LOGARITHMIC INTERPRETATION OF THE MAIN COMPONENT IN TORIC HILBERT SCHEMES

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1. Statement of theorem

We work over a base field k, except in the last section.

1.1. Fix a surjective morphism of fine monoids $\pi : P \to Q$, and let $j : A_Q \hookrightarrow A_P$ be the corresponding closed immersion, where $A_P := \mathrm{Spec}(k[P])$ and $A_Q := \mathrm{Spec}(k[Q])$. Assume further that the associated groups P^{gp} and Q^{gp} are torsion free, and that Q is sharp. Let T_P (resp. T_Q) denote the torus associated to the group P^{gp} (resp. Q^{gp}) so that T_P acts on A_P and T_Q acts on A_Q. The map π induces an inclusion of tori $\pi_T : T_Q \to T_P$, and the closed immersion j is compatible with the action of T_Q. Define a function $h : Q^{gp} \to \mathbb{N}$ by

$$h(q) := \begin{cases} 1 & \text{if } q \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

We can then consider the multigraded Hilbert functor of Haiman and Sturmfels [H-S]

$$\mathscr{H}^h_{A_P,T_Q} : (\text{schemes})^{\text{op}} \to \text{Set}$$

sending a scheme S to the set of T_Q-invariant closed immersions $Z \hookrightarrow A_{P,S}$ over S such that if $g : Z \to S$ is the structure morphism then for every $q \in Q^{gp}$ the q-eigenspace $(g_*\mathcal{O}_Z)_q$ of $g_*\mathcal{O}_Z$ is a finitely presented projective $\mathcal{O}_S$-module of rank $h(q)$. By [H-S, 1.1] the functor $\mathscr{H}^h_{A_P,T_Q}$ is representable by a quasi-projective scheme. To ease notation in what follows we write simply $\mathscr{H}$ for $\mathscr{H}^h_{A_P,T_Q}$ (since the above data will be fixed throughout the paper).

A (scheme-valued) point $u \in T_P(S)$ defines a map $u^* : \mathscr{H}(S) \to \mathscr{H}(S)$ by sending $i : Z \hookrightarrow A_{P,S}$ to the composite

$$Z \xrightarrow{i} A_{P,S} \xrightarrow{u} A_{P,S}.$$

Note that since Z is T_Q-invariant this map u^* depends only on the image of u in $T_P/T_Q = T_K$, where $K := \mathrm{Ker}(P^{gp} \to Q^{gp})$. We therefore obtain an action of T_K on $\mathscr{H}$.

There is a distinguished point $p_0 \in \mathscr{H}(\mathrm{Spec}(k))$ given by $j : A_Q \hookrightarrow A_P$. This defines a T_K-equivariant morphism (where T_K acts on itself by translation)

$$z : T_K \to \mathscr{H}, \quad u \mapsto u * p_0.$$

By [C, 3.4] the map z is an open immersion. We define the *main component* of $\mathscr{H}$ to be the scheme-theoretic closure of the image $z(T_K)$. Let $\mathscr{S}_{P\to Q}$ denote the normalization of the main component of $\mathscr{H}$.

The problem we propose a solution to in this note is the following:

Problem 1.2. *Define a reasonable (i.e. not the functor of points) functor which is represented by $\mathscr{S}_{P\to Q}$.*

1.3. The scheme $\mathscr{S}_{P\to Q}$ with its action of T_K is a normal toric variety and therefore has a natural fs log structure $M_{\mathscr{S}_{P\to Q}}$ such that $(\mathscr{S}_{P\to Q}, M_{\mathscr{S}_{P\to Q}})$ is log smooth over k. We will show that the fs log scheme $(\mathscr{S}_{P\to Q}, M_{\mathscr{S}_{P\to Q}})$ represents a logarithmic version of the multigraded Hilbert scheme.

1.4. Let M_{A_P} (resp. M_{A_Q}) denote the fine log structure on A_P (resp. A_Q) induced by the natural map $P \to k[P]$ (resp. $Q \to k[Q]$) so that the inclusion $A_Q \hookrightarrow A_P$ extends to a closed immersion of fine log schemes

$$(A_Q, M_{A_Q}) \hookrightarrow (A_P, M_{A_P}). \tag{1.4.1}$$

Note that the action of T_Q (resp. T_P) extends naturally to an action on (A_Q, M_{A_Q}) (resp. (A_P, M_{A_P})) and the closed immersion 1.4.1 is T_Q-equivariant.

1.5. Define a functor (the *logarithmic toric Hilbert functor*)

$$\mathscr{H}^{\log} : (\text{fs log schemes}/k)^{\text{op}} \to \text{Set}$$

by associating to any fs-log scheme (S, M_S) over k the set of commutative diagrams

$$\begin{array}{ccc} (Z, M_Z) & \xrightarrow{i} & (A_P, M_{A_P}) \times (S, M_S) \\ & {\scriptstyle g}\searrow & \downarrow \\ & & (S, M_S), \end{array} \tag{1.5.1}$$

such that the following hold:

(i) i is a T_Q-invariant closed immersion of fine log schemes;
(ii) g is log smooth and integral (this implies in particular that the underlying morphism $Z \to S$ is flat);
(iii) For every $q \in Q^{gp}$ the q-eigenspace $(g_*\mathcal{O}_Z)_q$ is a finitely presented projective $\mathcal{O}_S$-module of rank $h(q)$.
(iv) The map
$$P \to M_{(Z,M_Z)/(S,M_S)} := \text{Coker}(g^*M_S \to M_Z)$$
induced by i factors through Q.

Remark 1.6. In the case when the log structure on S is trivial, condition (iii) enables one to recover the log structure M_Z on Z from the underlying scheme. Indeed the monoid Q is obtained from the condition that $q \in Q^{gp}$ is in Q if and only if the eigenspace $(g_*\mathcal{O}_Z)_q$ is nonzero. Since each module $(g_*\mathcal{O}_Z)_q$ is locally free of rank 1, there is a T_Q-invariant open subscheme $Z^* \subset Z$ obtained by inverting local generators of the modules $(g_*\mathcal{O}_Z)_q$ for $q \neq 0$. Condition (ii) implies that for every geometric point $\bar{s} \to S$ the fiber $Z^*_{\bar{s}} \subset Z_{\bar{s}}$ is dense, and in fact equal to the maximal open T_Q-invariant subset in the toric variety $Z_{\bar{s}}$. From this it also follows that if $h : Z^* \to S$ is the projection, then $(h_*\mathcal{O}_{Z^*})_q$ is a projective module of rank 1 for all $q \in Q^{gp}$. Let $q \in Q$ be an element and let $p_1, p_2 \in P$ be two liftings of q to P. Let $\beta_1, \beta_2 \in \mathcal{O}_Z$ be the images of p_1 and p_2. Then β_1 and β_2 map to units in $h_*\mathcal{O}_{Z^*}$ and $\beta_1 \cdot \beta_2^{-1}$ is T_Q-invariant. Since the map $g_*\mathcal{O}_Z \to h_*\mathcal{O}_{Z^*}$ is injective, it follows that there exists a unique unit $u \in \mathcal{O}_S^*$ such that $\beta_1 = u\beta_2$. As above let K denote the kernel of $P^{gp} \to Q^{gp}$.

The preceding discussion implies that the map $P \to \mathcal{O}_Z$ induces a map $P_K \to \mathcal{O}_Z$ from the localization P_K of P, and the associated log structure is equal to M_Z.

It follows from this that the composite functor

$$(\text{schemes})^{\text{op}} \xrightarrow{S \mapsto (S, \mathcal{O}_S^*)} (\text{fs log schemes})^{\text{op}} \xrightarrow{\mathscr{H}^{\log}} \text{Set}$$

is representable by the open image of $z : T_K \to \mathscr{H}$.

A basic idea in the above definition of $\mathscr{H}^{\log}$ is that by allowing nontrivial log structures M_S on the base S, we also include specializations (i.e. degenerations) of points in $z(T_K)$, but the log deformation theory only allows deformations in the main component.

The main result of this paper, whose proof occupies the subsequent sections, is the following (see 2.9 for the morphism of functors relating $\mathscr{H}^{\log}$ and $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$):

Theorem 1.7. *The functor $\mathscr{H}^{\log}$ is represented by $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$.*

Note that the definition of $\mathscr{H}^{\log}$ in 1.5 makes sense not just over a field k but over $\mathbb{Z}$. In fact the only time in the above discussion where the base field is relevant is in the definition of the main component (since scheme-theoretic closure does not commute with base change), and the fact that the normalization of the main component is a toric variety. Write

$$\mathscr{H}_{\mathbb{Z}}^{\log} : (\text{fs log schemes}) \to \text{Set}$$

for this functor, so that for the field k above the functor $\mathscr{H}^{\log}$ is obtained by restricting $\mathscr{H}_{\mathbb{Z}}^{\log}$ to the category of fs log schemes over k. In the last section we explain how to generalize 1.7 to show that $\mathscr{H}_{\mathbb{Z}}^{\log}$ is representable by an integral version of $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$.

Remark 1.8. As pointed out to us by Sturmfels, one of the very appealing aspects of the multigraded Hilbert scheme is that tangent spaces of points can be computed by a simple formula. We discuss this and the relationship with the logarithmic tangent space of the toric variety $\mathcal{S}_{P\to Q}$ in 4.5.

1.9. (*Notation*). For a group G, we write $D(G)$ for the diagonalizable group scheme associated to G.

For a fine monoid P and ring R, we denote by $R[P]$ the monoid algebra on P, and for $p \in P$ we write $e_p \in R[P]$ for the image of p under the canonical map $P \to R[P]$.

1.10. (*Acknowledgements*). The author is grateful to Bernd Sturmfels for suggesting the problem in this paper, and Arthur Ogus for helpful conversations.

This work was partially supported by NSF grant DMS-0714086 and an Alfred P. Sloan Research Fellowship. Much of this work was done while the author visited the Institute Mittag-Leffler (Djursholm, Sweden). The author is grateful for the Institute's hospitality and excellent working conditions.

2. Logarithmic structure on the family over $\mathcal{S}_{P\to Q}$

2.1. The morphism $\mathcal{S}_{P\to Q} \to \mathscr{H}$ corresponds to a closed subscheme $j : \mathcal{Z} \hookrightarrow A_P \times \mathcal{S}_{P\to Q}$. Let $M_{A_P\times\mathcal{S}_{P\to Q}}$ be the product log structure on $A_P \times \mathcal{S}_{P\to Q}$ (given by $p_1^* M_{A_P} \oplus_{\mathcal{O}^*} p_2^* M_{\mathcal{S}_{P\to Q}}$ with its canonical map to the structure sheaf).

2.2. To construct the log structure on $\mathcal{Z}$, consider first the case when k is algebraically closed. In this case the toric variety $\mathcal{S}_{P\to Q}$ is given by a fan in $K_{\mathbb{R}}$ (recall that K is the kernel of $P^{gp}\to Q^{gp}$). Let us describe this fan in more detail (this description is derived from [C]). Let $L\subset K$ be a submonoid such that $\mathrm{Spec}(k[L])$ is an open subset of $\mathcal{S}_{P\to Q}$. The restriction of $\mathcal{Z}$ to $\mathrm{Spec}(k[L])$ can then be described as follows.

Let $E\subset Q\oplus P^{\mathrm{gp}}$ be the submonoid of elements (q,p) such that $\pi(p)=q$.

Lemma 2.3. *The restriction of $\mathcal{Z}\hookrightarrow A_P\times\mathcal{S}_{P\to Q}$ to $\mathrm{Spec}(k[K])\subset\mathcal{S}_{P\to Q}$ is equal to the closed immersion*

$$\mathrm{Spec}(k[E])\hookrightarrow\mathrm{Spec}(k[P\oplus K])$$

induced by the morphism of monoids $\gamma:P\oplus K\to E\subset Q\oplus P^{gp}$ sending (p,l) to $(\pi(p),p+l)$.

Proof. Let R be a ring and let $\rho_0:K\to R^*$ be the constant morphism sending all elements to 1. Then the point $p_0\in\mathscr{H}(\mathrm{Spec}(R))$ is given by the surjection

$$R[P]\simeq R\otimes_{\rho_0,R[K]}R[P\oplus K]\twoheadrightarrow R\otimes_{\rho_0,R[K]}R[E]\simeq R[Q].$$

Now let $\rho:K\to R^*$ be another homomorphism. We show that the R-valued point

$$(\mathcal{Z}_R\hookrightarrow A_P\times\mathrm{Spec}(R))\in\mathscr{H}(\mathrm{Spec}(R))$$

defined by the base change of $\mathcal{Z}$ along $\rho:k[K]\to R$ is equal to $\rho*p_0$.

To verify this we may replace R by a finite flat extension, and hence we may assume there exists an extension $\tilde{\rho}:P\to R^*$ of ρ. This extension also induces a morphism $\rho_E:E\to R^*$ by taking the composite

$$E\subset Q\oplus P^{gp}\xrightarrow{\mathrm{pr}}P^{gp}\xrightarrow{\tilde{\rho}}R^*.$$

The lemma then follows by noting that there is a commutative diagram

$$\begin{array}{ccc} R[P] & \twoheadrightarrow & R\otimes_{\rho_0,k[K]}k[E] \\ {\scriptstyle e_p\mapsto\tilde{\rho}(p)e_p}\uparrow & & \uparrow{\scriptstyle e_l\mapsto\rho_E(l)\otimes e_l} \\ R[P] & \twoheadrightarrow & R\otimes_{\rho,k[K]}k[E]. \end{array}$$

□

2.4. Let R be the coordinate ring of $\mathcal{Z}\times_{\mathcal{S}_{P\to Q}}\mathrm{Spec}(k[L])$ so we have a commutative diagram of rings

$$\begin{array}{ccc} k[P\oplus L] & \twoheadrightarrow & R \\ \uparrow & \searrow^{h} & \downarrow \\ k[L] & & \\ \downarrow & & \\ k[K] & \longrightarrow & k[E], \end{array}$$

where the map h is induced by the map of monoids γ. It follows that if $E_L\subset E$ denotes the image of $P\oplus L$, then R is equal to the monoid algebra $k[E_L]$. The morphism of monoids $L\to E_L$ is integral since this is equivalent to the flatness of the map of rings $k[L]\to k[E_L]$

[Og1, 4.3.7]. Conversely, if we start with a submonoid $L \subset K$ and the resulting map $L \to E_L$ is integral, then

(2.4.1)
$$\begin{array}{ccc} \mathrm{Spec}(k[E_L]) & \hookrightarrow & \mathrm{Spec}(k[P \oplus L]) \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(k[L]) \end{array}$$

defines a T_K-equivariant morphism $\mathrm{Spec}(k[L]) \to \mathcal{S}_{P\to Q}$, and therefore L lies in a cone of the fan defining $\mathcal{S}_{P\to Q}$.

Corollary 2.5. *The fan $\Sigma(P \to Q)$ defining $\mathcal{S}_{P\to Q}$ is characterized by the condition that a submonoid $L \subset K$ lies in a cone of $\Sigma(P \to Q)$ if and only if $L \to E_L$ is integral.*

2.6. Note that this characterization makes no reference to the ground field k. Furthermore, if k is not algebraically closed and $\mathcal{S}'_{P\to Q}$ temporarily denotes the toric variety over k associated to $\Sigma(P \to Q)$, then the families 2.4.1 define maps to $\mathcal{S}_{P\to Q}$ over the torus invariant open subsets of $\mathcal{S}'_{P\to Q}$. These maps are compatible by construction so we get a morphism $\mathcal{S}'_{P\to Q} \to \mathcal{S}_{P\to Q}$. This map of toric varieties becomes an isomorphism after base change to $\bar{k}$, and therefore is already an isomorphism over k. Thus even when k is not algebraically closed the toric variety $\mathcal{S}_{P\to Q}$ is defined by the fan $\Sigma(P \to Q)$.

2.7. This description also enables us to define a log structure $M_{\mathcal{Z}}$ on the universal family $\mathcal{Z} \subset A_P \times \mathcal{S}_{P\to Q}$. Let $L \subset K$ be a submonoid such that $\mathrm{Spec}(k[L])$ defines an open subset of $\mathcal{S}_{P\to Q}$. The restriction of $\mathcal{Z}$ to $\mathrm{Spec}(k[L])$ is then equal to $\mathrm{Spec}(k[E_L])$. Define $M_{\mathcal{Z}}$ to be the log structure associated to the map $E_L \to k[E_L]$. The morphism $\mathrm{Spec}(E_L \to k[E_L]) \to \mathrm{Spec}(L \to k[L])$ is log smooth since the map $K = L^{gp} \to E_L^{gp} = E^{gp}$ is injective with cokernel equal to Q^{gp} (which is torsion free by assumption). The following lemma implies that this construction is compatible with inclusions $L \hookrightarrow L'$ of submonoids of K. It therefore globalizes and we get a log structure $M_{\mathcal{Z}}$ on $\mathcal{Z}$.

Lemma 2.8. *Let $L \subset L' \subset K$ be submonoids whose associated groups are equal to K. Then the diagram*

$$\begin{array}{ccc} L & \longrightarrow & E_L \\ \downarrow & & \downarrow \\ L' & \longrightarrow & E_{L'} \end{array}$$

is cocartesian.

Proof. Let N denote the pushout $L' \oplus_L E_L$ so we have a map $N \to E_{L'}$ which we want to show is an isomorphism. The surjectivity is clear by the definition of $E_{L'}$. For the injectivity, note that the functor of taking the group associated to a monoid commutes with pushouts, and therefore $N^{gp} \simeq K \oplus_K E_L^{gp} \simeq E_L^{gp}$. This implies that the map $E_L^{gp} \to E_{L'}^{gp}$ is injective (since both are subgroups of $Q^{gp} \oplus P^{gp}$), and therefore $N \to E_{L'}$ is also injective. □

2.9. Let

$$\mathscr{H}_{\mathcal{S}} : (\text{fs log schemes})^{\mathrm{op}} \to \mathrm{Set}$$

denote the functor of points of the log scheme $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$. The diagram

$$\begin{array}{ccc} (\mathcal{Z}, M_{\mathcal{Z}}) & \hookrightarrow & (A_P, M_{A_P}) \times (\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}}) \\ & \searrow & \downarrow \\ & & (\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}}) \end{array}$$

defines a $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$-valued point of $\mathscr{H}^{\log}$ and therefore a morphism of functors

$$F : \mathscr{H}_{\mathcal{S}} \to \mathscr{H}^{\log}. \qquad (2.9.1)$$

We show in the following sections that this morphism of functors is an isomorphism.

3. Calculation of the log differentials

Let k be a field, L a fine sharp monoid, and let M_k denote the log structure on $\mathrm{Spec}(k)$ defined by the map $L \to k$ sending all nonzero elements to 0 (so $M_k = \mathcal{O}^*_{\mathrm{Spec}(k)} \oplus L$).

Recall that a log scheme (T, M_T) is called *hollow* if for any local section $m \in M_T$ the image of m in $\mathcal{O}_T$ is either a unit or zero. This implies that the sheaf $\overline{M}_T$ is a constant sheaf on each connected component of T (and the converse holds if T is reduced).

Proposition 3.1. *Let (T, M_T) be a hollow log scheme. Then any log smooth morphism $f : (T, M_T) \to (\mathrm{Spec}(k), M_k)$ is strict. In particular the underlying morphism of schemes $T \to \mathrm{Spec}(k)$ is smooth and $\Omega^1_{(T,M_T)/(k,M_k)} = \Omega^1_{T/k}$.*

Proof. Fix a log smooth morphism $f : (T, M_T) \to (\mathrm{Spec}(k), M_k)$. The assertion that f is strict is étale local on T. Furthermore, it suffices to verify the proposition after making a field extension of k, so we may assume that k is algebraically closed. By [Og1, 3.3.1] there exists étale locally on T an injective map of monoids $\gamma : L \to N$ such that the order of the torsion part of N^{gp}/L^{gp} is invertible in k and a chart $\beta : N \to M_T$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{f^b} & M_T \\ & {\scriptstyle\gamma}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & N & \end{array}$$

commutes, and such that the induced morphism

$$T \to \mathrm{Spec}(k \otimes_{k[L]} k[N])$$

is étale. Let $t \in T(k)$ be a closed point (note that T/k is locally of finite type since f is log smooth). The chart β defines a map $\beta_t : N \to k$. Let F denote the face $\beta_t^{-1}(k^*)$, and let N_F denote the localization of N at F. Then the image of t in $\mathrm{Spec}(k \otimes_{k[L]} k[N])$ is contained in the open subset $\mathrm{Spec}(k \otimes_{k[L]} k[N_F])$. Replacing N by N_F and shrinking correspondingly around t on T, we may assume that $F \subset N$ is in fact equal to N^*.

Let $(k \otimes_{k[L]} k[N])^\wedge$ denote the completion of $k \otimes_{k[L]} k[N]$ along the maximal ideal $\mathfrak{m}$ generated by the elements $\beta(n) - e_n$ ($n \in N^*$) and e_n ($n \in N - N^*$). Since $T \to \mathrm{Spec}(k \otimes_{k[L]} k[N])$ is étale and T is hollow, we must have that the image of any $n \in N - N^*$ in $(k \otimes_{k[L]} k[N])^\wedge$ is zero. On the other hand, consider the quotient of $k \otimes_{k[L]} k[N]$ by the ideal $\mathfrak{p}$ generated by

$\mathfrak{m}^2$ and the elements $\beta(n) - e_n$ for $n \in N^*$. As a k-vector space the quotient $k \otimes_{k[L]} k[N]/\mathfrak{p}$ has basis given by e_0 and e_n for a set of liftings $n \in N$ of the irreducible elements in N/N^* which are not in the image of L. Since the image of N in this ring is also zero we must have that the map $L \to N/N^*$ is an isomorphism, and so $N = L \oplus N^*$ and f is strict. □

Proposition 3.2. *Let* $(\mathrm{Spec}(k), M_k)$ *be as above, and let*

$$((Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (\mathrm{Spec}(k), M_k)) \in \mathscr{H}^{\log}(\mathrm{Spec}(k), M_k)$$

be an object.

(i) Z *is geometrically connected.*

(ii) The rank of the locally free sheaf $\Omega^1_{(Z,M_Z)/(k,M_k)}$ *is equal to the rank of the group* Q^{gp}.

Proof. We may without loss of generality assume that k is algebraically closed.

Let $R = \oplus_{q\in Q} R_q$ denote the coordinate ring of Z, where R_q is the 1-dimensional vector space on which T_Q acts through the character q. Statement (i) follows from the fact that R_0 is 1-dimensional.

Let $Z_i \subset Z$ be an irreducible component, which we view as a subscheme with the reduced structure, and let $R^{(i)}$ denote the coordinate ring of Z_i so we have a surjection

$$\pi_i : R \to R^{(i)}.$$

Since T_Q is connected, the action of T_Q on Z preserves Z_i, and the map π_i is compatible with the T_Q-actions. Let $Q^{(i)} \subset Q$ denote the subset of characters $q \in Q$ for which the q-eigenspace in $R^{(i)}$ is nonzero. Then $Q^{(i)} \subset Q$ is a submonoid, and Z_i is a toric variety with action of the torus $T_{Q^{(i)}}$. Fix a $T_{Q^{(i)}}$-equivariant embedding $j_i : T_{Q^{(i)}} \hookrightarrow Z_i$. Then the image of any element of P under the composite

$$P \to \Gamma(Z, M_Z) \to \Gamma(Z_i, \mathcal{O}_{Z_i}) \to k[Q^{(i),gp}]$$

is either a unit or zero, since the image of any $p \in P$ in $k[Q^{(i),gp}]$ is contained in an eigenspace for the action of T_Q. Therefore the restriction of M_Z to $T_{Q^{(i)}} \subset Z_i$ is hollow. By 3.1 it follows that the rank of $\Omega^1_{(Z,M_Z)/(k,M_k)}$ is equal to the rank of the group $Q^{(i),gp}$. It therefore suffices to show that for some i the rank of $Q^{(i),gp}$ is equal to the rank of Q^{gp}.

Let $J \subset R$ be the nil-radical. The nonzero eigenspaces for the T_Q-action on R/J are given by the union $\cup_i Q^{(i)} \subset Q$. The R/J-module $\mathrm{gr}_J(R)^+ := \oplus_{n\geq 1} J^n/J^{n+1}$ is finitely generated, so there exists elements $q_1, \dots, q_r \in Q$ such that every nonzero eigenspace occurring in $\mathrm{gr}_J(R)^+$ is of the form $q^{(i)} + q_j$ for some $q^{(i)} \in Q^{(i)}$ and some q_j. Since for every $q \in Q$ the eigenspace R_q is nonzero, it follows that the union of the $Q^{(i)}$ and the finitely many translates $Q^{(i)} + q_j$ is equal to all of Q. Therefore some $Q^{(i),gp}$ must have rank equal to the rank of Q^{gp}. □

Example 3.3. Let $P = \mathbb{N}^2$ with generators e_1 and e_2, $Q = \mathbb{N}$, and let $\pi : P \to Q$ be the map sending e_1 to 2 and e_2 to 1. The map on tori $T_Q \to T_P$ is then the map $\mathbb{G}_m \to \mathbb{G}_m^2$ sending u to (u^2, u). The kernel K of $P^{gp} \to Q^{gp}$ is generated by the element $(-1, 2)$. Let $L \subset K$ be the submonoid generated by $(-1, 2)$, and let $E_L \subset \mathbb{N} \oplus \mathbb{Z}^2$ be as in 2.4. The map $L \to E_L$ is automatically integral since L is generated by a single element. The monoid $E_L \subset \mathbb{N} \oplus \mathbb{Z}^2$ is the submonoid generated by the elements $(2, 1, 0)$, $(1, 0, 1)$, and $(0, -1, 2)$. From this it follows that $\mathrm{Spec}(k[E_L]) \to \mathrm{Spec}(k[L])$ is isomorphic to the family

$$\mathrm{Spec}(k[t,x,y]/(y^2 = tx)) \to \mathrm{Spec}(k[t]).$$

Note that the fiber over $t = 0$ with the reduced structure is isomorphic to $k[x]$, and the corresponding submonoid of $Q = \mathbb{N}$ is the submonoid generated by 2. Thus in general the $Q^{(i),gp} \subset Q^{gp}$ occurring in the proof of 3.2 may be strictly smaller than Q^{gp} though they have the same rank.

Corollary 3.4. *For any fs log scheme (S, M_S) and object 1.5.1 of $\mathscr{H}^{\log}(S, M_S)$, the rank of the locally free sheaf $\Omega^1_{(Z,M_Z)/(S,M_S)}$ is equal to the rank of Q^{gp}.*

Proof. The rank of $\Omega^1_{(Z,M_Z)/(S,M_S)}$ can be calculated fiber-by-fiber so it suffices to consider the case when S is the spectrum of a field. □

3.5. For an fs-log scheme and object 1.5.1 of $\mathscr{H}^{\log}(S, M_S)$, let $\mathbb{L}_i$ denote the logarithmic cotangent complex (in the sense of [Ol1]) of the morphism i. The distinguished triangle

$$i^*\Omega^1_{(A_P,M_{A_P})\times(S,M_S)/(S,M_S)} \to \Omega^1_{(Z,M_Z)/(S,M_S)} \to \mathbb{L}_i$$

shows that $\mathbb{L}_i$ is a complex concentrated in degree -1 and that $\mathscr{H}^{-1}(\mathbb{L}_i)$ is a locally free sheaf of rank equal to $\mathrm{rk}(P^{gp}) - \mathrm{rk}(Q^{gp}) = \mathrm{rk}(K)$.

Proposition 3.6. *For any fs log scheme (S, M_S) and object 1.5.1 of $\mathscr{H}^{\log}(S, M_S)$, the map*

$$P^{gp} \otimes \mathcal{O}_Z \simeq i^*\Omega^1_{(A_P,M_{A_P}\times(S,M_S)/(S,M_S)} \to \Omega^1_{(Z,M_Z)/(S,M_S)}$$

induces an isomorphism

$$Q^{gp} \otimes \mathcal{O}_Z \simeq \Omega^1_{(Z,M_Z)/(S,M_S)}. \quad (3.6.1)$$

Equivalently, the map

$$\mathscr{H}^{-1}(\mathbb{L}_i) \to P^{gp} \otimes \mathcal{O}_Z$$

identifies $\mathbb{L}_i$ with $K \otimes \mathcal{O}_Z[1]$.

Proof. We may assume that S is affine. To prove the proposition, it suffices to show that the (surjective) map on duals

$$\mathrm{Hom}(P^{gp}, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_Z}(\mathscr{H}^{-1}(\mathbb{L}_i), \mathcal{O}_Z) \quad (3.6.2)$$

factors through $\mathrm{Hom}(K, \mathcal{O}_Z)$, for then the map

$$\mathrm{Hom}(K, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_Z}(\mathscr{H}^{-1}(\mathbb{L}_i), \mathcal{O}_Z)$$

is a surjective morphism of projective $\mathcal{O}_Z$-modules of the same rank whence an isomorphism.

To verify that 3.6.2 factors through $\mathrm{Hom}(K, \mathcal{O}_Z)$, it suffices to show that the composite

$$\mathrm{Hom}(P^{gp}, \mathcal{O}_S) \hookrightarrow \mathrm{Hom}(P^{gp}, \mathcal{O}_Z) \to \mathrm{Hom}_{\mathcal{O}_Z}(\mathscr{H}^{-1}(\mathbb{L}_i), \mathcal{O}_Z) \quad (3.6.3)$$

factors through $\mathrm{Hom}(K, \mathcal{O}_S)$.

For a log scheme (T, M_T) let $(T[\epsilon], M_{T[\epsilon]})$ denote the log scheme whose underlying scheme $T[\epsilon]$ is the scheme of dual numbers of T obtained as the spectrum over T of the ring $\mathcal{O}_T[\epsilon]/(\epsilon^2)$. The log structure $M_{T[\epsilon]}$ is induced by the composite map

$$M_T \longrightarrow \mathcal{O}_T \xrightarrow{f \mapsto f} \mathcal{O}_T[\epsilon].$$

There is a commutative triangle

$$\begin{array}{ccc} (T, M_T) & \stackrel{j}{\hookrightarrow} & (T[\epsilon], M_{T[\epsilon]}) \\ & \searrow{=} & \downarrow \\ & & (T, M_T), \end{array}$$

where j is the exact closed immersion defined by setting ϵ equal to 0.

This construction defines a lifting

$$\begin{array}{ccc} (Z[\epsilon], M_{Z[\epsilon]}) & \stackrel{i[\epsilon]}{\hookrightarrow} & (A_P, M_{A_P}) \times (S[\epsilon], M_{S[\epsilon]}) \\ & \searrow & \downarrow \\ & & (S[\epsilon], M_{S[\epsilon]}) \end{array} \tag{3.6.4}$$

of 1.5.1 to $\mathscr{H}^{\log}((S[\epsilon], M_{S[\epsilon]}))$. Let $\mathscr{D}_\epsilon$ denote the set of commutative diagrams

$$\begin{array}{ccc} (Z', M_{Z'}) & \stackrel{i'}{\hookrightarrow} & (A_P, M_{A_P}) \times (S[\epsilon], M_{S[\epsilon]}) \\ & \searrow & \downarrow \\ & & (S[\epsilon], M_{S[\epsilon]}), \end{array}$$

reducing to 1.5.1 modulo ϵ, where i' is a (not necessarily T_Q-equivariant) closed immersion, and $(Z', M_{Z'}) \to (S[\epsilon], M_{S[\epsilon]})$ is log smooth. By [Ol1, 5.2] the lifting 3.6.4 identifies the set $\mathscr{D}_\epsilon$ with $\mathrm{Hom}_{\mathcal{O}_Z}(\mathscr{H}^{-1}(\mathbb{L}_i), \mathcal{O}_Z)$. On the other hand, we can also view

$$\mathrm{Hom}(P^{gp}, \mathcal{O}_S) \simeq \mathrm{Ker}(T_P(S[\epsilon]) \to T_P(S))$$

as the S-valued points of the Lie algebra $\mathrm{Lie}(T_P)$ of T_P. Unwinding the definitions, one finds that with these identifications the map 3.6.3 sends an element $u \in \mathrm{Lie}(T_P)(S)$ to the element of $\mathscr{D}_\epsilon$

$$\begin{array}{ccc} (Z[\epsilon], M_{Z[\epsilon]}) & \stackrel{u \circ i[\epsilon]}{\hookrightarrow} & (A_P, M_{A_P}) \times (S[\epsilon], M_{S[\epsilon]}) \\ & \searrow & \downarrow \\ & & (S[\epsilon], M_{S[\epsilon]}) \end{array}$$

obtained from 3.6.4 by composing the inclusion i with the infinitesimal automorphism u of $(A_P, M_{A_P}) \times (S[\epsilon], M_{S[\epsilon]})$. Since $i[\epsilon]$ is T_Q-equivariant, it follows that 3.6.3 sends $\mathrm{Hom}(Q^{gp}, \mathcal{O}_S) = \mathrm{Lie}(T_Q)(S)$ to zero and therefore factors through $\mathrm{Hom}(K, \mathcal{O}_S)$. □

4. Deformation theory

4.1. Let B be a ring, $I \subset B$ a square zero ideal, and set $B_0 := B/I$. Let M_B be a fs log structure on $\mathrm{Spec}(B)$, and let M_{B_0} be the pullback to $\mathrm{Spec}(B_0)$, so we have an exact closed immersion

$$(\mathrm{Spec}(B_0), M_{B_0}) \hookrightarrow (\mathrm{Spec}(B), M_B).$$

Let

$$(4.1.1)\qquad \begin{array}{ccc} (Z_0, M_{Z_0}) & \xrightarrow{i_0} & (A_P, M_{A_P}) \times (\mathrm{Spec}(B_0), M_{B_0}) \\ & \searrow^{g_0} & \downarrow{\scriptstyle f_0} \\ & & (\mathrm{Spec}(B_0), M_{B_0}) \end{array}$$

be an element of $\mathscr{H}^{\log}((\mathrm{Spec}(B_0), M_{B_0}))$.

Proposition 4.2. *There exists a lifting of 4.1.1 to an element of* $\mathscr{H}^{\log}((\mathrm{Spec}(B), M_B))$*.*

Proof. By [Ol1, 5.6], the obstruction to finding a commutative diagram of fine log schemes (not necessarily T_Q-invariant)

$$(4.2.1)\qquad \begin{array}{ccc} (Z_0, M_{Z_0}) & \xrightarrow{\quad j \quad} & (Z, M_Z) \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle i} \\ (A_P, M_{A_P}) \times (\mathrm{Spec}(B_0), M_{B_0}) & \longrightarrow & (A_P, M_{A_P}) \times (\mathrm{Spec}(B), M_B), \end{array}$$

with $(Z, M_Z) \to (\mathrm{Spec}(B), M_B)$ log smooth, is a class in the group

$$\mathrm{Ext}^2(\mathbb{L}_{i_0}, I \otimes \mathcal{O}_{Z_0}) \simeq H^1(Z_0, \mathscr{H}^{-1}(\mathbb{L}_{i_0})^* \otimes I) = 0.$$

Note that [Ol1, 5.6] is not stated in sufficient generality, but the proof gives also this result. Therefore we can find a diagram 4.2.1. Let $\mathscr{D}$ denote the set of diagrams 4.2.1. By [Ol1, 5.6] the set $\mathscr{D}$ admits a natural torsorial action of

$$\mathrm{Ext}^1(\mathbb{L}_{i_0}, I \otimes \mathcal{O}_{Z_0}) \simeq H^0(Z_0, \mathscr{H}^{-1}(\mathbb{L}_{i_0})^* \otimes I) = \mathscr{H}^{-1}(\mathbb{L}_{i_0})^* \otimes I.$$

There is an action

$$T_Q \times \mathscr{D} \to \mathscr{D}, \quad (u, d) \mapsto u * d$$

of T_Q on the set $\mathscr{D}$. A point $u \in T_Q(B)$ sends the diagram 4.2.1 to the diagram

$$\begin{array}{ccc} (Z_0, M_{Z_0}) & \xrightarrow{\quad j \circ u_0^{-1} \quad} & (Z, M_Z) \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle u \circ i} \\ (A_P, M_{A_P}) \times (\mathrm{Spec}(B_0), M_{B_0}) & \longrightarrow & (A_P, M_{A_P}) \times (\mathrm{Spec}(B), M_B), \end{array}$$

where u_0 denotes the image of u in $T_Q(B_0)$.

Fix an element $d_0 \in \mathscr{D}$. We then obtain a map

$$(4.2.2)\qquad T_Q(B) \to \mathscr{H}^{-1}(\mathbb{L}_{i_0})^* \otimes I, \quad u \mapsto [u * d_0 - d_0].$$

This map in fact extends to a morphism of sheaves

$$(4.2.3)\qquad T_Q \to \mathscr{H}^{-1}(\mathbb{L}_{i_0})^* \otimes I$$

on the big étale site of $\mathrm{Spec}(B)$. Indeed to define such a map of sheaves it suffices to specific a distinguished element of $(\mathscr{H}^{-1}(\mathbb{L}_{i_0}) \otimes I)(T_Q)$. This we obtain by making the base change $T_Q \times \mathrm{Spec}(B) \to \mathrm{Spec}(B)$ and noting that the construction of the map 4.2.2 is compatible with smooth base change on $\mathrm{Spec}(B)$. We then obtain the distinguished element of $(\mathscr{H}^{-1}(\mathbb{L}_{i_0})\otimes I)(T_Q)$ by taking the image of the identity map in $T_Q(T_Q\times\mathrm{Spec}(B))$ under the map

$$T_Q(T_Q\times\mathrm{Spec}(B))\to\mathscr{H}^{-1}(\mathbb{L}_{i_0})^*\otimes I\otimes\mathcal{O}_{T_Q}$$

obtained from 4.2.2 by taking $\mathrm{Spec}(B)$ equal to $T_Q\times\mathrm{Spec}(B)$ in the above discussion.

It follows from the construction that the map 4.2.3 is compatible with the action of T_Q on $\mathscr{H}^{-1}(\mathbb{L}_{i_0})^*\otimes I$ induced by the action on $\mathbb{L}_{i_0}$ coming from functoriality of the log cotangent complex in the sense that for any local section $u,v\in T_Q$ we have

$$\rho(u\cdot v)=\rho(v)^u+\rho(u).$$

By [SGA3, I.5.2.1] this map defines an element in $H^1(T_Q,\mathscr{H}^{-1}(\mathbb{L}_{i_0})^*\otimes I)$ (cohomology of the $T_Q-\mathcal{O}_B$-module in the language of [SGA3, I]). By [SGA3, I.5.3.3] we have $H^1(T_Q,\mathscr{H}^{-1}(\mathbb{L}_{i_0})^*\otimes I)=0$. Therefore there exists an element $e\in\mathscr{H}^{-1}(\mathbb{L}_{i_0})^*\otimes I$ such that $\rho(u)=e^u-e$ for all local sections $u\in T_Q$. Replacing our choice of lifting d_0 by d_0-e we obtain a T_Q-invariant lifting. □

Corollary 4.3. *The set of liftings of 4.1.1 to* $\mathscr{H}^{\log}((\mathrm{Spec}(B),M_B))$ *is canonically a torsor under* $\mathrm{Hom}(K,I)$*.*

Proof. The proof of 4.2 also shows that the set of liftings of 4.1.1 is a torsor under

$$\mathrm{Hom}(\mathscr{H}^{-1}(\mathbb{L}_{i_0}),I\otimes\mathcal{O}_Z)^{T_Q}=\mathrm{Hom}(K,I).$$

□

Corollary 4.4. *The diagram*

$$\begin{array}{ccc}\mathscr{H}_{\mathcal{S}}(B,M_B) & \longrightarrow & \mathscr{H}^{\log}(B,M_B)\\ \downarrow & & \downarrow\\ \mathscr{H}_{\mathcal{S}}(B_0,M_{B_0}) & \longrightarrow & \mathscr{H}^{\log}(B_0,M_{B_0})\end{array}$$

is cartesian.

Remark 4.5. Consider an object 1.5.1 with $S=\mathrm{Spec}(k)$ for some field k. Let $J\subset k[P]$ be the ideal defining the closed immersion $Z\hookrightarrow A_{P,k}$. By [H-S, 1.6], the tangent space of the multigraded Hilbert scheme $\mathscr{H}$ at the k-valued point $[Z]\in\mathscr{H}(k)$ defined by Z is canonically isomorphic to

$$\mathrm{Hom}_{\mathscr{O}_Z}(J/J^2,\mathscr{O}_Z)^{T_Q}.$$

On the other hand, the differential $d:J\to\Omega^1_{(A_{P,k},M_{A_{P,k}}\times M_S)/(S,M_S)}\simeq P^{gp}\otimes_{\mathbb{Z}}k[P]$ induces a map $\bar{d}:J/J^2\to P^{gp}\otimes\mathscr{O}_Z$ whose composite with the projection

$$P^{gp}\otimes\mathscr{O}_Z\to Q^{gp}\otimes\mathscr{O}_Z\simeq\Omega^1_{(Z,M_Z)/(S,M_S)}$$

is zero. The map $\bar{d}$ therefore factors through a map $J/J^2\to K\otimes\mathscr{O}_Z$, which we again denote by $\bar{d}$. Dualizing we obtain a map

$$\mathrm{Hom}(K,k)\simeq\mathrm{Hom}_{\mathscr{O}_Z}(K\otimes\mathscr{O}_Z,\mathscr{O}_Z)^{T_Q}\xrightarrow{\bar{d}^*}\mathrm{Hom}_{\mathscr{O}_Z}(J/J^2,\mathscr{O}_Z)^{T_Q}.$$

Unwinding the definitions and granting for the moment 1.7, one finds that this map is identified with the map on tangent spaces induced by the map $\mathcal{S}_{P\to Q}\hookrightarrow\mathscr{H}$.

5. Injectivity of $\mathscr{H}_{\mathcal{S}} \to \mathscr{H}^{\log}$

Proposition 5.1. *Let (T, M_T) be an fs log scheme. Then the map $\mathscr{H}_{\mathcal{S}}(T, M_T) \to \mathscr{H}^{\log}(T, M_T)$ is injective.*

Proof. Let

$$f_1, f_2 : (T, M_T) \to (\mathcal{S}, M_{\mathcal{S}})$$

be two morphisms defining the same element of $\mathscr{H}^{\log}(T, M_T)$. To prove that $f_1 = f_2$, we may by a standard limit argument assume that T is of finite type over k. In this case it further suffices to verify that $f_1 = f_2$ at the completion of T at a point, and for this in turn it suffices to verify that $f_1 = f_2$ modulo each power of the maximal ideal. We may therefore assume that T is the spectrum of an artinian local ring. By 4.4 it then suffices to consider the case when T is the spectrum of a field k' and there exists an fs monoid L and a morphism $L \to M_T$ inducing an isomorphism $L \simeq \overline{M}_T$.

Let R denote the completion of $k'[L]$ at the maximal ideal obtained by setting the nonzero elements of L to 0, let M_R denote the log structure induced by $L \to R$, and let $\mathfrak{m} \subset R$ be the maximal ideal. Since $(\mathcal{S}, M_{\mathcal{S}})$ is log smooth there exists a morphism

$$\tilde{f}_1 : (\mathrm{Spec}(R), M_R) \to (\mathcal{S}, M_{\mathcal{S}})$$

lifting f_1. Then by 4.4 there exists a unique morphism

$$\tilde{f}_2 : (\mathrm{Spec}(R), M_R) \to (\mathcal{S}, M_{\mathcal{S}})$$

such that for each integer $n \geq 0$ the elements of $\mathscr{H}^{\log}(R_n, M_{R_n})$ defined by $\tilde{f}_1$ and $\tilde{f}_2$ are equal, where R_n denotes the quotient of R by $\mathfrak{m}^{n+1}$. Since M_R is a subsheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$, to verify that $\tilde{f}_1 = \tilde{f}_2$ it suffices to show that the underlying morphisms of schemes $\mathrm{Spec}(R) \to \mathcal{S}$ are equal. For this in turn it suffices to show that the composites to the multigraded Hilbert scheme

$$\mathrm{Spec}(R) \to \mathcal{S} \to \mathscr{H} \tag{5.1.1}$$

are equal, since the generic point of $\mathrm{Spec}(R)$ maps to the open subset of $\mathcal{S}$ where $\mathcal{S} \to \mathscr{H}$ is an open immersion. But the statement that the two morphisms 5.1.1 are equal is clear, because by assumption the morphisms

$$\mathrm{Spec}(R_n) \to \mathscr{H}$$

obtained by reduction modulo each power of $\mathfrak{m}$ are equal. □

6. The functor $\mathscr{H}^{\log}$ is limit preserving

Proposition 6.1. *Let*

$$\cdots \to (\mathrm{Spec}(R_{i+1}), M_{R_{i+1}}) \to (\mathrm{Spec}(R_i), M_{R_i}) \to \cdots$$

be a filtering projective system of noetherian affine fs log schemes with strict transition maps, and set

$$(\mathrm{Spec}(R), M_R) := \varprojlim (\mathrm{Spec}(R_i), M_{R_i})$$

Then the map

$$\varinjlim \mathscr{H}^{\log}(\mathrm{Spec}(R_i), M_{R_i}) \to \mathscr{H}^{\log}(\mathrm{Spec}(R), M_R) \tag{6.1.1}$$

is a bijection.

Proof. The injectivity of 6.1.1 follows from the fact that the classifying stacks of log structures defined in [Ol2] are locally of finite type over $\mathbb{Z}$. Moreover, given an object $(Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (\mathrm{Spec}(R), M_R)$ one can find a T_Q-equivariant closed immersion

$$(Z_i, M_{Z_i}) \hookrightarrow (A_P, M_{A_P}) \times (\mathrm{Spec}(R_i), M_{R_i})$$

over some R_i inducing (Z, M_Z), such that such that the underlying closed immersion $Z_i \hookrightarrow A_P \times \mathrm{Spec}(R_i)$ is an element of $\mathscr{H}(R_i)$. The only issue then is to show that after possibly replacing i by some bigger index, the map $(Z_i, M_{Z_i}) \to (\mathrm{Spec}(R_i), M_{R_i})$ is log smooth. This follows from the following lemma applied to the closed subset $W_i \subset Z_i$ where $(Z_i, M_{Z_i}) \to (\mathrm{Spec}(R_i), M_{R_i})$ fails to be log smooth. □

Lemma 6.2. *Let S be a scheme, and $Z \hookrightarrow A_{P,S}$ an S-valued point of $\mathscr{H}$. Then for any T_Q-invariant closed subset $W \subset Z$ the image of W in S is closed.*

Proof. View W as a scheme with the reduced structure. Let $\pi : Z \to S$ be the affine projection. The ideal sheaf $J \subset \mathcal{O}_Z$ defining W corresponds to a T_Q-invariant sheaf of ideals $\pi_* J \subset \pi_* \mathcal{O}_Z$ and is therefore given by $\mathcal{O}_S$-submodules $(\pi_* J)_q \subset (\pi_* \mathcal{O}_Z)_q$. The image of W in S is then equal to the closed subscheme defined by the ideal $(\pi_* J)_0 \subset \mathcal{O}_S$. □

7. Surjectivity of $\mathscr{H}_{\mathfrak{S}} \to \mathscr{H}^{\log}$

The following result completes the proof of 1.7.

Proposition 7.1. *Let (T, M_T) be an fs log scheme. Then the map*

$$\mathscr{H}_{\mathfrak{S}}(T, M_T) \to \mathscr{H}^{\log}(T, M_T)$$

is surjective.

Proof. Consider first the case when $T = \mathrm{Spec}(k)$ and M_T is given by an fs monoid L and a map $L \to k$ sending all nonzero elements to zero.

Let $(Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (\mathrm{Spec}(k), M_k)$ be an object of $\mathscr{H}^{\log}(\mathrm{Spec}(k), M_k)$. By assumption the map $P \to H^0(Z, M_{(Z,M_Z)/(k,M_k)})$ factors through Q.

Lemma 7.2. *Let $g : (Z, M_Z) \to (\mathrm{Spec}(k), M_k)$ be the projection. Then the natural map $k^* \oplus L^{gp} \to H^0(Z, g^* M_k^{gp})^{T_Q}$ is an isomorphism, where $H^0(Z, g^* M_k^{gp})^{T_Q} \subset H^0(Z, g^* M_k^{gp})$ denotes the sections invariant under T_Q.*

Proof. Note first of all that $g^{-1}\overline{M}_k^{gp}$ is the constant sheaf defined by L^{gp}, and since Z is connected we therefore have $H^0(Z, g^{-1}\overline{M}_k^{gp}) = L^{gp}$. Consideration of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(Z, \mathcal{O}_Z^*) & \longrightarrow & H^0(Z, \mathcal{O}_Z^*) \oplus L^{gp} & \longrightarrow & L^{gp} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \simeq & & \\
0 & \longrightarrow & H^0(Z, \mathcal{O}_Z^*) & \longrightarrow & H^0(Z, g^* M_k^{gp}) & \longrightarrow & H^0(Z, g^{-1}\overline{M}_k^{gp}) & &
\end{array}$$

shows that the map

$$H^0(Z, \mathcal{O}_Z^*) \oplus L^{gp} \to H^0(Z, g^* M_k^{gp})$$

is an isomorphism. Since $H^0(Z, \mathcal{O}_Z)^{T_Q} = k$, this implies the lemma. $\square$

Since $K \subset P^{gp}$ maps to zero in $H^0(Z, M^{gp}_{(Z,M_Z)/(k,M_k)})$ and T_Q acts trivially on the image of K in $H^0(Z, M_Z^{gp})$, we obtain a map $\rho : K \to k^* \oplus L^{gp} = H^0(Z, g^*M_k^{gp})^{T_Q}$. Let $\Sigma \subset K$ denote $\rho^{-1}(k^* \oplus L)$, so that $\Sigma \subset K$ is a submonoid. We then get a map

$$i^* + \rho : P^{gp} \oplus K \to H^0(Z, M_Z^{gp}) \tag{7.2.1}$$

which sends $P \oplus \Sigma$ to $H^0(Z, M_Z)$. Let $E_\Sigma \subset Q \oplus P^{gp}$ denote the image of $P \oplus \Sigma$ as in 2.4. Since the composite

$$K \longrightarrow P^{gp} \oplus K \longrightarrow H^0(Z, M_Z^{gp})$$

is by construction zero, where the first arrow is $k \mapsto (k, -k)$, the map 7.2.1 descends to E_Σ and we obtain a commutative diagram

$$\begin{array}{ccc} \Sigma & \longrightarrow & E_\Sigma \\ \downarrow{\scriptstyle \rho} & & \downarrow \\ k^* \oplus L & \longrightarrow & H^0(Z, M_Z). \end{array} \tag{7.2.2}$$

Let $(V_\Sigma, M_{V_\Sigma}) \subset (A_P, M_{A_P}) \times (\mathrm{Spec}(k), M_k)$ denote the closed immersion obtained from the fiber product

$$\begin{array}{ccc} & & \mathrm{Spec}(E_\Sigma \to \mathrm{Spec}(k[E_\Sigma]) \\ & & \downarrow \\ (\mathrm{Spec}(k), M_k) & \xrightarrow{\rho} & \mathrm{Spec}(\Sigma \to k[\Sigma]). \end{array}$$

The diagram 7.2.2 then induces a morphism of log schemes compatible with the T_Q-actions

$$r : (Z, M_Z) \to (V_\Sigma, M_\Sigma) \tag{7.2.3}$$

which is a closed immersion since both are closed in $(A_P, M_{A_P}) \times (\mathrm{Spec}(k), M_k)$. We claim that r is an isomorphism.

Lemma 7.3. *The morphism r is log étale.*

Proof. Both (Z, M_Z) and (V_Σ, M_{V_Σ}) are log smooth over $(\mathrm{Spec}(k), M_k)$. It therefore suffices to show that the pullback map

$$r^*\Omega^1_{(V_\Sigma, M_{V_\Sigma})/(\mathrm{Spec}(k)M_k)} \to \Omega^1_{(Z,M_Z)/(\mathrm{Spec}(k),M_k)}$$

is an isomorphism. This is clear because this map is canonically identified with the identity map

$$\mathcal{O}_Z \otimes Q^{gp} \to \mathcal{O}_Z \otimes Q^{gp}.$$

$\square$

Lemma 7.4. *Let $f : (X, M_X) \to (Y, M_Y)$ be a log étale closed immersion, and let $(X^{\mathrm{sat}}, M_{X^{\mathrm{sat}}})$ (resp. $(Y^{\mathrm{sat}}, M_{Y^{\mathrm{sat}}})$) be the saturation of (X, M_X) (resp. (Y, M_Y)). Then the map*

$$f^{\mathrm{sat}} : (X^{\mathrm{sat}}, M_{X^{\mathrm{sat}}}) \to (Y^{\mathrm{sat}}, M_{Y^{\mathrm{sat}}})$$

is a strict closed and open immersion.

Proof. Let $\bar{x} \to X$ be a geometric point, and choose a a finitely generated group G and a map $G \to M^{gp}_{Y,\bar{x}}$ such that the induced map $G \to \overline{M}^{gp}_{Y,\bar{x}}$ is surjective. Since f is a closed immersion the composite map

$$G \to \overline{M}^{gp}_{Y,\bar{x}} \to \overline{M}^{gp}_{X,\bar{x}}$$

is also surjective. Define

$$P_Y := G \times_{M^{gp}_{Y,\bar{x}}} M_{Y,\bar{x}}, \quad P_X := G \times_{M^{gp}_{X,\bar{x}}} M_{X,\bar{x}}$$

so we have an inclusion $i : P_Y \subset P_X$. As in [Og1, 2.2.11], after replacing Y by an étale neighborhood of $\bar{x}$, the maps $P_Y \to M_{Y,\bar{x}}$ and $P_X \to M_{X,\bar{x}}$ extend to charts so we have a commutative diagram

$$\begin{array}{ccc} (X, M_X) & \xrightarrow{\quad f \quad} & (Y, M_Y) \\ \downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \beta_Y} \\ \mathrm{Spec}(P_X \to \mathbb{Z}[P_X]) & \xrightarrow{\ i\ } & \mathrm{Spec}(P_Y \to \mathbb{Z}[P_Y]), \end{array}$$

where the maps β_X and β_Y are strict. Now observe that since $\overline{M}^{gp}_{Y,\bar{x}} \to \overline{M}^{gp}_{X,\bar{x}}$ is surjective with kernel a finite group (since f is log étale), the map $P_Y \to P_X$ induces an isomorphism on saturations. Since

$$X^{\mathrm{sat}} = \mathrm{Spec}(\mathbb{Z}[P_X^{\mathrm{sat}}]) \times_{\mathrm{Spec}(\mathbb{Z}[P_X])} X \quad (\text{resp. } Y^{\mathrm{sat}} = \mathrm{Spec}(\mathbb{Z}[P_Y^{\mathrm{sat}}]) \times_{\mathrm{Spec}(\mathbb{Z}[P_Y])} Y)$$

with $M_{X^{\mathrm{sat}}}$ (resp. $M_{Y^{\mathrm{sat}}}$) induced by the natural map $P_X^{\mathrm{sat}} \to \mathcal{O}_{X^{\mathrm{sat}}}$ (resp. $P_Y^{\mathrm{sat}} \to \mathcal{O}_{Y^{\mathrm{sat}}}$), this implies that f^{sat} is a strict closed immersion.

On the other hand, from the commutative diagram

$$\begin{array}{ccc} (X^{\mathrm{sat}}, M_{X^{\mathrm{sat}}}) & \xrightarrow{f^{\mathrm{sat}}} & (Y^{\mathrm{sat}}, M_{Y^{\mathrm{sat}}}) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ (X, M_X) & \xrightarrow{\quad f \quad} & (Y, M_Y) \end{array}$$

where a, b, and f are log étale, we deduce that f^{sat} is also log étale, and hence since f^{sat} is strict the map $X^{\mathrm{sat}} \to Y^{\mathrm{sat}}$ is étale in the usual sense. It follows that f^{sat} is also an open immersion. □

Lemma 7.5. *Let* $(\mathrm{Spec}(k), M_k)$ *be as in the beginning of the proof of 7.1 (so we have a chart* $L \to M_k$*), and let* $L \to N$ *be a morphism of fine monoids with* N^{gp} *torsion free. Set* $(W, M_W) := \mathrm{Spec}(N \to k \otimes_{k[L]} k[N])$*. Then any log étale closed immersion* $(Z, M_Z) \hookrightarrow (W, M_W)$ *is an open and closed immersion.*

Proof. Let $(\mathcal{W}, M_{\mathcal{W}})$ denote $\mathrm{Spec}(N \to k[N])$, so that there is a strict closed immersion $(W, M_W) \hookrightarrow (\mathcal{W}, M_{\mathcal{W}})$. Let $z \in Z$ be a k-valued closed point and let $\widehat{\mathcal{O}}_{\mathcal{W},z}$ denote the completion of the local ring of $\mathcal{W}$ at z. We have $(\mathcal{W}^{\mathrm{sat}}, M_{\mathcal{W}^{\mathrm{sat}}})$ equal to $\mathrm{Spec}(N^{\mathrm{sat}} \to k[N^{\mathrm{sat}}])$. Let $\widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}$ denote the coordinate ring of the scheme $\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{W},z}) \times_{\mathcal{W}} \mathcal{W}^{\mathrm{sat}}$, so that $\widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}$ is a finite $\widehat{\mathcal{O}}_{\mathcal{W},z}$-algebra. Let (Z_z, M_{Z_z}) denote the fiber product $(Z, M_Z) \times_{(\mathcal{W}, M_{\mathcal{W}})} (\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{W},z}), M_{\widehat{\mathcal{O}}_{\mathcal{W},z}})$.

Since $(Z, M_Z) \hookrightarrow (W, M_W)$ is log étale, (Z_z, M_{Z_z}) lifts uniquely to a log étale closed immersion $(\mathcal{Z}_z, M_{\mathcal{Z}_z}) \hookrightarrow (\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{W},z}), M_{\widehat{\mathcal{O}}_{\mathcal{W},z}})$. On the other hand, since N^{gp} is torsion free, both $\widehat{\mathcal{O}}_{\mathcal{W},z}$ and $\widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}$ are integral domains. By 7.4 we conclude that the map

$$(\mathcal{Z}_z^{\mathrm{sat}}, M_{\mathcal{Z}_z^{\mathrm{sat}}}) \to (\mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}), M_{\widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}})$$

is an isomorphism. Since $\widehat{\mathcal{O}}_{\mathcal{W},z} \to \widehat{\mathcal{O}}_{\mathcal{W}^{\mathrm{sat}},z}$ is injective, it follows that $\mathcal{Z}_z \to \mathrm{Spec}(\widehat{\mathcal{O}}_{\mathcal{W},z})$ is étale at z. We conclude that $Z \to W$ is an open and closed immersion.

To check that the map $M_W \to M_Z$ is an isomorphism, we show that the map $\overline{M}_{\widehat{\mathcal{O}}_{\mathcal{W},z}} \to \overline{M}_{Z,z}$ is an isomorphism. Let S denote the localization of N at the face of elements mapping to units in $k(z)$, and let $\beta : S \to k(z)$ be the map defining z. Then $\widehat{\mathcal{O}}_{\mathcal{W},z}$ is equal to the completion $k[[S]]$ of $k[S]$ along the ideal defined by the elements $\beta(s) - e_s$, and $\overline{M}_{\widehat{\mathcal{W}},z}$ is equal to S/S^*. Let $s_1, s_2 \in S$ be two elements whose images in $\overline{M}_{Z,z}$ are equal. Then there exists an element $w \in k[[S]]^*$ such that $e_{s_1} = w \cdot e_{s_2}$. If $\overline{S}$ denotes S/S^* then the torus $D(\overline{S}^{gp})$ acts on the reduction $k[[S]]/\mathfrak{m}^{n+1}$ of $k[[S]]$ modulo each power of the maximal ideal. Now observe that since $\overline{S}$ is sharp, if

$$k[[S]]/\mathfrak{m}^{n+1} = \oplus_{\bar{s} \in \overline{S}} T_{\bar{s}}^{(n+1)}$$

denotes the character decomposition of $k[[S]]/\mathfrak{m}^{n+1}$ with respect to the $D(\overline{S}^{gp})$-action, then each element of $T_{\bar{s}}^{(n+1)}$ for $\bar{s} \neq 0$ is in the maximal ideal. On the other hand any point $u \in D(\overline{S}^{gp})(k)$ sends w to $u(s_1 - s_2) \cdot w$. Consequently, $s_1 - s_2 \in S^*$ which implies that s_1 and s_2 have the same image in $\overline{S}$. □

Applying 7.5 to 7.2.3, we conclude that our morphism r is an isomorphism (since V_Σ is connected). By A.1 this also implies that the map $\Sigma \to E_\Sigma$ is integral.

This completes the proof of 7.1 in the case when T is the spectrum of a field. From 4.4 we then deduce that 7.1 also holds when T is the spectrum of an artinian local ring.

Next we consider the case when $T = \mathrm{Spec}(R)$ is the spectrum of a complete noetherian local ring R. Let $\mathfrak{m} \subset R$ be the maximal ideal, and for any $n \geq 0$ let R_n denote the quotient $R/\mathfrak{m}^{n+1}$. Let $(Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (T, M_T)$ be an element of $\mathscr{H}^{\log}(T, M_T)$. For every $n \geq 0$, the reduction of this object modulo $\mathfrak{m}^{n+1}$ defines a unique morphism of fine log schemes $f_n : (\mathrm{Spec}(R_n), M_{R_n}) \to (\mathcal{S}_{P \to Q}, M_{\mathcal{S}_{P \to Q}})$, by the case of an artinian local ring already considered. It follows that we obtain a morphism $f : (\mathrm{Spec}(R), M_R) \to (\mathcal{S}_{P \to Q}, M_{\mathcal{S}_{P \to Q}})$ inducing the reduction of (Z, M_Z) modulo each power of the maximal ideal. In particular, the pullback of the scheme $\mathcal{Z} \to \mathcal{S}_{P \to Q}$ along f is equal to Z. On Z we therefore obtain a second quotient $i^* M_{A_P \times T} \to M'_Z$ such that modulo each power of the maximal ideal this quotient is equal to M_Z. We wish to show that M'_Z is equal to M_Z. For a geometric point $\bar{z} \to Z$ with image in the closed fiber, this follows from looking first at the completion of Z at z and then at the reductions modulo the maximal ideal. By specialization it follows that the two quotients are equal at any point of z whose closure meets the closed fiber of Z. Finally note that both quotients are T_Q-equivariant so to verify that the two quotients agree at a point $z \in Z$ it suffices to verify that they agree at some other point in the T_Q-orbit of z. By 6.2 any point $z \in Z$ contains a point $z' \in Z$ in its orbit such that the closure of z' meets the closed fiber of Z. This therefore completes the proof of 7.1 in the case of a complete noetherian local ring.

We now finish the proof of 7.1 by applying the Artin approximation theorem. So let (T, M_T) be any fs log scheme and $(Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (T, M_T)$ an element of $\mathscr{H}^{\log}(T, M_T)$. By the injectivity of $\mathscr{H}_{\mathcal{S}} \to \mathscr{H}^{\log}$, to prove that this element of $\mathscr{H}^{\log}(T, M_T)$ is in the image of $\mathscr{H}_{\mathcal{S}}(T, M_T)$ we may work étale locally on T. We may therefore assume that T is an affine scheme, and by 6.1 of finite type over k. Define

$$F : (T\text{-schemes})^{\mathrm{op}} \to \mathrm{Set}$$

to be the functor which to any $v : T' \to T$ associates the set of morphisms $f : (T', v^*M_T) \to (\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$ such that the image of f under the map

$$\mathscr{H}_{\mathcal{S}}(T', v^*M_T) \to \mathscr{H}^{\log}(T', v^*M_T)$$

is equal to the restriction to (T', v^*M_T) of our given object $(Z, M_Z) \hookrightarrow (A_P, M_{A_P}) \times (T, M_T)$. Using 6.1 one sees that F is a limit preserving functor. By the case of a complete noetherian local ring and the Artin approximation theorem [A, 1.12], we conclude that there exists an étale covering $T' \to T$ such that $F(T')$ is nonempty. This gives the desired morphism to $(\mathcal{S}_{P\to Q}, M_{\mathcal{S}_{P\to Q}})$ and therefore completes the proof of 7.1. $\square$

8. Representability of $\mathscr{H}^{\log}_{\mathbb{Z}}$

8.1. Let $(\mathcal{S}_{P\to Q,\mathbb{Z}}, M_{\mathcal{S}_{P\to Q,\mathbb{Z}}})$ denote the fs log scheme over $\mathrm{Spec}(\mathbb{Z})$ defined by the fan in 2.5. The constructions in 2.2 and 2.7 define an object of $\mathscr{H}^{\log}_{\mathbb{Z}}((\mathcal{S}_{P\to Q,\mathbb{Z}}, M_{\mathcal{S}_{P\to Q,\mathbb{Z}}}))$, and hence a morphism of functors

$$\mathscr{H}_{\mathcal{S},\mathbb{Z}} \to \mathscr{H}^{\log}_{\mathbb{Z}}, \tag{8.1.1}$$

where $\mathscr{H}_{\mathcal{S},\mathbb{Z}}$ denotes the functor on the category of fs log schemes represented by $(\mathcal{S}_{P\to Q,\mathbb{Z}}, M_{\mathcal{S}_{P\to Q,\mathbb{Z}}})$.

Proposition 8.2. *The map $\mathcal{S}_{P\to Q,\mathbb{Z}} \to \mathscr{H}$ from $\mathcal{S}_{P\to Q,\mathbb{Z}}$ to the multigraded Hilbert scheme (over $\mathbb{Z}$) defined by the family over $\mathcal{S}_{P\to Q,\mathbb{Z}}$ is proper.*

Proof. We verify the valuative criterion for properness. Let V be a complete discrete valuation ring with algebraically closed residue field k and fraction field F. Suppose given a commutative diagram of solid arrows

$$\begin{array}{ccc}
\mathrm{Spec}(F) & \xrightarrow{g_F} & \mathcal{S}_{P\to Q,\mathbb{Z}} \\
\downarrow & \overset{g_V}{\nearrow} & \downarrow \\
\mathrm{Spec}(V) & \xrightarrow{s} & \mathscr{H}.
\end{array}$$

We need to show that there exists a dotted arrow g_V filling in the diagram. Moreover, it suffices to consider the case when g_F factors through the dense open subset

$$\mathrm{Spec}(\mathbb{Z}[K]) \subset \mathcal{S}_{P\to Q,\mathbb{Z}}.$$

Let $\rho : K \to F^*$ be the map defining g_F, and let $V[P] \twoheadrightarrow R$ be the T_Q-equivariant surjection defining s. We then have a commutative diagram

$$\begin{array}{ccc} V[P] & \twoheadrightarrow & R \\ \downarrow & & \downarrow \\ F[P] & \longrightarrow & F \otimes_{\rho, F[K]} F[E], \end{array}$$

where $E \subset Q \oplus P^{gp}$ is defined as in 2.2. Let $L \subset K$ be the submonoid of elements $n \in K$ with $\rho(n) \in V$, and let $\rho_V : L \to V$ be the map induced by ρ. Then the map $P \oplus L \to R$ induced by the surjection $V \otimes_{\rho_V, V[L]} V[P \oplus L] \simeq V[P] \to R$ factors through $E_L \subset L$ since for this it suffices to show that the composite

$$P \oplus L \to R \to F \otimes_{\rho, F[K]} F[E]$$

factors through E which is clear. We therefore get a commutative diagram

$$\begin{array}{ccc} & V \otimes_{\rho_V, V[L]} V[P \oplus L] \simeq V[P] & \\ \swarrow & & \searrow \\ V \otimes_{\rho_V, V[L]} V[E_L] & \xrightarrow{\quad\gamma\quad} & R \end{array}$$

Lemma 8.3. *The map $L \to E_L$ is integral.*

Proof. The cokernel of the map $L \to E_L$ is equal to Q. Let $\pi : E_L \to Q$ be the projection. It suffices to show that for any $q \in Q$ there exists a lifting $\tilde{q}_0 \in \pi^{-1}(q)$ such that for any element $\tilde{q} \in \pi^{-1}(q)$ there exists an element $l \in L$ such that $\tilde{q} = \tilde{q}_0 + l$.

For this let $R_q \subset R$ be the q-eigenspace of R with respect to the T_Q-action. By assumption, R_q is a free V-module of rank 1. Since the map $V \otimes_{\rho_V, V[L]} V[E_L] \to R$ is surjective, there exists an element $\tilde{q}_0 \in \pi^{-1}(q)$ such that $\tilde{q}_0$ maps to a basis for R_q. If $\tilde{q} \in \pi^{-1}(q)$ is any other element, then $\tilde{q} - \tilde{q}_0$ is an element of K and $\rho(\tilde{q} - \tilde{q}_0)$ is in V since $\tilde{q}$ maps to a multiple of the image of $\tilde{q}_0$ in R_q. It follows that $\tilde{q} - \tilde{q}_0 \in L$. □

It follows that $V \to V \otimes_{\rho_V, V[L]} V[E_L]$ is flat, and therefore the map γ is an isomorphism, since for every q the map on eigenspaces

$$\gamma_q : (V \otimes_{\rho_V, V[L]} V[E_L])_q \to R_q$$

is a surjective map of rank 1 free V-modules, and therefore an isomorphism. From this we see that s is obtained by pullback along the map $\rho_V : \mathbb{Z}[L] \to V$ from the family over $\mathrm{Spec}(\mathbb{Z}[L]) \subset \mathcal{S}_{P \to Q, \mathbb{Z}}$. □

Corollary 8.4. *The morphism $\mathcal{S}_{P \to Q, \mathbb{Z}} \to \mathscr{H}$ identifies $\mathcal{S}_{P \to Q, \mathbb{Z}}$ with the normalization of the closure of the image of the map*

$$T_Q \to \mathscr{H}, \quad u \mapsto u * p_0,$$

where $p_0 \in \mathscr{H}(\mathbb{Z})$ is the point defined by $j : A_Q \hookrightarrow A_P$.

Proof. The projection $\mathcal{S}_{P \to Q, \mathbb{Z}} \to \mathscr{H}$ is quasi-finite, as this can be verified after base change to a field valued point of $\mathrm{Spec}(\mathbb{Z})$, where we already saw the result. Since the morphism is also proper, it is finite. Since $\mathcal{S}_{P \to Q, \mathbb{Z}}$ is normal, it follows that $\mathcal{S}_{P \to Q, \mathbb{Z}}$ is equal to the normalization of the closure in $\mathscr{H}$ of the image of the map $\mathcal{S}_{P\to Q,\mathbb{Z}} \times \mathrm{Spec}(\mathbb{Q}) \to \mathscr{H} \times \mathrm{Spec}(\mathbb{Q})$, where we again know the result by the case of a field. □

Theorem 8.5. *The map 8.1.1 is an isomorphism.*

Proof. Note first of all that the results of sections 3, 4, and 6 hold also over an arbitrary base scheme by exactly the same arguments.

To see that for any fs log scheme (T, M_T) the map

$$\mathscr{H}_{\mathcal{S},\mathbb{Z}}(T, M_T) \to \mathscr{H}_{\mathbb{Z}}^{\log}(T, M_T)$$

is injective, note that by the same argument used at the beginning of the proof of 5.1 it suffices to consider the case when T is the spectrum of a field, and in this case we have already shown the result.

The surjectivity of $\mathscr{H}_{\mathcal{S},\mathbb{Z}} \to \mathscr{H}_{\mathbb{Z}}^{\log}$ is also reduced to the case of a field by the same argument used in the proof of 7.1. □

Appendix A. A criterion for integrality

Proposition A.1. *Let L be a fine monoid and assume that $(L/L^*)^{gp}$ and L^{gp} are torsion free. Let $L \to P$ an injective morphism of fine monoids. Let k be a field and assume that the cokernel of the map $L^{gp} \to P^{gp}$ has order invertible in k. Let $\beta : L \to k$ be a morphism of monoids such that $\beta^{-1}(k^*) = L^*$. Then $L \to P$ is integral if and only if the morphism of log schemes*

(A.1.1) $$\mathrm{Spec}(P \to k \otimes_{\beta,k[L]} k[P]) \to \mathrm{Spec}(\ L \xrightarrow{\beta} k\ )$$

is integral.

Proof. We may without loss of generality assume that k is algebraically closed.

Let $\beta_0 : L \to k$ be the map obtained by composing the projection $L \to L/L^*$ with the map $L/L^* \to k$ sending all nonzero elements to 0. Let $\gamma : L^* \to k^*$ be the map induced by β. Since $(L/L^*)^{gp}$ is torsion free, there exists a map $\tilde{\gamma} : L^{gp} \to k^*$ extending γ. This map $\tilde{\gamma}$ defines an isomorphism of log schemes

$$\tilde{\gamma}, : \mathrm{Spec}(\ L \xrightarrow{\beta} k\ ) \to \mathrm{Spec}(\ L \xrightarrow{\beta_0} k\ )$$

which is the identity on $\mathrm{Spec}(k)$ and for which the map on log structures is induced by the map

$$L \to k^* \oplus L, \quad \ell \mapsto (\tilde{\gamma}(\ell)^{-1}, \ell).$$

Since k is algebraically closed and $L^{gp} \to P^{gp}$ is injective, we can also choose a map $\epsilon : P^{gp} \to k^*$ extending $\tilde{\gamma}$. We then obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(P \to k \otimes_{\beta,k[L]} k[P]) & \xrightarrow{\epsilon} & \mathrm{Spec}(P \to k \otimes_{\beta_0,k[L]} k[P]) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\ L \xrightarrow{\beta} k\ ) & \longrightarrow & \mathrm{Spec}(\ L \xrightarrow{\beta_0} k\ ), \end{array}$$

where the top horizontal arrow is the isomorphism induced by the map

$$P \to k^* \oplus P, \quad p \mapsto (\epsilon^{-1}(p), p).$$

It follows that it suffices to consider the case when $\beta = \beta_0$, which we assume for the rest of the proof.

Let $z_0 \in \mathrm{Spec}(k[L])$ denote the point defined by β.

The "only if" direction of the proposition is immediate.

For the "if" direction, note that it suffices to show that the morphism of rings $k[L] \to k[P]$ is flat. Let G denote P^{gp}/L^{gp} so we have an exact sequence of diagonalizable group schemes

$$0 \to D(G) \to D(P^{gp}) \to D(L^{gp}) \to 0.$$

The group $D(P^{gp})$ acts on $\mathrm{Spec}(k[P])$ covering the action of $D(L^{gp})$ on $\mathrm{Spec}(k[L])$. In particular, restricting the $D(P^{gp})$-action to $D(G)$ we obtain an action of $D(G)$ on $k[P]$ over $k[L]$. Let $k[P] = \oplus_{g \in G} M_g$ be the character decomposition of $k[P]$ with respect to this action. We need to show that each of the $k[L]$-modules M_g is flat over $k[L]$. Note that because $D(P^{gp})$ is commutative, the action of $D(P^{gp})$ induces an action of $D(P^{gp})$ on each M_g covering the action of $D(P^{gp})$ on $k[L]$ induced by the projection $D(P^{gp}) \to D(L^{gp})$. It follows that the closed set $Z_g \subset \mathrm{Spec}(k[L])$ of points $z \in \mathrm{Spec}(k[L])$ where the stalk of M_g is not flat over $\mathcal{O}_{\mathrm{Spec}(k[L]),z}$ is $D(L^{gp})$ invariant.

Lemma A.2. *Any nonempty $D(L^{gp})$-invariant closed subset $Z \subset \mathrm{Spec}(k[L])$ contains z_0.*

Proof. Let $z \in Z(k)$ be a closed point defined by a morphism $\gamma : L \to k$. Since L/L^* is sharp and $(L/L^*)^{gp}$ is torsion free, there exists a morphism of monoids $h : L/L^* \to \mathbb{N}$ such that $h^{-1}(0) = 0$. Consider the morphism

$$\rho : \mathbb{A}^1_k \times D(L^{gp}) \to \mathrm{Spec}(k[L])$$

defined by the map $L \to k[t, L^{gp}]$ sending l to $\gamma(l)t^{h(l)}e_l$. Then $\rho(1,1) = z$ and if $\tilde{\gamma} \in D(L^{gp})$ is an element extending $\gamma|_{L^*}$, then $\rho(0, \tilde{\gamma}^{-1}) = z_0$. Since the restriction of ρ to $\mathbb{G}_m \times D(L^{gp})$ is equal to the composite

$$\mathbb{G}_m \times D(L^{gp}) \xrightarrow{h+\mathrm{id}} D(L^{gp}) \xrightarrow{u \mapsto u*z} Z$$

and Z is closed, it follows that the image of ρ is contained in Z. □

It follows that to prove the integrality of the map $L \to P$, it suffices to show that the stalks of the M_g at the point $z_0 \in \mathrm{Spec}(k[L])$ are flat, and for this in turn it suffices to prove that the reductions modulo each power of the maximal ideal $\mathfrak{m}_{z_0}$ of $k[L]$ are flat. By assumption the morphism A.1.1 is integral. It follows that for each $n \geq 0$ the morphism of log schemes

$$\mathrm{Spec}(P \to (k[L]/\mathfrak{m}_{z_0}^{n+1}) \otimes_{k[L]} k[P]) \to \mathrm{Spec}(L \to k[L]/\mathfrak{m}_{z_0}^{n+1})$$

is log smooth and integral. Since a log smooth and integral morphism is flat, this concludes the proof that $L \to P$ is integral. □

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