet}))), \tag{10.7.7}$$

where $I^\bullet|_{\mathcal{X}_{\mathrm{et}}} \to J^{\bullet\bullet}$ is an injective resolution in the category of abelian sheaves on $\mathcal{X}_{\mathrm{et}}$ and $\mathrm{Tot}(J^{\bullet\bullet}))$ denotes the total complex of the double complex $J^{\bullet\bullet}$. But since restriction from $\mathcal{X}_{\mathrm{Et}}$ to $\mathcal{X}_{\mathrm{et}}$ is an exact functor with an exact left adjoint (since there is a morphism of topoi $\mathcal{X}_{\mathrm{Et}} \to \mathcal{X}_{\mathrm{et}}$), the complex $I^\bullet|_{\mathcal{X}_{\mathrm{et}}}$ is a complex of injective sheaves, and hence in the above we can take $J^{\bullet\bullet} = I^\bullet|_{\mathcal{X}_{\mathrm{et}}}$.

∎

**Corollary 10.8.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi–compact morphism of algebraic stacks. Then for any $\mathcal{M} \in D^+_{\text{qcoh}}(\mathcal{X}_{\text{Et}})$ the restriction to $\mathcal{Y}_{\text{lis-et}}$ of $Rf^{\text{Et}}_*(\mathcal{M})$ lies in $D^+_{\text{qcoh}}(\mathcal{Y})$ (here we extend the restriction functor to $D^+_{\text{qcoh}}(\mathcal{X}_{\text{Et}})$ by observing that it is an exact functor).*

*Proof.* Considerations of the distinguished triangles associated the truncations of $\mathcal{M}$ reduces the Corollary to the case when $\mathcal{M} \in Qcoh^{\text{Et}}(\mathcal{X})$ in which case the result follows from (10.7) and (6.20). □

**10.9.** To understand cohomology of sheaves in the flat topology, note first that for any algebraic stack $\mathcal{X}$ there is a morphism of ringed topoi $\eta : \mathcal{X}_{\text{fpqc}} \to \mathcal{X}_{\text{Et}}$ since the flat topology is a finer topology than the étale topology on the category of $\mathcal{X}$–schemes.

**Lemma 10.10.** *Let $F$ be a sheaf of $\mathcal{O}_{\mathcal{X}_{\text{fpqc}}}$–modules on $\mathcal{X}_{\text{fpqc}}$ such that for every $\mathcal{X}$–scheme $U$ the restriction $F_U$ of $F$ to $U_{\text{et}}$ is a quasi–coherent sheaf and such that for any flat $\mathcal{X}$–morphism $g : U' \to U$ the map $g^*F_U \to F_{U'}$ is an isomorphism. Then $R^i\eta_*(F) = 0$ for every $i > 0$.*

*Proof.* The assertion is local on $\mathcal{X}$ so it suffices to consider the case when $\mathcal{X}$ is a scheme. In this case the result follows from the same argument used in the proof of ([Mi], III.3.7). □

**Lemma 10.11.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi–compact morphism of algebraic stacks, and let $f^{\text{fpqc}} : \mathcal{X}_{\text{fpqc}} \to \mathcal{Y}_{\text{fpqc}}$ be the induced morphism of ringed topoi. The for any quasi–coherent sheaf $\mathcal{M}$ on $\mathcal{X}_{\text{fpqc}}$ and $i \geq 0$ the sheaf $F := R^if^{\text{fpqc}}_*(\mathcal{M})$ satisfies the hypotheses of* (10.10).

*Proof.* The assertion is local on $\mathcal{Y}$ so we can without loss of generality assume that $\mathcal{Y}$ is a quasi–compact scheme. Furthermore, if $P : X \to \mathcal{X}$ is a smooth cover by a quasi–compact scheme $X$ with associated 0–coskeleton $X_\bullet \to \mathcal{X}$, then as in (2.9) we have a spectral sequence

(10.11.1) $$E_1^{st} = R^tf^{s,\text{fpqc}}_*(\mathcal{M}|_{X_{s,\text{fpqc}}}) \implies R^{s+t}f^{\text{fpqc}}_*(\mathcal{M}),$$

where $f^s : X_s \to \mathcal{Y}$ denotes the projection. From this we conclude that it suffices to prove the result for each $X_s$ which reduces the proof to the case when $\mathcal{X}$ is an algebraic space. Repeating the reduction argument with an étale cover of $\mathcal{X}$ we further reduce to the case when $\mathcal{X}$ is a scheme.

In this case the result can be seen as follows. The sheaf $F$ is the sheaf associated to the presheaf which to any affine $\mathcal{Y}$–scheme $U$ associates the $\Gamma(U, \mathcal{O}_U)$–module $H^i((\mathcal{X} \times_{\mathcal{Y}} U)_{\text{fpqc}}, \mathcal{M})$. On the other hand, by ([Mi], III.3.7) this sheaf is naturally isomorphic to the sheaf associated to the presheaf which to any $U$ associates $H^i((\mathcal{X} \times_{\mathcal{Y}} U)_{\text{Zar}}, \mathcal{M}|_{(\mathcal{X}\times_{\mathcal{Y}}U)_{\text{Zar}}})$. The result therefore follows from ([SGA6], IV.3.1.0). □

**Corollary 10.12.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi–compact morphism of algebraic stacks and $\mathcal{M} \in Qcoh^{\text{fpqc}}(\mathcal{X})$ a quasi–coherent sheaf. Then for any $i \geq 0$ the sheaf $R^if_*(\mathcal{M}|_{\mathcal{X}_{\text{lis-et}}})$ is naturally isomorphic to the restriction of $R^if^{\text{fpqc}}_*(\mathcal{M})$ to $\mathcal{Y}_{\text{lis-et}}$.*

*Proof.* Consider the commutative diagram of ringed topoi

(10.12.1) $$\begin{array}{ccc} \mathcal{X}_{\text{fpqc}} & \xrightarrow{\eta_{\mathcal{X}}} & \mathcal{X}_{\text{Et}} \\ {\scriptstyle f^{\text{fpqc}}}\downarrow & & \downarrow{\scriptstyle f^{\text{Et}}} \\ \mathcal{Y}_{\text{fpqc}} & \xrightarrow{\eta_{\mathcal{Y}}} & \mathcal{Y}_{\text{Et}}, \end{array}$$

and let $\Lambda : \mathcal{X}_{\text{fpqc}} \rightarrow \mathcal{Y}_{\text{Et}}$ denote $\eta_{\mathcal{Y}} \circ f^{\text{fpqc}} = f^{\text{Et}} \circ \eta_{\mathcal{X}}$. By ([SGA4], V.5.4) there is a Leray spectral sequence

$$E_2^{pq} = R^p\eta_{\mathcal{Y}*}R^q f_*^{\text{fpqc}}(\mathcal{M}) \implies R^{p+q}\Lambda_*(\mathcal{M}), \tag{10.12.2}$$

which by (10.10) and (10.11) degenerates to give an isomorphism $\eta_{\mathcal{Y}*}R^if_*^{\text{fpqc}}(\mathcal{M}) \simeq R^i\Lambda_*(\mathcal{M})$ for every $i$. By (10.10) there is also a natural isomorphism $R^i\Lambda_*(\mathcal{M}) \simeq R^if_*^{\text{Et}}(\eta_{\mathcal{X}*}(\mathcal{M}))$ so we have

$$\eta_{\mathcal{Y}*}R^if_*^{\text{fpqc}}(\mathcal{M}) \simeq R^i\Lambda_*(\mathcal{M}) \simeq R^if_*^{\text{Et}}(\eta_{\mathcal{X}*}(\mathcal{M})). \tag{10.12.3}$$

The Corollary therefore follows from (10.7). □

**Theorem 10.13** ([LM-B], 15.6 (iv), [Fa], [Ol]). *Let $f : \mathcal{X} \rightarrow \mathcal{Z}$ be a proper morphism between locally noetherian algebraic stacks. Then for any coherent sheaf $\mathcal{M}$ on $\mathcal{X}$ and $i \geq 0$, the sheaves $R^if_*\mathcal{M}$ on $\mathcal{Z}_{\text{lis-et}}$ are coherent. More generally, the functor $Rf_*$ induces a functor*

$$Rf_* : D^+_{\text{coh}}(\mathcal{X}) \longrightarrow D^+_{\text{coh}}(\mathcal{Z}). \tag{10.13.1}$$

*Proof.* First observe that we may work locally on $\mathcal{Z}$ and hence can assume that $\mathcal{Z}$ is a quasi–compact scheme. This observation also implies that the theorem holds when $f$ is representable by the corresponding result for algebraic spaces ([Kn], IV.4.1).

Assuming that $\mathcal{Z}$ is a quasi–compact scheme, we now prove the Theorem using the "devissage Lemma" (6.7). Conditions (6.7 (i)) and (6.7 (ii)) hold trivially. Hence it suffices to prove that if $I : \mathcal{Y} \hookrightarrow \mathcal{X}$ is an integral closed substack and $\mathcal{N}$ is a coherent $\mathcal{O}_{\mathcal{Y}}$–module with support all of $\mathcal{Y}$, then there exists an object $N \in D^+_{\text{coh}}(\mathcal{Y})$ such that the Theorem holds for $I_*N$ and a morphism $\alpha : \mathcal{N} \rightarrow N$ which is an isomorphism over some dense open substack of $\mathcal{Y}$. For this choose using ([Ol], 1.1) a proper surjective map $p : Y \rightarrow \mathcal{Y}$ with $Y$ a scheme, and let $p_\bullet : Y_\bullet \rightarrow \mathcal{Y}$ by the 0–coskeleton of $p$. Consider the sheaf $p_\bullet^*\mathcal{N}$ on $Y_{\bullet,\text{Et}}$ whose restriction to $Y_{n,\text{Et}}$ is the quasi–coherent sheaf obtained by pullback from the quasi–coherent sheaf $\mathcal{N}$ (identified with a sheaf on the big étale site via (10.5)). Let

$$p_{\bullet*} : Y_{\bullet,\text{Et}} \rightarrow \mathcal{Y}_{\text{Et}} \tag{10.13.2}$$

be the natural morphism of topoi.

**Lemma 10.14.** *Let $N$ denote the restriction of $Rp_{\bullet*}p_\bullet^*\mathcal{N}$ to $\mathcal{Y}_{\text{lis-et}}$. Then $N$ is in $D^+_{\text{coh}}(\mathcal{Y})$ and the map $\alpha : \mathcal{N} \rightarrow N$ induced by adjunction is an isomorphism over some dense open substack of $\mathcal{Y}$.*

*Proof.* Using the argument of (2.9) and the fact that restriction is an exact functor, there is a spectral sequence

$$E_1^{st} = (R^tp_{s*}p_s^*\mathcal{N})|_{\mathcal{Y}_{\text{lis-et}}} \implies R^{s+t}p_{\bullet*}(p_\bullet^*\mathcal{N})|_{\mathcal{Y}_{\text{lis-et}}}, \tag{10.14.1}$$

where $p_s : Y_{s,\text{Et}} \rightarrow \mathcal{Y}_{\text{Et}}$ denotes the projection. Thus to prove that $N$ is in $D^+_{\text{coh}}(\mathcal{Y})$ it suffices to show that each $(R^tp_{s*}p_s^*\mathcal{N})|_{\mathcal{Y}_{\text{lis-et}}}$ is coherent. This follows from the comparison Theorem (10.7) and the representable case already considered.

Since $\mathcal{Y}$ is integral, there exists a dense open substack $\mathcal{U} \subset \mathcal{Y}$ over which $p$ is flat. We claim that the map $\mathcal{N} \rightarrow Rp_{\bullet*}p_\bullet^*\mathcal{N}$ becomes an isomorphism over $\mathcal{U}$. For this note that for any $i \geq 0$ the sheaf $R^ip_{\bullet*}p_\bullet^*\mathcal{N}|_{\mathcal{U}}$ is the sheaf associated to the presheaf which to any $\mathcal{U}$–scheme $U$ associates the group $H^i(Y_{U\bullet,\text{Et}}, \mathcal{N}|_{Y_{U_\bullet,\text{Et}}})$, where $Y_{U\bullet}$ denotes the base change of $Y_\bullet$ to $U$. Since restriction to the small étale site is an exact functor that takes injectives to injectives (since it is induced by a morphism of topoi $Y_{U\bullet,\mathrm{Et}} \to Y_{U\bullet,\mathrm{et}}$ and hence has an exact left adjoint), this sheaf is isomorphic to the sheaf associated to the presheaf which associates to $U$ the group $H^i(Y_{U\bullet,\mathrm{et}}, \mathcal{N}|_{Y_{U_\bullet,\mathrm{et}}})$. By ([SGA4], $\mathrm{V}^{\mathrm{bis}}$.4.2.1), if $U$ is affine this group is zero for $i > 0$ and isomorphic to $\mathcal{N}(U)$ for $i = 0$. □

Part (iii) of the following Lemma (which we state in slightly greater generality for other applications) combined with (6.20) implies that (10.13) holds for $I_*N$ and hence by the devissage Lemma (6.7) completes the proof of (10.13). □

**Lemma 10.15.** *Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi–compact morphism of algebraic stacks with $\mathcal{Z}$ a quasi–compact scheme, and let $Y \to \mathcal{Y}$ be a proper surjection with associated 0–coskeleton $p_\bullet : Y_\bullet \to \mathcal{Y}$. Let $g^{\mathrm{et}} : Y_{\bullet,\mathrm{et}} \to \mathcal{Z}_{\mathrm{et}}$ be the natural morphism of topoi, and $\mathcal{N}$ a quasi–coherent sheaf on $\mathcal{Y}$. Let $N$ denote the restriction of $Rp^{\mathrm{Et}}_{\bullet,*}(\mathcal{N}|_{Y_\bullet,\mathrm{Et}})$ to the lisse-étale site of $\mathcal{Y}$, where $p^{\mathrm{Et}}_\bullet : Y_{\bullet,\mathrm{Et}} \to \mathcal{Y}_{\mathrm{Et}}$ denote the morphism between the big étale topoi.*

(i) *For any integer $i$ there is a natural isomorphism*

$$R^i f_*(N)|_{\mathcal{Z}_{\mathrm{et}}} \simeq R^i g^{\mathrm{et}}_*(\mathcal{N}|_{Y_\bullet,\mathrm{et}}). \tag{10.15.1}$$

(ii) *There is a spectral sequence*

$$E_1^{st} = R^t g^{\mathrm{et}}_{s,*}(\mathcal{N}|_{Y_{s,\mathrm{et}}}) \implies R^i f_*(N)|_{\mathcal{Z}_{\mathrm{et}}}, \tag{10.15.2}$$

*where $g^{\mathrm{et}}_s : Y_{s,\mathrm{et}} \to \mathcal{Z}_{\mathrm{et}}$ denotes the morphism of topoi induced by the composite $g_s : Y_s \to \mathcal{Y} \to \mathcal{Z}$.*

(iii) *If the map $\mathcal{Y} \to \mathcal{Z}$ is proper, $\mathcal{Y}$ and $\mathcal{Z}$ are locally noetherian, and $\mathcal{N}$ is coherent, then for any integer $i$ the sheaf $R^i f_*(N)|_{\mathcal{Z}_{\mathrm{et}}}$ is coherent.*

*Proof.* Note first that (ii) follows immediately from (i) and the construction of (2.9). Furthermore, if $\mathcal{Y} \to \mathcal{Z}$ is proper, then each $g_s : Y_s \to \mathcal{Z}$ is proper and each term $R^t g^{\mathrm{et}}_{s,*}(\mathcal{N}|_{Y_{s,\mathrm{et}}})$ in the spectral sequence (10.15.2) is coherent by ([Kn], IV.4.1). Hence (iii) follows from (ii).

To prove (i), proceed as follows. First note that we have

$$R^i g^{\mathrm{et}}_*(\mathcal{N}|_{Y_\bullet,\mathrm{et}}) \simeq R^i g^{\mathrm{Et}}_*(\mathcal{N}|_{Y_\bullet,\mathrm{Et}})|_{\mathcal{Z}_{\mathrm{et}}} \simeq (R^i f^{\mathrm{Et}}_*(Rp^{\mathrm{Et}}_{\bullet,*}(\mathcal{N}|_{Y_\bullet,\mathrm{Et}})))|_{\mathcal{Z}_{\mathrm{et}}}, \tag{10.15.3}$$

where $f^{\mathrm{Et}} : \mathcal{Y}_{\mathrm{Et}} \to \mathcal{Z}_{\mathrm{Et}}$ denotes the morphism between the big étale topoi (here we use the fact that restriction from the big étale site to the small étale site is exact and takes injectives to injectives since it has an exact left adjoint). Write $\widetilde{N}$ for $Rp^{\mathrm{Et}}_{\bullet,*}(\mathcal{N}|_{Y_\bullet,\mathrm{Et}})$, and let $\mathrm{res}_{\mathcal{Z}}$ be the restriction functor from sheaves on $\mathcal{Z}_{\mathrm{Et}}$ to $\mathcal{Z}_{\mathrm{et}}$. Since $\mathrm{res}_{\mathcal{Z}}$ is exact, we have isomorphisms

$$R\mathrm{res}_{\mathcal{Z}} \circ Rf^{\mathrm{Et}}_*(-) \simeq R(\mathrm{res}_{\mathcal{Z}} \circ f^{\mathrm{Et}}_*)(-), \tag{10.15.4}$$

and

$$R\widetilde{\mathrm{res}}_{\mathcal{Z}} \circ Rf_*(-) \simeq R(\widetilde{\mathrm{res}}_{\mathcal{Z}} \circ f_*)(-), \tag{10.15.5}$$

where $\widetilde{\mathrm{res}}_{\mathcal{Z}}$ denotes the restriction from the lisse-étale site of $\mathcal{Z}$ to the étale site. If $\mathrm{res}'_{\mathcal{Y}}$ denotes the restriction functor from sheaves on $\mathcal{Y}_{\mathrm{Et}}$ to $\mathcal{Y}_{\text{lis-et}}$, then there is a natural isomorphism of functors

$$\mathrm{res}_{\mathcal{Z}} \circ f^{\mathrm{Et}}_* \simeq \widetilde{\mathrm{res}}_{\mathcal{Z}} \circ f_* \circ \mathrm{res}'_{\mathcal{Y}}. \tag{10.15.6}$$

By the universal property of derived functors we therefore obtain a map

$$R(\mathrm{res}_{\mathcal{Z}} \circ f_*^{\mathrm{Et}})(\widetilde{N}) \to R(\widetilde{\mathrm{res}}_{\mathcal{Z}} \circ f_*)(\mathrm{res}'_{\mathcal{Y}}(\widetilde{N})) \tag{10.15.7}$$

which via the isomorphisms (10.15.3), (10.15.4), and (10.15.5) give a map

$$Rg_*^{\mathrm{et}}(\mathcal{N}|_{Y_\bullet,\mathrm{et}}) \to R^i f_*(N)|_{\mathcal{Z}_{\mathrm{et}}}. \tag{10.15.8}$$

That this arrow is an isomorphism follows from the following sub–lemma applied to $\widetilde{N}$: □

**Sub-Lemma 10.16.** *Let $\mathcal{C}$ be the full subcategory of the category of sheaves on $\mathcal{Y}_{\mathrm{Et}}$ consisting of sheaves $\mathcal{M}$ whose restriction to $\mathcal{Y}_{\text{lis-et}}$ is quasi–coherent. Denote by $D^+_{\mathcal{C}}(\mathcal{Y}_{\mathrm{Et}})$ the full subcategory of the derived category of bounded below complexes of $\mathcal{O}_{\mathcal{Y}_{\mathrm{Et}}}$–modules whose cohomology sheaves are in $\mathcal{C}$. Then for any $\widetilde{N} \in D^+_{\mathcal{C}}(\mathcal{Y}_{\mathrm{Et}})$ the map* (10.15.7) *is an isomorphism.*

*Proof.* The category $\mathcal{C}$ is closed under kernels, cokernels, and extensions, and hence by considering the distinguished triangles associates to the truncations of $\widetilde{N}$ we are reduced to the case when $\widetilde{N}$ is an object of $\mathcal{C}$. Furthermore, the assertion is local on $\mathcal{Z}$ so it suffices to show that the map

$$H^i(\mathcal{Y}_{\mathrm{Et}}, \widetilde{N}) \to H^i(\mathcal{Y}_{\text{lis-et}}, \widetilde{N}|_{\mathcal{Y}_{\text{lis-et}}}) \tag{10.16.1}$$

is an isomorphism for every integer $i$. Let $U \to \mathcal{Y}$ be a smooth cover and $U_\bullet$ the 0–coskeleton. There is a commutative square

$$\begin{array}{ccc} H^i(\mathcal{Y}_{\mathrm{Et}}, \widetilde{N}) & \longrightarrow & H^i(\mathcal{Y}_{\text{lis-et}}, \widetilde{N}|_{\mathcal{Y}_{\text{lis-et}}}) \\ \downarrow & & \downarrow \\ H^i(U_{\bullet,\mathrm{Et}}, \widetilde{N}|_{U_{\bullet,\mathrm{Et}}}) & \xrightarrow{\gamma} & H^i(U_{\bullet,\mathrm{et}}, \widetilde{N}|_{U_{\bullet,\mathrm{et}}}), \end{array} \tag{10.16.2}$$

where the left vertical arrow is an isomorphism by ([SGA4], V.2.3.4) and the right vertical arrow is an isomorphism by (2.3) and (4.7). That the diagram commutes can be seen by noting that it commutes for $i = 0$ and that (10.16.2) is obtained from a diagram of cohomological $\delta$–functors

$$\begin{array}{ccc} \{H^i(\mathcal{Y}_{\mathrm{Et}}, (-))\} & \longrightarrow & \{H^i(\mathcal{Y}_{\text{lis-et}}, (-)|_{\mathcal{Y}_{\text{lis-et}}})\} \\ \downarrow & & \downarrow \\ \{H^i(U_{\bullet,\mathrm{Et}}, (-)|_{U_{\bullet,\mathrm{Et}}})\} & \longrightarrow & \{H^i(U_{\bullet,\mathrm{et}}, (-)|_{U_{\bullet,\mathrm{et}}})\}, \end{array} \tag{10.16.3}$$

where $\{H^i(\mathcal{Y}_{\mathrm{Et}}, (-))\}$ is universal.

Thus it suffices to show that the arrow $\gamma$ is an isomorphism. This follows from the observation that the restriction functor from $U_{\bullet,\mathrm{Et}}$ to $U_{\bullet,\mathrm{et}}$ is exact and takes injectives to injectives since it is obtained from a morphism of topoi $U_{\bullet,\mathrm{Et}} \to U_{\bullet,\mathrm{et}}$. □

## 11. The Grothendieck Existence Theorem, Grothendieck's Fundamental Theorem for Proper Morphisms, and Zariski's Connectedness Theorem

**Theorem 11.1.** *Let $A$ be a noetherian adic ring and let $\mathfrak{a} \subset A$ be an ideal of definition. Let $\mathcal{X}/A$ be a proper algebraic stack, and for every $n \geq 0$ let $\mathcal{X}_n$ denote the stack $\mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{a}^{n+1})$.*

(i) *The functor sending a coherent sheaf to its reductions defines an equivalence of categories between the category of coherent sheaves on* $\mathcal{X}$ *and the category of compatible systems* $\{\mathcal{M}_n\}_{n\geq 0}$ *of coherent sheaves on the* $\mathcal{X}_n$.

(ii) *If* $\mathcal{M}$ *is a coherent sheaf on* $\mathcal{X}$ *with reductions* $\{\mathcal{M}_n\}_{n\geq 0}$*, then for every* $i \geq 0$ *the natural map*

$$H^i(\mathcal{X}, \mathcal{M}) \longrightarrow \varprojlim_n H^i(\mathcal{X}_n, \mathcal{M}_n) \tag{11.1.1}$$

*is an isomorphism. If the finitely generated* $A$*–module* $H^i(\mathcal{X}, \mathcal{M})$ *is viewed as a topological* $A$*–module with the* $\mathfrak{a}$*–adic topology and the right hand side of* (11.1.1) *is viewed as a topological* $A$*–module with the inverse limit topology, then this is an isomorphism of topological* $A$*–modules.*

*Proof.* Statement (i) follows from ([Ol], 1.4). To prove (ii), we modify the proof of (loc. cit. 3.2). Let $U_\bullet \to \mathcal{X}$ by a hypercover ([Fr], 3.3) with each $U_n$ an affine $A$–scheme of finite type, and let $\widehat{U}_\bullet$ denote the simplicial formal scheme obtained by completing the $U_n$ along $\mathfrak{a}$. There is a natural morphism of ringed topoi $q : \widehat{U}_{\bullet,\mathrm{et}} \to U_{\bullet,\mathrm{et}}$. For a coherent sheaf $\mathcal{M}$ on $\mathcal{X}$ we write $\widehat{\mathcal{M}}$ for the sheaf on $\widehat{U}_{\bullet,\mathrm{et}}$ obtained by restricting $\mathcal{M}$ to $U_{\bullet,\mathrm{et}}$ and then pulling back to $\widehat{U}_{\bullet,\mathrm{et}}$. The morphism $q$ induces a map of cohomology groups

$$H^i(\mathcal{X}, \mathcal{M}) \longrightarrow H^i(\widehat{U}_\bullet, \widehat{\mathcal{M}}) \tag{11.1.2}$$

which by ([Ol], 3.2) is an isomorphism (actually in (loc. cit.) the hypercover $U_\bullet$ is assumed to be obtained as the 0–coskeleton of a morphism $U \to \mathcal{X}$ but the proof works for an arbitrary hypercover). Since each $U_n$ is assume to be affine, the cohomology group $H^i(\widehat{U}_\bullet, \widehat{\mathcal{M}})$ is equal to the $i$–th cohomology group of the complex

$$\cdots \to \Gamma(\widehat{U}_n, \widehat{\mathcal{M}}) \to \Gamma(\widehat{U}_{n+1}, \widehat{\mathcal{M}}) \to \cdots \tag{11.1.3}$$

obtained from the normalized complex of the cosimplicial $A$–module $[n] \mapsto \Gamma(\widehat{U}_n, \widehat{\mathcal{M}})$.

Define

$$\widehat{L}^n := \Gamma(\widehat{U}_n, \widehat{\mathcal{M}}), \quad L^n_k := \Gamma(U_n \otimes_A (A/\mathfrak{a}^{k+1}), \mathcal{M}_k|_{U_n \otimes_A (A/\mathfrak{a}^{k+1})}), \tag{11.1.4}$$

$$\widehat{K}^n := \mathrm{Ker}(\widehat{L}^n \to \widehat{L}^{n+1}), \quad K^n_k := \mathrm{Ker}(L^n_k \to L^{n+1}_k), \tag{11.1.5}$$

$$\widehat{I}^n := \mathrm{Im}(\widehat{L}^{n-1} \to \widehat{L}^n), \quad I^n_k := \mathrm{Im}(L^{n-1}_k \to L^n_k). \tag{11.1.6}$$

We then have $\widehat{L}^n = \varprojlim_k L^n_k$. Again since each $U_n$ is affine, we have $H^i(\mathcal{X}_k, \mathcal{M}_k) = H^i(L^\bullet_k)$ and the arrow (11.1.1) is identified with the natural map

$$H^i(\widehat{L}^\bullet) \to \varprojlim_k H^i(L^\bullet_k). \tag{11.1.7}$$

**Lemma 11.2.** *Let* $\mathcal{S}$ *denote the graded ring* $\oplus_{k\geq 0}\mathfrak{a}^k/\mathfrak{a}^{k+1}$*, and let* $i$ *be an integer.*

(i) *The graded* $\mathcal{S}$*–module* $\oplus_{k\geq 0} H^i(\mathcal{X}, \mathfrak{a}^k\mathcal{M}/\mathfrak{a}^{k+1}\mathcal{M})$ *is of finite type.*

(ii) *The projective system* $\{H^i(\mathcal{X}_k, \mathcal{M}_k)\}_k$ *satisfies the Mittag–Leffler condition (henceforth abbreviated ML condition).*

*Proof.* For (i), note that by ([A-M], 10.22 (iii)), the sheaf $\oplus_{k\geq 0}\mathfrak{a}^k\mathcal{M}/\mathfrak{a}^{k+1}\mathcal{M}$ on $\mathrm{Spec}(\mathcal{S})\times_{\mathrm{Spec}(A)}\mathcal{X}$ is coherent. Hence (i) follows from (10.13) applied to the proper morphism

$$\mathrm{Spec}(\mathcal{S})\times_{\mathrm{Spec}(A)}\mathcal{X}\rightarrow \mathrm{Spec}(\mathcal{S}). \tag{11.2.1}$$

Statement (ii) follows from (i) (applied with $i$ and $i+1$) and ([EGA], $0_{\mathrm{III}}$.13.7.7) applied to $\mathcal{S}$ and the projective system $\{\mathcal{M}_k\}$ of sheaves on $\mathcal{X}$. □

To prove that (11.1.7) is an isomorphism, it suffices to show that the systems $\{K^n_k\}_k$ and $\{I^n_k\}_k$ satisfy the ML condition and that the natural maps

$$\widehat{K}^n\rightarrow \varprojlim_k K^n_k,\quad \widehat{I}^n\rightarrow \varprojlim_k I^n_k \tag{11.2.2}$$

are isomorphisms. For if the systems $\{I^n_k\}_k$ satisfy the ML condition, then we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{I}^n & \longrightarrow & \widehat{K}^n & \longrightarrow & H^n(\mathcal{X},\mathcal{M}) & \longrightarrow & 0\\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & \varprojlim_k I^n_k & \longrightarrow & \varprojlim_k K^n_k & \longrightarrow & \varprojlim H^n(\mathcal{X}_k,\mathcal{M}_k) & \longrightarrow & 0, \end{array} \tag{11.2.3}$$

where the bottom row is exact since $\{I^n_k\}_k$ satisfies the ML condition. If the maps $a$ and $b$ are isomorphisms it follows that the map $c$ is an isomorphism as well.

Now observe that not only do we have $\widehat{L}^n\simeq\varprojlim L^n_k$, the system $\{L^n_k\}_k$ has the property that for every $k$ the map $\widehat{L}^n\rightarrow L^n_k$ is surjective since each $U_n$ is affine. This implies that the systems $\{I^n_k\}_k$ also satisfy the ML condition, and in fact have the stronger property that for any $k$ the map $I^n_{k+1}\rightarrow I^n_k$ is surjective. Now consider the exact sequences

$$0\longrightarrow I^n_k\longrightarrow K^n_k\longrightarrow H^n(\mathcal{X}_k,\mathcal{M}_k)\longrightarrow 0. \tag{11.2.4}$$

By (11.2 (ii)) the system $\{H^n(\mathcal{X}_k,\mathcal{M}_k)\}$ satisfies the ML condition. Therefore, for each $k$ there exists an integer $k_0\geq k$ such that the groups $H^n(\mathcal{X}_{k'},\mathcal{M}_{k'})$ for $k'\geq k_0$ all have the same image in $H^n(\mathcal{X}_k,\mathcal{M}_k)$. We claim that for $k'\geq k_0$ the groups $K^n_{k'}$ also all have the same image in $K^n_k$. To see this let $J_{k'}\subset K^n_k$ denote the image of $K^n_{k'}$. We show that $J_{k'}=J_{k'+1}$ for every $k'\geq k_0$. For this let $\eta\in J_{k'}$ be an element. Since the images of $J_{k'}$ and $J_{k'+1}$ in $H^n(\mathcal{X}_k,\mathcal{M}_k)$ are equal, there exists an element $\iota\in J_{k'+1}$ such that $\eta-\iota\in I^n_k$. Replacing $\eta$ by $\eta-\iota$ we may therefore assume that $\eta\in I^n_k\cap J_{k'}$. But in this case the result is clear because the map $I^n_{k'}\rightarrow I^n_k$ is surjective for all $k'\geq k$. In particular, $I^n_k$ is contained in $\cap_{k'\geq k}J_{k'}$.

We conclude that in particular the systems $\{K^n_k\}_k$ satisfy the ML condition. Since $\varprojlim_k$ is a left exact functor, it is also clear that the natural map $\widehat{K}^n\rightarrow\varprojlim_k K^n_k$ is an isomorphism. Now since the system $\{K^n_k\}_k$ satisfies the ML condition, we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{K}^n & \longrightarrow & \widehat{L}^n & \longrightarrow & \widehat{I}^{n+1} & \longrightarrow & 0\\ & & \downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_k K^n_k & \longrightarrow & \varprojlim_k L^n_k & \longrightarrow & \varprojlim_k I^{n+1}_k & \longrightarrow & 0 \end{array} \tag{11.2.5}$$

from which we conclude that the map $\widehat{I}^{n+1}\rightarrow\varprojlim_k I^{n+1}_k$ is an isomorphism. This completes the proof of the Theorem. □

**Theorem 11.3.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of locally noetherian algebraic stacks. Then $f_*\mathcal{O}_{\mathcal{X}}$ is a coherent sheaf on $\mathcal{Y}$. Let $\mathcal{Y}' := \underline{\mathrm{Spec}}_{\mathcal{Y}}(f_*\mathcal{O}_{\mathcal{X}})$ be the relative spectrum of $f_*\mathcal{O}_{\mathcal{X}}$ and let $f' : \mathcal{X} \to \mathcal{Y}'$ be the canonical factorization of $f$. Then $f'$ is proper, $f'_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}'}$, and the geometric fibers of $f'$ are nonempty and connected.*

*Proof.* The proof is essentially the same as in ([EGA], III.4.3.1). That $f_*\mathcal{O}_{\mathcal{X}}$ is a coherent sheaf follows from (10.13). That $f'$ is proper is clear because the map $f$ is proper and $\mathcal{Y}' \to \mathcal{Y}$ is separated. That $f'_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}'}$ is immediate from the construction since $\mathcal{Y}'$ is affine over $\mathcal{Y}$.

It remains only to see that the geometric fibers of $f'$ are non-empty and connected. For this we may as well replace $\mathcal{Y}$ by $\mathcal{Y}'$ and hence may assume that $f_*\mathcal{O}_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{Y}}$. Furthermore, note that the formation of $\mathcal{Y}'$ commutes with arbitrary flat base change $Y \to \mathcal{Y}$. Therefore, we can without loss of generality assume that $\mathcal{Y}$ is the spectrum of a complete noetherian local ring $A$ with separably closed residue field. Since $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{Y}}$ the map $\mathcal{X} \to \mathcal{Y}$ is dominant and since it is proper it is also surjective. Let $s \in \mathrm{Spec}(A)$ be the closed point and $\mathcal{X}_s$ the fiber. If $\mathcal{X}_s = \coprod_{i=1}^p \mathcal{Z}_i$, then for each $n \geq 0$ the reduction $\mathcal{X}_n = \mathcal{X} \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/\mathfrak{m}_s^{n+1})$ is naturally a disjoint union $\coprod_{i=1}^p \mathcal{Z}_{i,n}$ where each $\mathcal{Z}_{i,n}$ reduces to $\mathcal{Z}_i$. It follows that there is an isomorphism

$$\varprojlim H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}) \simeq \prod_{i=1}^p \Gamma(\mathcal{Z}_{i,n}, \mathcal{O}_{\mathcal{Z}_{i,n}}). \tag{11.3.1}$$

Since the natural map $A \to \varprojlim H^0(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ is an isomorphism by assumption and $A$ is local, this implies that $p = 1$ which proves the Theorem. □

The factorization $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ is called the *Stein factorization* of $f$.

## Appendix A. A Theorem of Gabber

The results of this appendix were communicated to us by Ofer Gabber and are entirely due to him. Though it is technically not used in the text, Theorem A.6 below is a natural result to consider when studying the lisse-étale topos.

**A.1.** Let $\mathcal{S}$ be a site with associated topos $\mathcal{S}^\sim$, and $\mathcal{S}' \subset \mathcal{S}$ a full subcategory. We view $\mathcal{S}'$ as a site with the topology induced by that on $\mathcal{S}$ ([SGA4], III.3.1) Assume that the following conditions hold:

(A.1.1) The final object of $\mathcal{S}^\sim$ is covered by objects of $\mathcal{S}'$.

(A.1.2) For every object $X \in \mathcal{S}'$ and covering family $\{Y_i \to X\}$ in $\mathcal{S}$, there exists a refinement $\{Z_{ij} \to X\}$ of $\{Y_i \to X\}$ such that the objects $Z_{ij}$ are in $\mathcal{S}'$.

(A.1.3) If $X \in \mathcal{S}'$ is an object and $\{Z_i \to X\}$ is a family of morphisms in $\mathcal{S}'$ which form a covering in $\mathcal{S}$, then for any $W \to X$ in $\mathcal{S}'$ the fiber product $Z_i \times_X W$ exists in $\mathcal{S}'$ and is also a fiber product in $\mathcal{S}$.

(A.1.4) There exists a full subcategory $\mathcal{S}'' \subset \mathcal{S}'$ satisfying (A.1.1)–(A.1.3) such that binary products in $\mathcal{S}''$ are representable and are also products in $\mathcal{S}$.

**Example A.2.** The case of interest in this paper is the following. Let $\mathcal{X}$ be an algebraic stack and $f : V \to \mathcal{X}$ a smooth morphism from a scheme $V$. Take $\mathcal{S}$ to be the site $\text{Lis-Et}(\mathcal{X})|_V$

consisting of $V$–schemes $U \to V$ for which the composite $U \to V \to \mathcal{X}$ is smooth, and let $\mathcal{S}' \subset \mathcal{S}$ be the full subcategory Lis-Et($V$) of smooth $V$–schemes. The topology on Lis-Et$(\mathcal{X})|_V$ is that induced by the topology on Lis-Et$(\mathcal{X})$.

**Lemma A.3.** *A family of morphisms $\{Z_i \to X\}$ in $\mathcal{S}'$ is a covering if and only if the induced family of morphisms in $\mathcal{S}$ is a covering. The inclusion $\mathcal{S}' \subset \mathcal{S}$ is cocontinuous in the sense of* ([SGA4], III.2.1).

*Proof.* By definition, the topology on $\mathcal{S}'$ is the finest topology making the inclusion $\mathcal{S}' \subset \mathcal{S}$ continuous. Since the topology on $\mathcal{S}'$ defined by the morphisms $\{Z_i \to X\}$ whose image in $\mathcal{S}$ are coverings is a topology making $\mathcal{S}' \subset \mathcal{S}$ continuous, every family of morphisms as in the Lemma is a covering family in $\mathcal{S}'$. The converse follows from ([SGA4], III.1.6).

The last statement follows from (A.1.2). □

**Lemma A.4.** *For any $F \in \mathcal{S}^\sim$, the natural map*

$$H^0(\mathcal{S}, F) \to H^0(\mathcal{S}', F|_{\mathcal{S}'}) \tag{A.4.1}$$

*is an isomorphism.*

*Proof.* By (A.1.1), there exists a family of objects $\{X_i\}$ in $\mathcal{S}''$ covering the initial object $e$ of $\mathcal{S}^\sim$. Let $R$ denote $\coprod X_i \times_e \coprod X_j$ which is a disjoint union of objects represented by objects of $\mathcal{S}''$ since binary products exist in $\mathcal{S}''$. We then have

$$H^0(\mathcal{S}, F) = \mathrm{Eq}(\prod F(X_i) \rightrightarrows F(R)) = H^0(\mathcal{S}', F|_{\mathcal{S}'}). \tag{A.4.2}$$

□

**A.5.** Since the restriction functor $\mathcal{S}^\sim \to \mathcal{S}'^\sim$ is exact by (A.1.2), the functors sending $F \in \mathcal{S}^\sim$ to $H^q(\mathcal{S}'^\sim, F|_{\mathcal{S}'})$ form a cohomological $\delta$–functor on the category of abelian sheaves on $\mathcal{S}$. The map (A.4.1) therefore extends to a unique morphism of cohomological $\delta$–functors

$$\{H^q(\mathcal{S}, -)\} \to \{H^q(\mathcal{S}', (-)|_{\mathcal{S}'})\}. \tag{A.5.1}$$

**Theorem A.6.** *The morphism of cohomological $\delta$–functors* (A.5.1) *is an isomorphism.*

For any object $X \in \mathcal{S}^\sim$ define $\mathcal{S}|_X$ and $\mathcal{S}'|_X$ to be the categories of objects over $X$. We view $\mathcal{S}|_X$ as a site as in ([SGA4], III.5.1).

**Lemma A.7.** *Let $X \in \mathcal{S}^\sim$ be a sheaf isomorphic to a coproduct of sheaves representable by objects of $\mathcal{S}''$. Then the subcategory $\mathcal{S}'|_X \subset \mathcal{S}|_X$ satisfies* (A.1.1)–(A.1.4). *In particular, for any $F \in (\mathcal{S}|_X)^\sim$ the natural map*

$$H^0(\mathcal{S}|_X, F) \to H^0(\mathcal{S}'|_X, F) \tag{A.7.1}$$

*is an isomorphism.*

*Proof.* Write $X = \coprod Z_i$ with $Z_i \in \mathcal{S}''$. Then the family $\{Z_i \to X\}$ defines a covering of the initial object of $(\mathcal{S}|_X)^\sim$. Axioms (A.1.2) and (A.1.3) follow immediately from the corresponding axioms for $\mathcal{S}' \subset \mathcal{S}$. For (A.1.4) consider the category of consisting of the $Z_i \to X$ and all morphisms $U \to X$ in $\mathcal{S}''$ which appear in a covering family $\{U_i \to Z_i\}$ for some $i$. Then properties (A.1.1)–(A.1.3) are immediate from the definition, and for two morphisms $U \to Z_i$ and $V \to Z_i$ the fiber product $U \times_{Z_i} V$ is representable by (A.1.3) and is the product in $\mathcal{S}|_X$. □

It follows that if (A.6) is true, then for any object $X \in \mathcal{S}^{\sim}$ isomorphic to a coproduct of sheaves representable by objects of $\mathcal{S}''$ and abelian sheaf $F \in (\mathcal{S}|_X)^{\sim}$ the natural map

$$H^j(\mathcal{S}|_X, F) \to H^j(\mathcal{S}'|_X, F) \tag{A.7.2}$$

is an isomorphism.

To prove (A.6), we proceed by induction on $q$. The case $q = 0$ has already been shown, so we prove the result for $q$ assuming the result holds in degrees $\leq q-1$. By the preceding observation, for any object $X \in \mathcal{S}^{\sim}$ isomorphic to a coproduct of sheaves represented by objects of $\mathcal{S}''$ the map (A.7.2) is an isomorphism for $j \leq q-1$.

Fix one such sheaf $X \simeq \coprod Z_i \in \mathcal{S}^{\sim}$ covering the initial object, and let $X_\bullet$ be the simplicial object in $\mathcal{S}^{\sim}$ obtained as the 0–coskeleton of $X \to e$. Since the subcategory $\mathcal{S}''$ has binary products, for every $n \in \mathbb{N}$ the sheaf $X_n$ is isomorphic to a coproduct of sheaves representable by objects of $\mathcal{S}''$. Consider the simplicial topoi $\mathcal{S}'^{\sim}|_{X_\bullet}$ and $\mathcal{S}^{\sim}|_{X_\bullet}$. By ([SGA4], V.2.3.4 (1)), for any abelian sheaf $F$ there is are natural isomorphisms (here and in what follows we simplify the notation by just writing $F$ instead of $F|_{\mathcal{S}|_{X_\bullet}}$ etc.)

$$H^q(\mathcal{S}|_{X_\bullet}, F) \simeq H^q(\mathcal{S}, F), \quad H^q(\mathcal{S}'|_{X_\bullet}, F) \simeq H^q(\mathcal{S}', F). \tag{A.7.3}$$

Furthermore, the natural map $H^q(\mathcal{S}|_{X_\bullet}, F) \to H^q(\mathcal{S}'|_{X_\bullet}, F)$ extends naturally to a morphism of spectral sequences (2.9.2)

$$\begin{array}{ccc} E_1^{st} = H^t(\mathcal{S}|_{X_s}, F) & \Longrightarrow & H^{s+t}(\mathcal{S}|_{X_\bullet}, F) \simeq H^{s+t}(\mathcal{S}, F) \\ & & \downarrow \\ E_1^{st} = H^t(\mathcal{S}'|_{X_s}, F) & \Longrightarrow & H^{s+t}(\mathcal{S}'|_{X_\bullet}, F) \simeq H^{s+t}(\mathcal{S}', F). \end{array} \tag{A.7.4}$$

Let $K_X \subset H^q(\mathcal{S}, F)$ (resp. $K'_X \subset H^q(\mathcal{S}', F)$) be the first step of the filtration on the abutment so there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_X & \longrightarrow & H^q(\mathcal{S}, F) & \longrightarrow & H^q(\mathcal{S}|_X, F) \\ & & {\scriptstyle a}\downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K'_X & \longrightarrow & H^q(\mathcal{S}', F) & \longrightarrow & H^q(\mathcal{S}'|_X, F) \end{array} \tag{A.7.5}$$

with exact rows. Since the map on $E_1^{st}$–terms $H^t(\mathcal{S}|_{X_s}, F) \to H^t(\mathcal{S}'|_{X_s}, F)$ is an isomorphism for $t \leq q-1$ by the induction hypothesis, for any $r \geq 1$ the maps on $E_r^{st}$–terms in the spectral sequences are also isomorphisms for $t \leq q-1$. This implies that the map $a : K_X \to K'_X$ is an isomorphism.

The construction of this morphism of spectral sequences is functorial with respect to morphisms $X' \to X$, where $X' \in \mathcal{S}^{\sim}$ is also isomorphic to a disjoint union of sheaves representable by objects of $\mathcal{S}''$. For any element $\alpha \in H^q(\mathcal{S}, F)$ there exists by (A.1.2) a covering $X' \to X$, where $X'$ is isomorphic to a disjoint union of objects of $\mathcal{S}''$, such that the image of $\alpha$ in $H^q(\mathcal{S}|_{X'}, F)$ is zero. This implies that the map $H^q(\mathcal{S}, F) \to H^q(\mathcal{S}', F)$ is injective since by the above discussion the map

$$\mathrm{Ker}(H^q(\mathcal{S}, F) \to H^q(\mathcal{S}|_{X'}, F)) \to \mathrm{Ker}(H^q(\mathcal{S}', F) \to H^q(\mathcal{S}'|_{X'}, F)) \tag{A.7.6}$$

is injective. Similarly, if $\beta \in H^q(\mathcal{S}', F)$ is a class, there exists by (A.1.2) an object $X$ isomorphic to a disjoint union of objects of $\mathcal{S}''$ such that $\beta$ lies in $K'_X$. Since the map $a$ is an isomorphism it follows that $\beta$ is in the image of $H^q(\mathcal{S}, F)$. This completes the inductive step and hence the proof of (A.6). □

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