HOM–STACKS AND RESTRICTION OF SCALARS

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Abstract. Fix an algebraic space S, and let $\mathcal{X}$ and $\mathcal{Y}$ be separated Artin stacks of finite presentation over S with finite diagonals (over S). We define a stack $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ classifying morphisms between $\mathcal{X}$ and $\mathcal{Y}$. Assume that $\mathcal{X}$ is proper and flat over S, and fppf–locally on S there exists a finite finitely presented flat cover $Z \to \mathcal{X}$ with Z an algebraic space. Then we show that $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack with quasi–compact and separated diagonal.

1. Statements of results

Fix an algebraic space S, let $\mathcal{X}$ and $\mathcal{Y}$ be separated Artin stacks of finite presentation over S with finite diagonals. Define $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ to be the fibered category over the category of S–schemes, which to any $T \to S$ associates the groupoid of functors $\mathcal{X}_T \to \mathcal{Y}_T$ over T, where $\mathcal{X}_T$ (resp. $\mathcal{Y}_T$) denotes $\mathcal{X} \times_S T$ (resp. $\mathcal{Y} \times_S T$).

Theorem 1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be finitely presented separated Artin stacks over S with finite diagonals. Assume in addition that $\mathcal{X}$ is flat and proper over S, and that locally in the fppf topology on S there exists a finite and finitely presented flat surjection $Z \to \mathcal{X}$ from an algebraic space Z. Then the fibered category $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack locally of finite presentation over S with separated and quasi–compact diagonal. If $\mathcal{Y}$ is a Deligne–Mumford stack (resp. algebraic space) then $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is also a Deligne–Mumford stack (resp. algebraic space).*

Remark 1.2. If S is the spectrum of a field and $\mathcal{X}$ is a Deligne–Mumford stack which is a global quotient stack and has quasi–projective coarse moduli space, then by ([15], 2.1) there always exists a finite flat cover $Z \to \mathcal{X}$.

Remark 1.3. When S is arbitrary and $\mathcal{X}$ is a twisted curve in the sense of ([1]), it is also true that étale locally on S there exists a finite flat cover $Z \to \mathcal{X}$ as in (1.1). This is shown in ([19]).

Remark 1.4. It is possible to prove that $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is an Artin stack under weaker assumptions on the diagonals of $\mathcal{X}$ and $\mathcal{Y}$ (though for the diagonal of $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ to have reasonable properties the assumptions of (1.1) seem necessary). This has recently been shown by Aoki.

Theorem 1.1 will be deduced from another result about pushforwards of stacks.

Let $\mathcal{S}/S$ be a separated Artin stack locally of finite presentation and with finite diagonal. For any morphism of algebraic spaces $f : S \to T$, define $f_*\mathcal{S}$ to be the fibered category over T which to any T'/T associates the groupoid $\mathcal{S}(T' \times_T S)$, with the natural notion of pullback. We call $f_*\mathcal{S}$ the *restriction of scalars* of $\mathcal{S}$ from S to T.

Theorem 1.5. *Let $f : S \to T$ be a proper, finitely presented, and flat morphism of algebraic spaces. Then the fibered category $f_*\mathcal{S}$ is an Artin stack locally of finite presentation over T*

with quasi–compact and separated diagonal. If $\mathcal{S}$ is a Deligne–Mumford stack (resp. algebraic space) then the stack $f_*\mathcal{S}$ is also a Deligne–Mumford stack (resp. algebraic space).

One can also consider variants of (1.1). Recall that a morphism of Artin stacks $g : \mathcal{U} \to \mathcal{V}$ is called *representable* if for any algebraic space V and morphism $V \to \mathcal{V}$ the fiber product $\mathcal{U} \times_{\mathcal{V}} V$ is an algebraic space. By ([16], 8.1.2) the morphism g is representable if and only if it is faithful as a functor between fibered categories. Another description of this notion is provided by ([5], 2.2.7) which shows that g is representable if and only if (it has separated diagonal and) for every geometric point $\bar{v} \to \mathcal{V}$ the fiber product $\mathcal{U} \times_{\mathcal{V}} \bar{v}$ is an algebraic space.

Corollary 1.6. *With the notation and hypothesis of* (1.1)*, let*

$$\underline{\mathrm{Hom}}_S^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y}) \subset \underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \tag{1.6.1}$$

be the fibered category of representable morphisms $\mathcal{X} \to \mathcal{Y}$. Then $\underline{\mathrm{Hom}}_S^{\mathrm{rep}}(\mathcal{X}, \mathcal{Y})$ is an open substack of $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$.

The proofs of (1.1), (1.5), and (1.6) are given in sections 3-5.

In the context of this paper it is also worth mentioning the following "boundedness" result which will be presented in a subsequent paper ([19]). Let $\mathcal{X}$ and $\mathcal{Y}$ be separated Deligne–Mumford stacks of finite presentation over S and define $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ as above. Assume that $\mathcal{X}$ is flat and proper over S, and that locally in the fppf topology on S, there exists a finite flat surjection $Z \to \mathcal{X}$ from an algebraic space Z. Let $\mathcal{Y} \to W$ be a finite surjection over S to a separated algebraic space W over S of finite presentation. By (1.1) we then have Deligne–Mumford stacks $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ and $\underline{\mathrm{Hom}}_S(\mathcal{X}, W)$.

Theorem 1.7. *The natural map*

$$\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}_S(\mathcal{X}, W) \tag{1.7.1}$$

is of finite type.

Note that in the case when S is the spectrum of a field, then $\mathcal{X}$ has a coarse moduli space $\pi : \mathcal{X} \to X$ and the formation of this moduli space commutes with arbitrary base change (see (2.11) below for a discussion of this). In this case the right side of (1.7.1) is canonically isomorphic to $\underline{\mathrm{Hom}}_S(X, W)$ by the universal property of coarse moduli spaces.

In the subsequent paper ([19]), we will also explain how the above result implies boundedness for the moduli space of twisted stable maps into a Deligne–Mumford stack ([1]).

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2. Preliminaries

In this section we collect for the convenience of the reader some standard facts used in what follows.

2.1. As in ([8], IV.8.2.2), let $(S_\alpha, u_{\alpha\beta})$ be a projective system of schemes with each morphism $u_{\alpha\beta} : S_\beta \to S_\alpha$ affine. Assume further that each S_α is quasi-compact and quasi-separated, and let S denote $\varprojlim_\alpha S_\alpha$.

Proposition 2.2. *Let $\mathcal{Q}$ be an Artin stack of finite presentation over S. Then there exists an Artin stack $\mathcal{Q}_\alpha$ of finite presentation over some S_α, and an isomorphism $\mathcal{Q}_\alpha \times_{S_\alpha} S \simeq \mathcal{Q}$.*

Consider the following properties of Artin stacks over S:

(i) *is Deligne–Mumford;*
(ii) *has finite diagonal;*
(iii) *is proper over S;*
(iv) *is separated over S;*
(v) *is flat over S.*

If $\mathcal{Q}$ has one or more of the properties (i)-(v)*, then $\mathcal{Q}_\alpha$ can also be chosen to have the same properties relative to S_α.*

Proof. Let

$$(U, R, \iota, m, p_1, p_2) \tag{2.2.1}$$

be a smooth groupoid presentation ([16], 4.3) for $\mathcal{Q}$ with U and R of finite presentation over S.

Let P be a property of morphisms of algebraic spaces occuring in the lists in ([8], IV.8.10.5) and ([8], IV.17.7.8) (in particular smooth or étale). If $f_\alpha : X_\alpha \to Y_\alpha$ is a morphism of schemes of finite presentation over some S_α such that the base change $f : X \to Y$ to S has a property P, then there exists some $\alpha' \geq \alpha$ such that the base change $f_{\alpha'} : X_{\alpha'} \to Y_{\alpha'}$ has property P. This follows from (loc. cit.). In particular, any smooth (resp. étale) groupoid in schemes over S can be approximated by a smooth (resp. étale) groupoid in schemes over some S_α.

Note also that if $\mathcal{S}_\alpha/S_\alpha$ is an Artin stack of finite presentation admitting a smooth groupoid presentation $R_\alpha \rightrightarrows U_\alpha$ with R_α and U_α schemes, and such that the base change $\mathcal{S}/S$ is an algebraic space, then there exists some $\alpha' \geq \alpha$ such that $\mathcal{S}_{\alpha'} := \mathcal{S}_\alpha \times_{S_\alpha} S_{\alpha'}$ is an algebraic space. Indeed consider the map $\sigma : R_\alpha \to U_\alpha \times_{S_\alpha} U_\alpha$, where $R_\alpha := U_\alpha \times_{\mathcal{S}_\alpha} U_\alpha$. By ([16], 8.1.1) the stack $\mathcal{S}_\alpha$ is an algebraic space if and only if σ is a monomorphism. The claim therefore follows from ([8], IV.8.10.5).

This proves in particular that algebraic spaces of finite presentation over S can be approximated. Since a morphism of algebraic spaces can be described by a morphism of groupoids, this also shows that morphisms of algebraic spaces of finite presentation over S can be approximated. Furthermore, by considering étale covers by schemes one sees that the analogues for algebraic spaces of ([8], IV.8.10.5 for properties (i)-(xi)) and ([8], IV.17.7.8) also hold.

Once we know that algebraic spaces of finite presentation over S and morphisms between them can be approximated, then by the same argument we can also approximate smooth (resp. étale) groupoids (2.2.1) with U and R algebraic spaces of finite presentation over S as well as morphisms of such groupoids. In this way we obtain the approximation $\mathcal{Q}_\alpha$ of $\mathcal{Q}$.

To see the statement about properties (i)–(v) proceed as follows.

The fact that a Deligne–Mumford stack $\mathcal{Q}$ can be approximated by a Deligne–Mumford stack $\mathcal{Q}_\alpha$ follows from the construction of $\mathcal{Q}_\alpha$ by choosing the groupoid (2.2.1) to be an étale groupoid and also approximating it by an étale groupoid.

For (iii) and (iv), consider first the case of algebraic spaces. Let $X_\alpha \to S_\alpha$ be an algebraic space of finite presentation with base change $X \to S$ to S, and choose an étale presentation $R_\alpha \rightrightarrows U_\alpha$ of X_α with R_α and U_α schemes. Then X_α is separated if and only if the diagonal map $z_\alpha : R_\alpha \to U_\alpha \times_{S_\alpha} U_\alpha$ is a closed immersion. Now if $X \to S$ is separated, then the base change $z : R \to U \times_S U$ of z_α to S is a closed immersion and therefore by the case of schemes ([8], IV.8.10.5 (iv)), there exists $\alpha' \geq \alpha$ such that the base change $z_{\alpha'} : R_{\alpha'} \to U_{\alpha'} \times_{S_{\alpha'}} U_{\alpha'}$ of z_α to $S_{\alpha'}$ is a closed immersion. This implies that the base change $X_{\alpha'}$ of X_α to $S_{\alpha'}$ is separated, and hence proves (iv) for algebraic spaces. Once (iv) has been established for algebraic spaces, statement (iii) for algebraic spaces follows from the same argument used in ([8], IV.8.10.5) using Chow's lemma for algebraic spaces ([14], IV.3.1).

For (ii) (resp. (iv)) for Artin stacks, let $\mathcal{Q}_\alpha$ be an approximation of $\mathcal{Q}$, let $U_\alpha \to \mathcal{Q}_\alpha$ be a smooth cover of finite presentation, and define $R_\alpha := U_\alpha \times_{\mathcal{Q}_\alpha} U_\alpha$. Then we need to show that if $\mathcal{Q}$ has finite (resp. proper) diagonal, then we can choose $\alpha' \geq \alpha$ such that the base change $R_{\alpha'} \to U_\alpha \times_{S_\alpha} U_\alpha$ of $R_\alpha \to U_\alpha \times_{S_\alpha} U_\alpha$ to $S_{\alpha'}$ is finite (resp. proper). This follows in the case when R_α and U_α for schemes from ([8], IV.8.10.5) and in the case of algebraic spaces from (iii) for algebraic spaces which enables one to carry through the proof of (loc. cit.) for algebraic spaces.

Statement (iii) follows from the same argument used for schemes in ([8], IV.8.10.5) using (iv) and Chow's lemma for stacks ([18], 1.1).

Finally for (v), let $\mathcal{Q}_\alpha$ be an Artin stack of finite presentation over S_α, and let $\mathcal{Q}$ denote the base change to S. Let $U_\alpha \to \mathcal{Q}_\alpha$ be a smooth cover with U_α a scheme of finite presentation over S_α. If $\mathcal{Q}$ is flat over S, then by ([8], IV.11.2.6) there exists some $\alpha' \geq \alpha$ such that the base change $U_{\alpha'}$ of U_α to $S_{\alpha'}$ is flat over $S_{\alpha'}$. This implies that $\mathcal{S}_{\alpha'} := \mathcal{S}_\alpha \times_{S_\alpha} S_{\alpha'}$ is also flat over $S_{\alpha'}$ which proves the statement about (v). □

Corollary 2.3. *In order to prove* (1.1), (1.6), *and* (1.7) *(resp.* (1.5)*) it suffices to consider the case when S (resp. T) is of finite type over an excellent Dedekind ring.*

Proof. The statements of (1.1), (1.6), and (1.7) are all local on S in the étale topology, and hence we may assume that S is an affine scheme, say $S = \mathrm{Spec}(A)$. Write A as a filtering inductive limit $A = \varinjlim A_\alpha$ where each A_α is a finite type $\mathbb{Z}$–algebra, and set $S_\alpha := \mathrm{Spec}(A_\alpha)$. Then by (2.2) we can approximate $\mathcal{X}$, $\mathcal{Y}$, and W in the above theorems by stacks over some S_α. The statement about (1.5) follows from the same argument. □

Next we gather together some facts about coarse moduli spaces. Let S be a scheme and $\mathcal{X}$ an Artin stack over S.

Definition 2.4. A *coarse moduli space for* $\mathcal{X}$ is a morphism $\pi : \mathcal{X} \to X$ from $\mathcal{X}$ to an algebraic space X such that the following conditions hold:

(i) If ξ the spectrum of an algebraically closed field, then the map π induces a bijection between the set of isomorphism classes of objects in $\mathcal{X}(\xi)$ and $X(\xi)$.
(ii) The map π is universal for maps from $\mathcal{X}$ to algebraic spaces.

Remark 2.5. In general, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are Artin stacks, the collection of morphisms $\mathcal{F}_1 \to \mathcal{F}_2$ form a category $\mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$. If $\mathcal{F}_2$ is an algebraic space, however, then this category is equivalent to a set. Property (2.4 (ii)) therefore implies that if a coarse moduli space for $\mathcal{X}$ exists then it is unique up to unique isomorphism. It also implies that there is a unique morphism $X \to S$ whose composite with π is equal to the structure morphism of $\mathcal{X}$, so X is canonically an S-space.

The basic existence result about coarse moduli spaces is the following:

Theorem 2.6 ([13])**.** *Assume that S is locally noetherian. Let $\mathcal{X}$ be an Artin stack of finite type over S with finite diagonal. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with X of finite type and separated over S. Furthermore the following properties hold:*

(i) *The morphism π is proper, quasi-finite, surjective, and the natural map $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.*
(ii) *If $X' \to X$ is an étale morphism and $\mathcal{X}' := X' \times_X \mathcal{X}$, then the projection $\pi' : \mathcal{X}' \to X'$ is a coarse moduli space for $\mathcal{X}'$. In particular, the formation of coarse moduli space commutes with étale base change $S' \to S$.*
(iii) *Let $q : \mathcal{X} \to Y$ be a proper S–morphism from $\mathcal{X}$ to an algebraic space Y separated and of finite type over S such that* (2.4 (i)) *holds for q, and such that the map $\mathcal{O}_Y \to q_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism. Then the map $h : X \to Y$ provided by* (2.4 (ii)) *is an isomorphism.*

Proof. The existence of the coarse moduli spaces is essentially ([13], 1.3). Strictly speaking it is only shown there that there exists a morphism $\pi : \mathcal{X} \to X$ with property (2.4 (i)) which is universal for morphisms to algebraic spaces locally of finite type over the base S. However, an inspection of the proof of the universal property ([13], 5.1) reveals that the same argument works without the locally of finite type assumption.

Statement (ii) is ([13], 3.2.1). For statement (i), note that by (ii) we may work étale locally on X. Now it follows from the arguments of ([13], 3.3) reviewed in the proof of (2.11) below (note that the argument given there only uses the existence of the coarse moduli space), that étale locally on the coarse space X there exists a finite surjection $U \to \mathcal{X}$. In this case, the map $U \to X$ is finite, surjective, and of finite type by ([13], 6.3) and hence by (2.7 (ii)) below the morphism $\mathcal{X} \to X$ is proper, quasi–finite, and surjective. The statement that $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$ follows from ([13], 1.3) which implies that ([13], 1.8 (F)) holds.

For (iii), note first that the morphism h is quasi–finite and surjective since it induces a bijection on geometric points by (2.4 (i)). It is also proper since $h \circ \pi = q$ is proper, h is separated and of finite type (being a morphism of separated S–spaces of finite type), and π is surjective (see (2.7 (ii)) below). Therefore h is quasi–finite and proper whence finite. Indeed to verify that h is finite we may work étale locally on Y, and hence the finiteness of h follows from ([14], II.6.16) which shows that the base change of X to an étale cover of Y by schemes is again a scheme and the result for schemes ([8], III.4.4.2). This implies that h is an isomorphism if and only if the morphism $\mathcal{O}_Y \to h_* \mathcal{O}_X$ is an isomorphism. Statement (iii) therefore follows from the fact that $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism. □

Lemma 2.7 (Stack version of ([8], II.5.4.3))**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be two separated morphisms of finite type of Artin stacks over S such that $g \circ f$ is proper. Then*

(i) *The morphism f is proper.*

(ii) *If f is surjective then g is proper.*

Proof. In case (ii), that g is universally closed and hence proper follows from the same argument used to prove ([8], II.5.4.3 (ii)).

Statement (i) follows from (ii) as follows. We can without loss of generality replace $\mathcal{Z}$ by a smooth cover of finite type over S, and hence may assume that $\mathcal{Z}$ is a scheme. Furthermore, we may work locally on S and hence may assume that S is quasi–compact. Using Chow's Lemma ([18], 1.1), choose a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{c} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y}, \end{array} \tag{2.7.1}$$

where X and Y are schemes and a and b are proper and surjective. Denote by $d : Y \to \mathcal{Z}$ the composite $g \circ b$. Since the composite of proper morphisms is proper, the morphism $d \circ c = (g \circ f) \circ a$ is proper. Hence by ([8], II.5.4.3) the morphism c is proper which implies that $b \circ c = f \circ a$ is also proper. Thus (i) follows from (ii) applied to $X \to \mathcal{X} \to \mathcal{Y}$. □

Corollary 2.8. *With notation as in* (2.6)*, if $\mathcal{X}$ is reduced (resp. integral) then the coarse moduli space X is also reduced (resp. integral).*

Proof. Let $\pi : \mathcal{X} \to X$ be the projection. Then $\mathcal{O}_X \to \pi_*\mathcal{O}_{\mathcal{X}}$ is an isomorphism by (2.6 (i)), and since $\mathcal{X}$ is reduced (resp. integral and π has connected geometric fibers) the sheaf $\pi_*\mathcal{O}_{\mathcal{X}}$ has no nonzero nilpotent sections (resp. no nonzero zero divisors). □

Corollary 2.9. *Let the notation be as in* (2.6)*, and assume further that $\mathcal{X}$ is reduced. Let $f : \mathcal{X} \to Z$ be a proper quasi-finite surjective morphism from $\mathcal{X}$ to an algebraic space Z separated and locally of finite type over S.*

(i) *If the morphism f satisfies* (2.4 (i)) *and also has a section, then f is a coarse moduli space for $\mathcal{X}$.*

(ii) *If Z is normal and $\mathcal{X}$ is integral and there exists a dense open subset $U \subset Z$ such that $f_U : \mathcal{X} \times_Z U \to U$ satisfies* (2.4 (i)) *and has a section, then $f : \mathcal{X} \to Z$ is a coarse moduli space for $\mathcal{X}$.*

Proof. The assertions of the corollary are local on Z, so we may assume that Z is of finite type over S. Let $\pi : \mathcal{X} \to X$ be the coarse moduli space of $\mathcal{X}$. The universal property (2.4 (ii)) provides a morphism $h : X \to Z$. Since π is proper and surjective and f is proper, the morphism h is also proper (2.7 (ii)).

To prove (i), note that since (2.4 (i)) holds for both X and Z, the morphism h is also radicial by ([8], I.3.5.5). Since f has a section, the morphism h also has a section $s : Z \to X$. Since h is separated this morphism s is a closed immersion, and it is also surjective since h is radicial. Therefore s is a closed immersion defined by a nilpotent ideal. On the other hand, X is reduced by (2.8) so s and h must be isomorphisms.

For (ii), note that $h : X \to Z$ is in any case a finite morphism since it is proper and quasi–finite. By (i) it also restricts to an isomorphism over U. Since Z is normal and X is integral by (2.8) it follows that h is an isomorphism. □

Proposition 2.10. *Let S be a locally noetherian scheme and $\mathcal{X}$ an Artin stack of finite type over S with finite diagonal. Let $\pi : \mathcal{X} \to X$ be its coarse moduli space. Then $\mathcal{X}$ is proper over S if and only if X is proper over S.*

Proof. The "if" direction is clear since π is a proper morphism, and the composite of two proper morphisms is again proper.

The "only if" direction follows from (2.7 (ii)), and the surjectivity of the morphism π. □

Proposition 2.11. *Let S be a noetherian scheme, and $\mathcal{X}$ an Artin stack of finite type over S with finite diagonal. Then there exists a finite stratification $\{S_i\}$ of S such that if $\mathcal{X}_i$ denotes the stack $\mathcal{X} \times_S S_i$ over S_i then the formation of the coarse moduli space of $\mathcal{X}_i$ commutes with arbitrary base change $V \to S_i$.*

Proof. We may without loss of generality assume that S is reduced. By noetherian induction it suffices to exhibit a dense open set $U \subset S$ such that the formation of the coarse space of $\mathcal{X}_U := \mathcal{X} \times_S U$ commutes with arbitrary base change $U' \to U$. For this let $\pi : \mathcal{X} \to X$ be the coarse space over S. Since X is quasi–compact and the formation of coarse moduli space commutes with étale base change $X' \to X$, it suffices to find the dense open set $U \subset S$ after replacing X by an étale cover.

We claim that there exists a quasi–finite flat separated surjection $W \to \mathcal{X}$. This is essentially the proof of ([13], 3.3), but we provide a slightly different argument for the convenience of the reader (note also that essentially the same argument can be found in ([7], V.7)). The assertion is evidently local on S, so we may assume that S is a noetherian affine scheme. Let $U \to \mathcal{X}$ be a smooth surjection with U a quasi–projective S–scheme, and let $s, t : R \rightrightarrows U$ be the resulting groupoid presentation of $\mathcal{X}$. Let $R' \to R$ be a quasi-compact etale cover of the quasi-compact algebraic space R by a scheme R'. Let $x \in \mathcal{X}$ be a geometric closed point in a fiber over S, and let $u \in U$ be a closed point over x. By ([7], V.7.2) applied to the two projections from R' to U over S we get a closed subscheme $W \subset U$ such that

(1) The map $U \times_{\mathcal{X}} W = R \times_{s,U} W \to U$ induced by t is flat at points over u.
(2) The set $s(s^{-1}(W) \cap t^{-1}(u))$ is finite and nonempty.

Statement (1) implies that the map $W \to \mathcal{X}$ is flat over x, and hence after replacing W by the maximal open subset $W_{fl} \subset W$ where the morphism $W \to \mathcal{X}$ is flat, we may assume that $W \to \mathcal{X}$ is flat. Furthermore, Statement (2) implies that the fiber $W \times_{\mathcal{X}} x$ is finite. After further shrinking on $\mathcal{X}$ we may therefore assume that the map $W \to \mathcal{X}$ is also quasi–finite ([8], IV.13.1.3).

Since the map $W \to \mathcal{X}$ is flat and hence has open image, we can after replacing $\mathcal{X}$ by the image of W assume that there exists quasi–finite flat separated surjection $W \to \mathcal{X}$. Let $\bar{x} \to X$ be a geometric point. Since $\pi : \mathcal{X} \to X$ is quasi–finite and proper, the map $w : W \to X$ is quasi–finite. If $X_{\bar{x}}$ denotes the spectrum of the strictly henselian local ring $\mathcal{O}_{X,\bar{x}}$, then $W \times_X X_{\bar{x}}$ breaks up as $P \coprod Q$, where $P \to X_{\bar{x}}$ is finite and surjective (since $W \to X$ is separated). By "spreading out" we see that after further étale localizing on X around $\bar{x}$, we can in fact choose a flat quasi–finite surjection $W \to \mathcal{X}$ such that $w : W \to X$ is finite. Then the map $W \to \mathcal{X}$ is proper (2.7 (i)). Let R denote $W \times_{\mathcal{X}} W$, and let $r : R \to X$ be the projection. The morphism r is also quasi–finite and proper, and hence finite since R and

X are algebraic spaces. Let $s, t : R \rightarrow W$ be the two projections, and let I (resp. J) denote the image (resp. cokernel) of the morphism of finite $\mathcal{O}_X$–modules $s - t : w_*\mathcal{O}_W \rightarrow r_*\mathcal{O}_R$ so that there are exact sequences

$$0 \rightarrow \mathcal{O}_X \rightarrow w_*\mathcal{O}_W \rightarrow I \rightarrow 0, \quad 0 \rightarrow I \rightarrow r_*\mathcal{O}_R \rightarrow J \rightarrow 0. \tag{2.11.1}$$

By (2.6 (i)), to find the desired open set it suffices to find an open set $U \subset S$ such that the restriction to X_U of the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow w_*\mathcal{O}_W \xrightarrow{s-t} r_*\mathcal{O}_R \tag{2.11.2}$$

remains exact after arbitrary base change $U' \rightarrow U$. For this in turn it suffices to find a dense open set $V \subset S$ over which all the coherent $\mathcal{O}_X$–modules $w_*\mathcal{O}_W$, I, J, and $r_*\mathcal{O}_R$ are flat. Let $U \subset X$ be the open subset where all these sheaves are S–flat, let Z be the complement of U, and let $\overline{Z} \subset S$ denote the image of S. The open set U contains all points of X lying over the generic points of S since S is reduced, and therefore $\overline{Z}$ does not contain any generic points. Since $\overline{Z}$ is also a constructible subset of S it follows that the complement of $\overline{Z}$ contains a dense open subset of S. □

In the case when $\mathcal{X}$ is Deligne–Mumford, the structure of the stack $\mathcal{X}$ over X can be described locally as follows:

Theorem 2.12 ([1])**.** *Let S be a locally noetherian scheme and $\mathcal{X}$ a separated Deligne–Mumford stack of finite type over S. Denote by $\pi : \mathcal{X} \rightarrow X$ the coarse moduli space of $\mathcal{X}$. Then for any geometric point $\bar{x} \rightarrow X$ there exists an étale neighborhood W of $\bar{x}$ such that the base change $\mathcal{W} := \mathcal{X} \times_X W$ is isomorphic to a quotient $[U/\Gamma]$, where U is a finite scheme over W and Γ is a finite group acting on U over W. In fact, we may choose Γ to be isomorphic to the stabilizer group of any geometric point of $\mathcal{X}$ lying over $\bar{x}$ (all such liftings have isomorphic stabilizer groups by* (2.4 (i))*).*

Proof. All but the last statement is in ([1], 2.2.3). The statement about choosing Γ to be equal to the stabilizer group of any lifting of $\bar{x}$ follows from the proof of (loc. cit.) as follows. Let X_{sh} denote the spectrum of the strict henselization $\mathcal{O}_{X,\bar{x}}$. Then the proof of (loc. cit.) shows that after replacing X by an étale neighborhood of $\bar{x}$, the stack $\mathcal{X}$ is isomorphic to a quotient $[U/\Gamma]$, where U is a finite X–scheme such that $U \times_X X_{\text{sh}}$ is connected (and hence the spectrum of a strictly henselian local ring) and Γ is a finite group acting on U over X. This implies that the fiber product $\mathcal{X}_{\bar{x}} := \bar{x} \times_X \mathcal{X}$ is isomorphic to $[U_{\bar{x}}/\Gamma]$, where $U_{\bar{x}} := \bar{x} \times_X U$. Since $U_{\bar{x}}$ is equal to the spectrum of an artinian local ring this implies that the stabilizer group of any geometric point of $\mathcal{X}$ over $\bar{x}$ is Γ. □

3. Reduction of (1.1) to (1.5)

3.1. It is clear that $\underline{\text{Hom}}_S(\mathcal{X}, \mathcal{Y})$ is a stack with respect to the fppf topology on S. By the following Lemma it therefore suffices to prove (1.1) after replacing S by an fppf cover.

Lemma 3.2. *Let $\mathcal{F}$ be a stack over S with respect to the fppf topology, and assume that there exists an fppf cover $S' \rightarrow S$ such that the base change $\mathcal{F}' := \mathcal{F} \times_S S'$ is an Artin stack with quasi–compact and separated diagonal. Then $\mathcal{F}$ is also an Artin stack with quasi–compact and separated diagonal.*

Proof. By ([16], 10.1) it suffices to verify that the diagonal of $\mathcal{F}$ is representable, quasi–compact, and separated, and that there exists an fppf surjection $X \to \mathcal{F}$ with X a scheme.

So consider an S–scheme T, a morphism $f : T \to \mathcal{F} \times_S \mathcal{F}$, and let I denote the fiber product $\mathcal{F} \times_{\mathcal{F}\times_S\mathcal{F},f} T$. Since $\mathcal{F}$ is a stack with respect to the fppf topology and $\mathcal{F}'$ is an Artin stack, I is an fppf stack fibered in discrete categories over T. Thus, by ([16], 10.4.1), to verify that I is an algebraic space separated and quasi–compact over T it suffices to prove that the base change $I \times_T (T \times_S S') \simeq I \times_S S'$ is an algebraic space separated and quasi–compact over $T \times_S S'$. This follows from the assumption that $\mathcal{F}'$ is an Artin stack with separated and quasi–compact diagonal.

To find an fppf cover of $\mathcal{F}$ by a scheme, simply choose an fppf cover $X' \to \mathcal{F}'$ by a scheme X', and note that the composite $X' \to \mathcal{F}' \to \mathcal{F}$ is an fppf cover of $\mathcal{F}$. □

3.3. We may now assume that there exists a finite flat surjection $V^0 \to \mathcal{X}$, where V^0 is a proper flat algebraic space over S. Since $\mathcal{X} \to S$ and $V^0 \to \mathcal{X}$ are proper and flat, the algebraic spaces $V^1 := V^0 \times_{\mathcal{X}} V^0$ and $V^2 := V^0 \times_{\mathcal{X}} V^0 \times_{\mathcal{X}} V^0$ are proper and flat over S.

Note also that if $f_i : V^i \to S$ is the structure morphism, then the fibered category $\underline{\mathrm{Hom}}_S(V^i, \mathcal{Y})$ is canonically isomorphic to $f_{i*}(\mathcal{Y} \times_S V^i)$. Thus (1.5) implies that each of the $\underline{\mathrm{Hom}}_S(V^i, \mathcal{Y})$ is an Artin stack with quasi–compact and separated diagonal. When $\mathcal{Y}$ is a Deligne–Mumford stack (resp. algebraic space) these stacks are even Deligne–Mumford stacks (resp. algebraic spaces). Let $\mathcal{P}$ be the fiber product of the diagram

$$\begin{array}{ccc} & \underline{\mathrm{Hom}}_S(V^0, \mathcal{Y}) & \\ & \Big\downarrow{\scriptstyle \mathrm{pr}_1 \times \mathrm{pr}_2} & \\ \underline{\mathrm{Hom}}_S(V^1, \mathcal{Y}) \times_S \underline{\mathrm{Hom}}_S(V^1, \mathcal{Y}) & \xleftarrow{\ \Delta\ } & \underline{\mathrm{Hom}}_S(V^1, \mathcal{Y}). \end{array} \tag{3.3.1}$$

For any S–scheme T, the groupoid $\mathcal{P}(T)$ is the groupoid of pairs (v, ι), where $v : V^0_T \to \mathcal{Y}$ is a 1-morphism and $\iota : \mathrm{pr}_1^* v \simeq \mathrm{pr}_2^* v$ is an isomorphism in $\mathcal{Y}(V^1_T)$. Let $\mathcal{A} \to \underline{\mathrm{Hom}}_S(V^2, \mathcal{Y})$ be the inertia stack of $\underline{\mathrm{Hom}}_S(V^2, \mathcal{Y})$. If T is an S–scheme, then $\mathcal{A}(T)$ is the groupoid of pairs (v, α), where $v \in \mathcal{Y}(V^2_T)$ and α is an automorphism of v. If $\mathcal{Y}$ is a Deligne–Mumford stack, then $\mathcal{A}$ is also a Deligne–Mumford stack. Furthermore, there is a natural morphism

$$\tau : \mathcal{P} \longrightarrow \mathcal{A}, \quad (v, \iota) \mapsto (\mathrm{pr}_3^* v, \mathrm{pr}_{12}^*(\iota) \circ \mathrm{pr}_{23}^*(\iota) \circ \mathrm{pr}_{13}^*(\iota)^{-1}), \tag{3.3.2}$$

and by fppf descent theory a cartesian square

$$\begin{array}{ccc} \mathcal{P} & \longleftarrow & \underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \\ {\scriptstyle \tau}\Big\downarrow & & \Big\downarrow \\ \mathcal{A} & \xleftarrow{\ s\ } & \underline{\mathrm{Hom}}_S(V^2, \mathcal{Y}), \end{array} \tag{3.3.3}$$

where s is the map sending $v \in \mathcal{Y}(V^2_T)$ to $(v, \mathrm{id}) \in \mathcal{A}(T)$. Consequently, (1.5) implies (1.1).

4. Proof of (1.6) granting (1.5)

Granting (1.5), to prove (1.6) it suffices to show that for a scheme T and a morphism $T \to \underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ corresponding to a morphism $f : \mathcal{X}_T \to \mathcal{Y}_T$ of stacks over T, the condition that f is representable is represented by an open subset of T.

For this note first that if f is representable then for any T–scheme $T' \to T$ the pullback $f' : \mathcal{X}_{T'} \to \mathcal{Y}_{T'}$ is also representable. This follows from ([5], 2.2.7). We can therefore define a functor F which associates to a T–scheme $T' \to T$ the unital set if the base change $f' : \mathcal{X}_{T'} \to \mathcal{Y}_{T'}$ of f to T' is representable and the empty–set otherwise. We claim that F is represented by an open subset of T.

For ease of notation, we replace T by S so f is a morphism over S. Let $V \to \mathcal{Y}$ be a smooth surjection with V a scheme. Then f is representable if and only if the base change $V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space. By ([5], 2.2.5 (2)) there exists a maximal open substack $U \subset V \times_{\mathcal{Y}} \mathcal{X}$ which is an algebraic space. By (loc. cit.) the open substack U is characterized by the condition that for every geometric point $\bar{x} \to U$ the stabilizer group scheme of $\bar{x}$ is trivial, and U is maximal with this property. Equivalently, a geometric point $\bar{x} \to V \times_{\mathcal{Y}} \mathcal{X}$ is in U if and only if the geometric point $\bar{x} \to \mathcal{X}$ obtained by composing with the projection to $\mathcal{X}$ has the property that the kernel of the morphism of group schemes $\underline{\mathrm{Aut}}_{\mathcal{X}}(\bar{x}) \to \underline{\mathrm{Aut}}_{\mathcal{Y}}(f(\bar{y}))$ over $\mathrm{Spec}(k(\bar{x}))$ is trivial. If $\pi : V \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ denotes the projection and $\mathcal{U} \subset \mathcal{X}$ denotes the image under π of U (an open substack of $\mathcal{X}$ since π is smooth), then it follows that U is equal to $\pi^{-1}(\mathcal{U})$. In particular, f is representable if and only if $\mathcal{U} = \mathcal{X}$. Note also that the formation of the substack $\mathcal{U} \subset \mathcal{X}$ is compatible with arbitrary base change $T' \to T$ since $\mathcal{U}$ is characterized by a condition on geometric points.

It follows that the functor F can be described as the functor which to any S–scheme T associates the unital set if the inclusion $\mathcal{U}_T \hookrightarrow \mathcal{X}_T$ is an isomorphism and the empty set otherwise. Let $\mathcal{Z} \subset \mathcal{X}$ denote the complement of $\mathcal{U}$. Since $\mathcal{X}$ is proper over S the image $Z \subset S$ of $\mathcal{Z}$ is closed, and it follows that F is represented by the complement of Z. □

5. Proof of (1.5)

Lemma 5.1. *Let S be a locally noetherian algebraic space, let X/S be a proper, flat algebraic space and Y/S a separated algebraic space of finite type. Then the functor $\underline{\mathrm{Hom}}_S(X, Y)$ is a separated algebraic space locally of finite type over S.*

Proof. Let $P := X \times_S Y$, and let $\mathrm{Hilb}_{P/S}$ denote the functor which to any T/S associates the set of closed algebraic subspaces of $P_T := P \times_S T$ which are finitely presented, proper, and flat over T.

Because X is proper and S–flat and Y is separated, the graph $\Gamma_f \subset P_T$ of any morphism $f : X_T \to Y_T$ defines an element in $\mathrm{Hilb}_{P/S}(T)$, and hence the association $f \mapsto \Gamma_f$ defines an inclusion of functors

$$\underline{\mathrm{Hom}}_S(X, Y) \subset \mathrm{Hilb}_{P/S}. \tag{5.1.1}$$

We claim that this inclusion is representable by open immersions.

To see this, let $Z \subset P_T$ be a finitely presented closed subspace flat and proper over T. The condition that Z is obtained from an element of $\underline{\mathrm{Hom}}_S(X, Y)(T)$ is equivalent to the statement that the projection map $\pi : Z \to X_T$ is an isomorphism. That (5.1.1) is representable by open immersions therefore follows from the following sub–lemma (which is implicit though not explicitly stated in ([9], section 4 (c))).

Sub-Lemma 5.2. *Let $\pi : Z \to X$ be a finitely presented morphism between algebraic spaces which are proper and flat over a base scheme S. Let Γ be the functor on S–schemes which to any T/S associates $\{*\}$ if $\pi_T : Z_T \to X_T$ is an isomorphism, and the empty set otherwise. Then Γ is represented by an open subset of S.*

Proof. Since both Z and X are flat over S, the map π is étale if and only if for every $s \in S$ the map on the fibers $Z_s \to X_s$ is étale ([8], IV.17.8.2). Let $R \subset Z$ be the closed subset (with the reduced structure) where π fails to be étale. Since Z/S is proper, the image of R in S is a closed set $\overline{R} \subset S$, and the functor Γ is a subfunctor of the open set $T - \overline{R}$. Therefore, replacing T by this open set we may assume that π is étale. The map π is also finite since Z/S is proper, and hence the map π is an isomorphism if and only if the locally free sheaf $\pi_*\mathcal{O}_Z$ has rank 1 on X. Let $Q \subset X$ denote the open and closed subspace where this fails to hold. Then the image $\overline{Q} \subset T$ of Q is closed since X/S is proper, and Γ is represented by $S - \overline{Q}$. □

To conclude the proof of (5.1), observe that by ([3], 6.2) the functor $\mathrm{Hilb}_{P/S}$ is a separated algebraic space locally of finite type over S (the separatedness is not explicitly stated in loc. cit., but follows from the separatedness assertion in ([3], 6.1); note also the correction to ([3], 6.1) in ([2], Appendix)). □

5.3. By (2.3), in order to prove (1.5) it suffices to consider the case when S is excellent and of finite type over the integers.

We first show that $f_*\mathcal{S}$ is an Artin stack (and in particular has quasi–compact diagonal) locally of finite type over T by verifying Artin's conditions ([2], 5.3). It is clear from the definition that $f_*\mathcal{S}$ is a stack with respect to the fppf topology. That $f_*\mathcal{S}$ is limit preserving in the sense of ([2], §1) follows from ([16], 4.18).

5.4 (Compatibility with completions)**.** What has to be shown is that if $\hat{A}$ is a complete noetherian local ring over T, then the natural functor

$$(f_*\mathcal{S})(\hat{A}) \longrightarrow \varprojlim (f_*\mathcal{S})(\hat{A}_n) \tag{5.4.1}$$

is an equivalence of categories, where $\hat{A}_n$ denotes $\hat{A}/\mathfrak{m}_{\hat{A}}^{n+1}$. By the definition of $f_*\mathcal{S}$, this is equivalent to showing that the natural functor

$$\mathcal{S}(S_{\hat{A}}) \longrightarrow \varprojlim \mathcal{S}(S_{\hat{A}_n}) \tag{5.4.2}$$

is an equivalence of categories. Note that since f is proper the algebraic space $S_{\hat{A}}$ is proper over $\hat{A}$. To see that (5.4.2) is fully faithful, let $u_1, u_2 \in \mathcal{S}(S_{\hat{A}})$ be two objects, and let I denote the $S_{\hat{A}}$–space $\underline{\mathrm{Isom}}(u_1, u_2)$. Observe that I is finite over $S_{\hat{A}}$ and in particular is a proper algebraic space over $\hat{A}$. The full faithfulness is then equivalent to the statement that

$$I(S_{\hat{A}}) \longrightarrow \varprojlim I(S_{\hat{A}_n}) \tag{5.4.3}$$

is bijective, which holds since I is proper over $\hat{A}$ and S/T is proper (when S and T are schemes this is ([8], III.5.4.1); the case of algebraic spaces follows from the same argument using the Grothendieck existence theorem for algebraic spaces ([14], V.6.3)). To see that (5.4.2) is essentially surjective, suppose $\{f_n : S_{\hat{A}_n} \to \mathcal{S}\}$ is a compatible family. Let $\mathcal{A}_n$ denote the sheaf of algebras $\Gamma_{f_n *}(\mathcal{O}_{S_{\hat{A}_n}})$ on $S_{\hat{A}_n} \times_S \mathcal{S}$, where Γ_{f_n} denotes the graph morphism. Because $\mathcal{S}$ is separated with finite diagonal, the graph morphisms Γ_{f_n} are finite. Since $\mathcal{S}$ is

S–separated, so $S_{\widehat{A}_n} \times_S \mathcal{S}$ is separated over $\mathrm{Spec}(\widehat{A}_n)$, Γ_{f_n} has image with $S_{\widehat{A}_n}$-proper support (as $S_{\widehat{A}_n}$ is $\widehat{A}_n$-proper, so (2.7 (ii)) applies). Therefore, the collection $\{\mathcal{A}_n\}$ is a compatible family of coherent sheaves of algebras on $S_{\hat{A}_n} \times_S \mathcal{S}$ with proper support, and hence by (A.1) is induced by a unique coherent sheaf of algebras $\mathcal{A}$ on $S_{\hat{A}} \times_S \mathcal{S}$ with proper support over $S_{\hat{A}}$. Hence the relative spectrum of $\mathcal{A}$, which we denote by $\mathcal{P}$, is a proper Artin stack with finite diagonal over $S_{\hat{A}}$. Constructing $f : S_{\hat{A}} \to \mathcal{S}$ inducing $\{f_n\}$ is therefore equivalent to showing that the projection map $\mathcal{P} \to S_{\hat{A}}$ is an isomorphism.

For this note first that $\mathcal{P}$ is an algebraic space. Indeed by ([5], 2.2.5 (2)) there exists a maximal open substack $\mathcal{P}_{\mathrm{rep}} \subset \mathcal{P}$ which is an algebraic space and whose geometric points are precisely the geometric points $\bar{x} \to \mathcal{P}$ such that the automorphism group scheme $\underline{\mathrm{Aut}}(\bar{x})$ is trivial. Since the reduction of $\mathcal{P}$ to $\mathrm{Spec}(\hat{A}/\mathfrak{m}_{\hat{A}})$ is by construction equal to the relative spectrum of $\mathcal{A}_0$ over $S_{\hat{A}} \times_S \mathcal{S}$, the projection map $\mathcal{P} \otimes_{\hat{A}} \hat{A}/\mathfrak{m}_{\hat{A}} \to S_{\hat{A}_0}$ is an isomorphism. In particular, the geometric points of the closed fiber all have trivial stabilizer group schemes, and hence $\mathcal{P}_{\mathrm{rep}}$ contains the closed fiber of $\mathcal{P}$. It follows that $\mathcal{P}_{\mathrm{rep}} = \mathcal{P}$, so $\mathcal{P}$ is an algebraic space. Now by the openness of the quasi–finite locus (in the case of schemes this is ([8], IV.13.1.4), and this reference generalizes immediately to algebraic spaces), the morphism $\mathcal{P} \to S_{\hat{A}}$ is quasi–finite. The morphism $\mathcal{P} \to S_{\hat{A}}$ is therefore a quasi–finite morphism between proper algebraic spaces over $\hat{A}$ and hence is finite. Furthermore, modulo each power of $\mathfrak{m}_{\hat{A}}$ the map $\mathcal{P} \to S_{\hat{A}}$ reduces to an isomorphism by construction. The map $\mathcal{P} \to S_{\hat{A}}$ is therefore also an isomorphism, by $\hat{A}$–properness of $\mathcal{P}$ and $S_{\hat{A}}$.

5.5 (Condition (S1$'$))**.** Suppose given a commutative diagram of noetherian rings over T

(5.5.1)
$$\begin{array}{ccccc} & & B & & \\ & & \downarrow & & \\ A' & \longrightarrow & A & \longrightarrow & A_0, \end{array}$$

for which

(5.5 (i)) The ring A_0 is reduced.

(5.5 (ii)) The maps $A' \to A$ and $A \to A_0$ are surjections with nilpotent kernels.

(5.5 (iii)) The composite $B \to A_0$ is surjective.

(5.5 (iv)) The kernel $I := \mathrm{Ker}(A' \to A)$ is an A_0–module of finite type.

Suppose further given an element $a \in f_*\mathcal{S}(A) = \mathcal{S}(S_A)$. For any morphism of schemes $g : S_A \to Y$ over S, let $\mathcal{S}_a(Y)$ denote the category of pairs (y, σ), where $y \in \mathcal{S}(Y)$ and $\sigma : g^*y \to a$ is an isomorphism in $\mathcal{S}(S_A)$. Then condition (S1$'$) in ([2]) is equivalent to the statement that

(5.5.2)
$$\mathcal{S}_a(S_{A' \times_A B}) \longrightarrow \mathcal{S}_a(S_{A'}) \times \mathcal{S}_a(S_B)$$

is an equivalence of categories. To prove this it is easiest to prove the following more general result omitting the properness assumption on f.

Lemma 5.6. *Let W be a flat algebraic space locally of finite type over $A' \times_A B$, and let $W_{A'}$ (resp. W_A, W_B) denote the base change to A' (resp. A, B). Then for any element*

$w \in \mathcal{S}(W_A)$, *the functor*

$$\mathcal{S}_w(W) \to \mathcal{S}_w(W_{A'}) \times \mathcal{S}_w(W_B) \tag{5.6.1}$$

is an equivalence of categories.

Proof. Since $\mathcal{S}$ is a stack with respect to the étale topology, one reduces immediately to the case when W is an affine scheme of finite type over $A' \times_A B$.

In this case, write $W = \mathrm{Spec}(R)$, and let $R_{A'}$ (resp. R_A, R_B) denote the coordinate ring of $W_{A'}$ (resp. W_A, W_B). Since R is flat over $A' \times_A B$, the natural map

$$R \to R_{A'} \times_{R_A} R_B \tag{5.6.2}$$

is an isomorphism. Thus the result in this case is a special case of ([21], 1.4.4). □

5.7 (Deformation modules)**.** Suppose $A' \to A$ is any surjection of noetherian rings over T with square–zero kernel I. Suppose further given a 1-morphism $a : S_A \to \mathcal{S}$ over S, corresponding to an object in $(f_*\mathcal{S})(A)$. Let $L_{\mathcal{S}/S}$ denote the cotangent complex of the morphism of $\mathcal{S} \to S$ ([17]).

By ([17], 1.4), the following hold:

(5.7 (i)) There is a canonical class

$$o(I) \in \mathrm{Ext}^1(a^* L_{\mathcal{S}/S}, I \otimes \mathcal{O}_{S_A}). \tag{5.7.1}$$

whose vanishing is necessary and sufficient for the existence of a lifting of a to a 1–morphism $a' : S_{A'} \to \mathcal{S}$.

(5.7 (ii)) If $o(I) = 0$, then the set of isomorphism classes of such liftings a' is naturally a torsor under

$$\mathrm{Ext}^0(a^* L_{\mathcal{S}/S}, I \otimes \mathcal{O}_{S_A}). \tag{5.7.2}$$

(5.7 (iii)) For any lifting a' of a, the group of automorphisms of a' inducing the identity on a is canonically isomorphic to

$$\mathrm{Ext}^{-1}(a^* L_{\mathcal{S}/S}, I \otimes \mathcal{O}_{S_A}). \tag{5.7.3}$$

From this and standard properties of cohomology, it follows that we have an obstruction theory for $f_*\mathcal{S}$ and that the conditions (1) and (3) in ([2], 5.3) hold.

5.8 (Condition on automorphisms)**.** What has to be shown is that if A_0 is an integral domain, $a_0 \in f_*\mathcal{S}(A_0)$ is an object, and ϕ is an automorphism of a_0 which induces the identity in $f_*\mathcal{S}(k)$ for a dense set of points $A_0 \to k$, then ϕ is the identity. Let $\underline{\mathrm{Aut}}(a_0)$ denote the algebraic space over S_{A_0} of automorphisms of a_0.

The space $\underline{\mathrm{Aut}}(a_0)$ is separated over S_{A_0} since $\mathcal{S}$ has separated diagonal, and therefore the fiber product Z of the diagram

$$\begin{array}{ccc} & & S_{A_0} \\ & & \big\downarrow{\scriptstyle e\times\phi} \\ \underline{\mathrm{Aut}}(a_0) & \xrightarrow{\Delta} & \underline{\mathrm{Aut}}(a_0) \times_{S_{A_0}} \underline{\mathrm{Aut}}(a_0) \end{array} \tag{5.8.1}$$

is a closed subspace of S_{A_0}. Let $U \subset S_{A_0}$ be the complement of Z. Let $f_0 : S_{A_0} \to \mathrm{Spec}(A_0)$ be the pullback of f. Since f_0 is flat and of finite presentation, the image $f_0(U)$ is an open subset of $\mathrm{Spec}(A_0)$. On the other hand, by assumption the complementary set $f_0(U)^c$ contains a dense set of points. Thus $f_0(U) = \emptyset$, and $U = \emptyset$. It follows that $Z \hookrightarrow S_{A_0}$ is defined by a nilpotent ideal $\mathcal{N} \subset \mathcal{O}_{S_{A_0}}$ which we claim is zero. Since S_{A_0} is flat over A_0, the sheaf $\mathcal{N}$ is zero if and only if it becomes zero on the generic fiber of $\mathrm{Spec}(A_0)$. Since the nilimmersion $Z \hookrightarrow S_{A_0}$ restricts to an isomorphism over a dense set of points of $\mathrm{Spec}(A_0)$, the maximal open subset of S_{A_0} where $\mathcal{N}$ is zero contains fibers over a dense subset of $\mathrm{Spec}(A_0)$. By properness of S_{A_0} over $\mathrm{Spec}(A_0)$ it therefore also contains the generic fiber. This completes the proof of the statement that $f_*\mathcal{S}$ is a locally quasi–separated Artin stack locally of finite type.

5.9. It remains to show that the diagonal map

$$\Delta : f_*\mathcal{S} \longrightarrow f_*\mathcal{S} \times_T f_*\mathcal{S} \tag{5.9.1}$$

is separated and quasi–compact, and that when $\mathcal{S}$ is a Deligne–Mumford stack (resp. algebraic space) the stack $f_*\mathcal{S}$ is a Deligne–Mumford stack (resp. algebraic space).

To see that the diagonal Δ is separated and quasi–compact, it suffices to show that if T'/T is a scheme and $s_1, s_2 \in \mathcal{S}(S_{T'})$, then the functor F on T'–schemes which to any W/T' associates the set of isomorphisms $s_1 \to s_2$ over S_W is representable by a quasi–compact and separated scheme over T'. Let I denote the $S_{T'}$–space $\underline{\mathrm{Isom}}(s_1, s_2)$; this is $S_{T'}$–finite since $\mathcal{S}$ has finite diagonal over S. Then F is isomorphic to the functor on the category of T'–schemes which to any W/T' associates the set of sections of the morphism $I_W \to S_W$ over W. That Δ is separated and quasi–compact therefore follows from the following proposition.

Proposition 5.10. *Let $f : S \to T$ be a proper morphism of locally noetherian algebraic spaces, and $\pi : I \to S$ a finite morphism. Let $\underline{\mathrm{Sec}}(I/S)$ denote the functor on the category of T–schemes which to any W/T associates the set of sections of the W–morphism $I_W \to S_W$ obtained by base change. Then $\underline{\mathrm{Sec}}(I/S)$ is a separated algebraic space of finite type over T. If f is a projective morphism then $\underline{\mathrm{Sec}}(I/S)$ is quasi-projective over T.*

Proof. The functor $\underline{\mathrm{Sec}}(I/S)$ is isomorphic to the fiber product of the diagram

$$\begin{array}{ccc} & & \underline{\mathrm{Hom}}_T(S, I) \\ & & \downarrow \\ T & \xrightarrow{\mathrm{id}} & \underline{\mathrm{Hom}}_T(S, S). \end{array} \tag{5.10.1}$$

By (5.1) the functor $\underline{\mathrm{Sec}}(I/S)$ is therefore an algebraic space locally of finite type over T. Since $\underline{\mathrm{Hom}}_T(S, I)$ and $\underline{\mathrm{Hom}}_T(S, S)$ are T–separated by (5.1), the space $\underline{\mathrm{Sec}}(I/S)$ is also T-separated.

To see the quasi–compactness of $\underline{\mathrm{Sec}}(I/S)$ over T, note first that it holds when f is projective. Indeed, if $\mathcal{L}$ is a relatively ample invertible sheaf on S, then since $I \to S$ is finite the pullback of $\mathcal{L}$ to I is also T–ample. Let $\mathrm{Hilb}^0_{I/T} \subset \mathrm{Hilb}_{I/T}$ be the open subscheme which associates to a T–scheme T' the set of flat and finitely presented closed subspaces $\Gamma \subset I_{T'}$ with Hilbert polynomial with respect to $\mathcal{L}|_I$ equal to the Hilbert polynomial of S. Then by the proof of (5.1), $\underline{\mathrm{Sec}}(I/S)$ is represented by an open subset of $\mathrm{Hilb}^0_{I/T}$ and hence is represented by a quasi–projective scheme over T by ([9], 3.2).

To prove the quasi–compactness of $\underline{\mathrm{Sec}}(I/S)$ over T in general, we reduce to the case when f is projective as follows.

First we give a lemma which we state in slightly greater generality for later use in the proof of (1.7) in ([19]). If T is a scheme, and $\mathcal{Q} \to X$ is a morphism over T from an Artin stack $\mathcal{Q}$ to a scheme X, let $\underline{\mathrm{Sec}}(\mathcal{Q}/X)$ denote the stack over T which to any T–scheme T' associates the groupoid of sections of the base change $\mathcal{Q} \times_T T' \to X \times_T T'$.

Lemma 5.11. *Let $X_0 \hookrightarrow X$ be a closed immersion defined by a square–zero ideal $\mathcal{J}$ of proper flat algebraic spaces over a noetherian base scheme T, and let $\mathcal{Q} \to X$ be a quasi–finite and proper morphism from a Deligne–Mumford stack. Denote by $\mathcal{Q}_0$ the base change $\mathcal{Q} \times_X X_0$, and assume that $\underline{\mathrm{Sec}}(\mathcal{Q}/X)$ and $\underline{\mathrm{Sec}}(\mathcal{Q}_0/X_0)$ are Deligne–Mumford stacks locally of finite type over T (we will see below that this is always the case). Then the natural map*

$$\underline{\mathrm{Sec}}(\mathcal{Q}/X) \to \underline{\mathrm{Sec}}(\mathcal{Q}_0/X_0) \tag{5.11.1}$$

is of finite type.

Proof. It suffices to show that if $T \to \underline{\mathrm{Sec}}(\mathcal{Q}_0/X_0)$ is a morphism corresponding to a section $s : X_0 \to \mathcal{Q}_0$, then the fiber product

$$\mathcal{P} := \underline{\mathrm{Sec}}(\mathcal{Q}/X) \times_{\underline{\mathrm{Sec}}(\mathcal{Q}_0/X_0)} T \tag{5.11.2}$$

is of finite type over T. Furthermore, we may assume that T is an integral noetherian affine scheme. The stack $\mathcal{P}$ associates to any $w : W \to T$ the groupoid of liftings $\tilde{s} : X_W \to \mathcal{Q}$ over X of the composite

$$X_{0,W} \to \mathcal{Q}_0 \to \mathcal{Q}, \tag{5.11.3}$$

where the first map is the one induced by s. Since $\mathcal{Q}$ is a Deligne–Mumford stack, there are no infinitesimal automorphisms of its objects, and hence $\mathcal{P}$ is actually an algebraic space. Furthermore, to prove that $\mathcal{P}$ is quasi–compact it suffices by Noetherian induction to exhibit a dense open set $U \subset T$ such that $\mathcal{P}_U$ is quasi–compact.

Let $L_{\mathcal{Q}_0/X_0}$ be the cotangent complex of $\mathcal{Q}_0/X_0$, and observe that by ([12], III.2.2.4) there is a canonical obstruction

$$o \in \mathrm{Ext}^1(s^* L_{\mathcal{Q}_0/X_0}|_{X_{0,W}}, \mathcal{J}_W) \tag{5.11.4}$$

whose vanishing is necessary and sufficient for the existence of a lifting $\tilde{s}$ as above. Here we write $\mathcal{J}_W$ for the pullback of $\mathcal{J}$ to $X_{0,W}$. Note that by the S–flatness of X_0, $\mathcal{J}_W$ is also equal to the ideal of $X_{0,W}$ in X_W. Furthermore, if $o = 0$, then by (loc. cit.), the set of isomorphism classes of liftings $\tilde{s}$ forms a torsor under the group

$$\mathrm{Ext}^0(s^* L_{\mathcal{Q}_0/X_0}|_{X_{0,W}}, \mathcal{J}_W). \tag{5.11.5}$$

Since X_0 is proper over T, there exists a dense open subset $U \subset S$ over which the modules

$$\mathrm{Ext}^j(s^* L_{\mathcal{Q}_0/X_0}|_{X_{0,W} \times_S U}, \mathcal{J}_W|_{X_{0,W} \times_S U}), \quad j = 0, 1, \tag{5.11.6}$$

are locally free and such that the formation of these modules commutes with arbitrary base change. From the above discussion we conclude that $\mathcal{P}_U$ is represented by an fppf torsor under the vector bundle (5.11.5) restricted to a closed subscheme of U. In particular, $\mathcal{P}_U$ is quasi–compact. □

To prove that $\underline{\mathrm{Sec}}(I/S)$ is of finite type over T, it suffices by noetherian induction to find a nonempty open subset $U \subset T$ such that the restriction $\underline{\mathrm{Sec}}(I_U/S_U) \to U$ is of finite type. Furthermore we may assume that T is quasi–compact and reduced.

To find the open subset $U \subset T$, we can further reduce to the case when S is reduced as follows. Since S is noetherian there exists a sequence of closed subschemes

$$S_{\mathrm{red}} = S_0 \subset S_1 \subset \cdots \subset S_r = S \tag{5.11.7}$$

with $S_i \subset S_{i+1}$ defined by a square–zero ideal. For each i let I_i denote $I \times_S S_i$. Since T is reduced, we can after replacing T by a dense open subset assume that each S_i is flat over T. In this case the functors $\underline{\mathrm{Sec}}(I_i/S_i)$ are algebraic spaces locally of finite type over S_i by (5.1) and (5.10.1), and by (5.11) the morphisms

$$\underline{\mathrm{Sec}}(I_{i+1}/S_{i+1}) \to \underline{\mathrm{Sec}}(I_i/S_i) \tag{5.11.8}$$

are all of finite type. This therefore reduces the proof to the case when S and T are both reduced.

By Chow's lemma ([14], IV.3.1), there exists a proper birational morphism $h : S' \to S$ with S' a projective T–scheme. After shrinking on T we may assume that S' is flat over T. Let I' denote the fiber product $I \times_S S'$. The morphism $\mathcal{O}_S \to h_*\mathcal{O}_{S'}$ is injective since S is reduced. Let M denote the cokernel. Since h is proper, the sheaf M is a coherent sheaf of $\mathcal{O}_S$–modules.

The following is a variant of the usual cohomology and base change results ([8], III.7).

Lemma 5.12. *Let S be a locally noetherian scheme and $f : \mathcal{X} \to \mathcal{Y}$ a representable morphism of Artin stacks of finite type over S. Let $\mathcal{F}$ be a quasi–coherent sheaf on $\mathcal{X}$ flat over S, and assume that the quasi-coherent sheaves $R^if_*\mathcal{F}$ on $\mathcal{Y}$ are flat over S for all $i \geq 0$. Then for any morphism of schemes $S' \to S$ and $i \geq 0$ the base change morphism*

$$g^*R^if_*\mathcal{F} \to R^if'_*(h^*\mathcal{F}) \tag{5.12.1}$$

is an isomorphism, where $f' : \mathcal{X}' \to \mathcal{Y}'$ is the base change of f to S' and $g : \mathcal{Y}' \to \mathcal{Y}$ and $h : \mathcal{X}' \to \mathcal{X}$ are the projections.

Proof. Note first that the sheaves $R^if_*\mathcal{F}$ are quasi–coherent by ([16], 13.2.6 (iii)) (note also the correction in ([20], 6.20)).

The assertion of the Lemma is local in the smooth topology on S and $\mathcal{Y}$. We may therefore assume that S and $\mathcal{Y}$ are both affine schemes. Since f is representable the stack $\mathcal{X}$ is an algebraic space in this case. It further suffices to consider morphisms $S' \to S$ of affine schemes.

If $S' \to S$ is such a morphism of affine schemes, then $h : \mathcal{X}' \to \mathcal{X}$ is affine, so

$$R\Gamma(\mathcal{X}', h^*\mathcal{F}) \simeq R\Gamma(\mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}). \tag{5.12.2}$$

We therefore need to show that for all i the morphism

$$\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} R^i\Gamma(\mathcal{X}, \mathcal{F}) \to R^i\Gamma(\mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}) \tag{5.12.3}$$

is an isomorphism. This is a standard descending induction beginning with an integer i so large that $R^j\Gamma(\mathcal{X}, \mathcal{F}) = 0$ and $R^j\Gamma(\mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})$ are zero for all $j \geq i$ and all $\mathcal{F}$. □

After shrinking on T, we may assume that all the sheaves $R^i h_* \mathcal{O}_{S'}$ are flat over T, and hence by (5.12) the formation of $h_* \mathcal{O}_{S'}$ commutes with arbitrary base change $W \to T$. After further shrinking T we may also assume that M is flat over T. In this case, for any morphism $W \to T$ the base change $h_W : S'_W \to S_W$ has the property that the map $\mathcal{O}_{S_W} \otimes_{\mathcal{O}_S} h_* \mathcal{O}_{S'} \to h_{W*} \mathcal{O}_{S'_W}$ is an isomorphism and the map $\mathcal{O}_{S_W} \to h_{W*} \mathcal{O}_{S'_W}$ is injective. The following lemma and the projective case already completed now completes the proof that $\underline{\mathrm{Sec}}(I/S)$ is of finite type over T, and hence also completes the proof that the diagonal (5.9.1) is quasi–compact. □

Lemma 5.13. *Assuming that the formation of $h_* \mathcal{O}_{S'}$ commutes with arbitrary base change $W \to T$ and that the morphism $\mathcal{O}_S \to h_* \mathcal{O}_{S'}$ also remains injective after such base changes, the natural map*

$$\underline{\mathrm{Sec}}(I/S) \to \underline{\mathrm{Sec}}(I'/S') \tag{5.13.1}$$

is of finite type.

Proof. Since I is finite over S it corresponds to a coherent sheaf of $\mathcal{O}_S$–algebras $\mathcal{A}$. Let $\mathcal{B}$ denote the coherent sheaf of $\mathcal{O}_S$–algebras $h_* \mathcal{O}_{S'}$. Since by assumption the formation of $h_* \mathcal{O}_{S'}$ commutes with arbitrary base change, the functor $\underline{\mathrm{Sec}}(I'/S')$ is isomorphic to the functor over T which to any $W \to T$ associates the set of morphisms $\mathcal{A}_W \to \mathcal{B}_W$ of $\mathcal{O}_{S_W}$–algebras on S_W, where $\mathcal{A}_W$ (resp. $\mathcal{B}_W$) denotes the pullback of $\mathcal{A}$ (resp. $\mathcal{B}$) to S_W. The functor $\underline{\mathrm{Sec}}(I/S)$ is isomorphic to the functor which to any $W \to T$ associates the set of morphisms of $\mathcal{O}_{S_W}$–algebras $\mathcal{A}_W \to \mathcal{O}_{S_W}$. Since the map $\mathcal{O}_{S_W} \to \mathcal{B}_W$ is injective by assumption, it is clear that $\underline{\mathrm{Sec}}(I/S)$ is a subfunctor of $\underline{\mathrm{Sec}}(I'/S')$.

Let $W \to \underline{\mathrm{Sec}}(I'/S')$ be a morphism corresponding to a morphism $\varphi : \mathcal{A}_W \to \mathcal{B}_W$, and as above let M_W denote the cokernel of $\mathcal{O}_S \to \mathcal{B}_W$. Let $\bar{\varphi} : \mathcal{A}_W \to M_W$ be the morphism of coherent $\mathcal{O}_S$–modules induced by φ. Then the fiber product

$$\underline{\mathrm{Sec}}(I/S) \times_{\underline{\mathrm{Sec}}(I'/S')} W \tag{5.13.2}$$

is the subfunctor of W which to any $W' \to W$ associates the unital set if the pullback of $\bar{\varphi}$ to $S_{W'}$ is the zero map, and the empty set otherwise. As usual, by noetherian induction to prove that (5.13.2) is of finite type it suffices to find a dominant morphism $U \to W$ of finite type such that the pullback of (5.13.2) is of finite type over U.

Renaming W and S_W and T and S respectively, we can assume that $W = T$. We may also assume T is reduced. Let Q be the cokernel of $\bar{\varphi}$, and let K be the kernel of the map $M_T \to Q$. After replacing T by a dense open subset, we may assume that Q is flat over T in which case the sequence

$$0 \to K \to M_T \to Q \to 0 \tag{5.13.3}$$

remains exact after arbitrary base change $W \to T$. Let $Z \subset S$ denote the support of K. Since $S \to T$ is proper the image of Z in T is a closed subset whose complement represents (5.13.2) for $W = T$. □

5.14. To show that $f_* \mathcal{S}$ is a Deligne–Mumford stack when $\mathcal{S}$ is Deligne–Mumford, it suffices to verify that the objects of $f_* \mathcal{S}$ admit no infinitesimal automorphisms ([16], 8.1). But this is clear, for if $T'_0 \hookrightarrow T'$ is a closed immersion of T–schemes defined by a nilpotent ideal, then $S_{T'_0} \hookrightarrow S_{T'}$ is also a closed immersion defined by a nilpotent ideal, and since $\mathcal{S}$ is a Deligne–Mumford stack over S, there are no nontrivial automorphisms of objects in $\mathcal{S}(S_{T'})$

inducing the identity in $\mathcal{S}(S_{T_0'})$. Similarly, if $\mathcal{S}$ is an algebraic space then to prove that $f_*\mathcal{S}$ is an algebraic space it suffices by ([16], 8.1.1) to prove that the objects of $f_*\mathcal{S}$ admit no nontrivial automorphisms which is immediate.

Appendix A. Grothendieck Existence Theorem revisited

In ([18], 1.4) a version of the Grothendieck Existence Theorem was proven for stacks. Unfortunately, the Theorem there was not stated in sufficient generality for the purposes of this paper, so we explain in this appendix how to deduce the following stronger result from (loc. cit.). In the case of schemes (resp. algebraic spaces) this is the version proven by Grothendieck (resp. Knutson).

Theorem A.1. *Let $\hat{A}$ be a noetherian adic ring with defining ideal $\mathfrak{a} \subset \hat{A}$, and let $\mathcal{X}$ be a separated Artin stack of finite type over $\hat{A}$. Then reduction induces an equivalence of categories between the category of coherent sheaves on $\mathcal{X}$ with proper support over $\hat{A}$ and the category of compatible systems $\{\mathcal{F}_n\}$ of coherent sheaves $\mathcal{F}_n$ on $\mathcal{X}_n := \mathcal{X} \otimes_{\hat{A}} \hat{A}/\mathfrak{m}^n_{\hat{A}}$ with proper support over $\hat{A}/\mathfrak{m}^n_{\hat{A}}$.*

Remark A.2. Note that the theorem is also true with $\mathcal{X}$ separated and locally of finite type over $\hat{A}$ by a trivial reduction to the quasi–compact case.

For a locally noetherian Artin stack $\mathcal{X}$ and coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, the *support of* $\mathcal{F}$ is the closed substack of $\mathcal{X}$ defined by the vanishing of the ideal sheaf $J := \mathrm{Ker}(\mathcal{O}_{\mathcal{X}} \to \mathcal{H}om(\mathcal{F}, \mathcal{F}))$.

Proof. First note that the reduction functor is fully faithful. Indeed if $\mathcal{F}$ and $\mathcal{G}$ are two coherent sheaves with proper support over $\hat{A}$ and $\{\mathcal{F}_n\}$ and $\{\mathcal{G}_n\}$ denote the compatible families obtained by reduction, then to verify that the map

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \varprojlim \mathrm{Hom}(\mathcal{F}_n, \mathcal{G}_n) \tag{A.2.1}$$

is bijective, it suffices to do so after replacing $\mathcal{X}$ by the union of the supports of $\mathcal{F}$ and $\mathcal{G}$. The full faithfulness therefore follows from the case when $\mathcal{X}$ is proper over $\hat{A}$.

For the essential surjectivity, let $\{\mathcal{F}_n\}$ be a compatible family of coherent sheaves on the reductions $\mathcal{X}_n$ with proper support. To prove that $\{\mathcal{F}_n\}$ is induced by a coherent sheaf on $\mathcal{X}$ with proper support, it suffices by the proper case ([18], 1.4) to find a closed substack $\mathcal{Z} \subset \mathcal{X}$ with $\hat{A}$–proper support such that each $\mathcal{F}_n$ is supported on the reduction $\mathcal{Z}_n$ of $\mathcal{Z}$.

To construct this substack $\mathcal{Z} \subset \mathcal{X}$, let $p : X \to \mathcal{X}$ be a proper surjection with X a separated $\hat{A}$–scheme (such a morphism p exists by ([18], 1.1)). For an integer n, let X_n denote the reduction of X modulo $\mathfrak{a}^n$, and let $\mathcal{F}'_n$ be the pullback of $\mathcal{F}_n$ to X_n. The sheaves $\{\mathcal{F}'_n\}$ form a compatible collection of coherent sheaves with $\hat{A}$–proper support on the reductions X_n, and hence by the Grothendieck Existence Theorem for schemes ([8], III.5.1.4) the system $\{\mathcal{F}'_n\}$ is induced by a unique coherent sheaf $\mathcal{F}'$ on X with $\hat{A}$–proper support. Let $Z \subset X$ be the support of $\mathcal{F}'$, and let $\mathcal{W} \subset \mathcal{X}$ be the scheme–theoretic image of Z (if $p_Z : Z \to \mathcal{X}$ is the projection, then $\mathcal{W}$ is the closed substack of $\mathcal{X}$ defined by the kernel of the map $\mathcal{O}_{\mathcal{X}} \to p_{Z*}\mathcal{O}_Z$). Since p is a proper morphism, the image of Z under p is closed. This implies that the induced map $Z \to \mathcal{W}$ is separated and surjective, whence by (2.7 (ii)) the Artin stack $\mathcal{W}$ is proper over $\hat{A}$.

Let $\mathcal{J} \subset \mathcal{O}_{\mathcal{X}}$ be the sheaf of ideals defining $\mathcal{W}$, and for $r \geq 1$ let $\mathcal{W}^{(r)} \subset \mathcal{X}$ denote the closed substack defined by $\mathcal{J}^r$. We claim that there exists an integer $r \geq 1$ such that for every n the sheaf $\mathcal{F}_n$ is supported on the reduction $\mathcal{W}_n^{(r)}$ of $\mathcal{W}^{(r)}$ modulo $\mathfrak{a}^n$. This will certainly complete the proof of (A.1).

Let $\mathrm{Spec}(R) \to \mathcal{X}$ be a smooth morphism, $\widehat{R}$ the $\mathfrak{a}$–adic completion of R, and let $q : Y \to \mathrm{Spec}(\widehat{R})$ denote the base change $X \times_{\mathcal{X}} \mathrm{Spec}(\widehat{R})$. Let $J_{\widehat{R}}$ denote the pullback of the ideal $\mathcal{J}$ to $\mathrm{Spec}(\widehat{R})$, and let $P \subset Y$ denote the inverse image of Z. For every n, let Y_n denote the reduction of Y modulo $\mathfrak{a}^n$, let F_n denote the pullback of $\mathcal{F}_n$ to $\mathrm{Spec}(\widehat{R}/\mathfrak{a}^n)$, and let F' denote the pullback of $\mathcal{F}'$ to Y with reduction F'_n to Y_n. Since the morphism $Y \to X$ is flat, the subspace $P \subset Y$ is equal to the support of F' (this follows immediately from the definition of the support of a coherent sheaf). Also note that since $\widehat{R}$ is $\mathfrak{a}$–adically complete, the system $\{F_n\}$ is induced by a unique coherent $\widehat{R}$–module F. Furthermore, the pullback q^*F is canonically isomorphic to F' since the compatible systems $\{q^*F|_{Y_n}\}$ and $\{F'_n\}$ are canonically isomorphic. Since $\mathcal{X}$ is quasi–compact, to find the integer r such that the $\mathcal{F}_n$ are supported on the reductions $\mathcal{W}_n^{(r)}$ it suffices to show that there exists an integer r such that the support of F is contained in $\mathrm{Spec}(\widehat{R}/J^r_{\widehat{R}})$.

Since the morphism $\mathrm{Spec}(\widehat{R}) \to \mathcal{X}$ is flat and the formation of scheme–theoretic image commutes with flat base change (in the case of schemes this is ([8], IV.2.3.2), the case of stacks follows from a standard reduction to this case), the ideal $J_{\widehat{R}}$ defines the scheme–theoretic image of P under q. Let $S \subset \mathrm{Spec}(\widehat{R})$ denote the support of F. If $W \subset \mathrm{Spec}(\widehat{R})$ denotes the closed subscheme defined by $J_{\widehat{R}}$ (which is also equal to the inverse image of $\mathcal{W}$), then it follows that a field valued point $x : \mathrm{Spec}(K) \to \mathrm{Spec}(\widehat{R})$ factors through W if and only if the pullback x^*F is not the zero–module. It follows that the underlying topological spaces of S and W are equal. Since $\widehat{R}$ is noetherian, this implies that there exists an integer $r \geq 1$ such that $S \subset \mathrm{Spec}(\widehat{R}/J^r_{\widehat{R}})$ as desired. □

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