A GEOMETRIC CONSTRUCTION OF SEMISTABLE EXTENSIONS OF CRYSTALLINE REPRESENTATIONS

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Abstract. We study unipotent fundamental groups for open varieties over p-adic fields with base point degenerating to the boundary. In particular, we show that the Galois representations associated to the étale unipotent fundamental group are semistable.

1. Introduction

1.1. The purpose of this paper is to explain how p-adic Hodge theory for the unipotent fundamental group provides examples of extensions of crystalline representations which are semistable but not crystalline, and where the monodromy operator has a clear geometric intepretation.

We will use a p-adic analogue of the following construction in the complex analytic situation. Let $X/\mathbb{C}$ be a smooth proper scheme, let $D \subset X$ be a divisor with normal crossings, and let X° denote $X - D$. Let $x \in D(\mathbb{C})$ be a point of D. Set

$$\Delta := \{z \in \mathbb{C} : |z| < 1\},$$

and let Δ^* denote $\Delta - \{0\}$. Choose a holomorphic map $\delta : \Delta \to X_{\mathrm{an}}$ sending 0 to x, and such that $\delta^{-1}(X^\circ) = \Delta^*$. This defines a holomorphic family of pointed complex analytic varieties

$$\begin{tikzcd} X_{\mathrm{an}} \times \Delta^* \arrow[d, "\mathrm{pr}_2"] \\ \Delta^* \arrow[u, bend left=60, "\delta \times \mathrm{id}"] \end{tikzcd}$$

and we can consider the assignment that sends a point $y \in \Delta^*$ to the group $\pi_1(X_{\mathrm{an}}, \delta(Y))$. Using for example the universal cover of Δ^* one sees that these fundamental groups of the fibers form a local system on Δ^*. If $y_0 \in \Delta^*$ is a point then the corresponding representation

$$\mathbb{Z} \simeq \pi_1(\Delta^*) \to \mathrm{Aut}(\pi_1(X_{\mathrm{an}}, \delta(y_0))$$

is given by sending the generator $1 \in \mathbb{Z}$ to conjugation by the image under $\delta_* : \pi_1(\Delta^*, y_0) \to \pi_1(X_{\mathrm{an}}, \delta(y_0))$ of $1 \in \mathbb{Z} \simeq \pi_1(\Delta^*, y_0)$.

1.2. We will consider this construction in the p-adic context replacing Δ^* by a p-adic field, and using p-adic Hodge theory for the fundamental group developed by Shiho and others. The technical differential graded algebra ingredients come from our earlier study of p-adic Hodge theory for the fundamental group in [8]. Let us review the main result of that paper, in the simplest case of constant coefficients.

Let p be a prime, and k a perfect field of characteristic p. Let V denote the ring of Witt vectors of k and let K be the field of fractions of V. Fix an algebraic closure $K \hookrightarrow \overline{K}$.

The ring V comes equipped with a lift of Frobenius $\sigma : V \to V$, which also induces an automorphism of K, which we denote by the same letter.

Let X/V be a smooth proper scheme, and let $D \subset X$ be a divisor with normal crossings relative to V. Denote by $X^\circ \subset X$ the complement of D in X, and by X_K, X°_K etc., the generic fibers. Let M_X denote the log structure on X defined by D. For any point $x \in X^\circ(V)$, we can then consider various realizations of the unipotent completion of the fundamental group of X°_K:

Etale realization $\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)$*:* This is the Tannaka dual of the category of unipotent étale $\mathbb{Q}_p$-local systems on the geometric generic fiber of X°. The group $\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)$ is a pro-unipotent group scheme over $\mathbb{Q}_p$ with action of the Galois group G_K of $\overline{K}$ over K.

De Rham realization $\pi_1^{\mathrm{dR}}(X^\circ_K, x)$*:* This is the Tannaka dual of the category of unipotent modules with integrable conection on X°_K/K. It is a pro-unipotent group scheme over K.

Crystalline realization $\pi_1^{\mathrm{crys}}(X^\circ_k, x)$*:* This is the Tannaka dual of the category of unipotent log isocrystals on (X_k, M_{X_k}) over V. It is a pro-unipotent group scheme over K with a semi-linear Frobenius automorphism φ.

The main result of [8] in the present situation is then that there is a canonical isomorphism of group schemes

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V) \simeq \pi_1^{\mathrm{crys}}(X^\circ_k, x) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V),$$

compatible with the Galois and Frobenius automorphisms. Here $\mathrm{B}_{\mathrm{cris}}(V)$ denotes Fontaine's ring of crystalline periods This implies in particular that the coordinate ring $\mathcal{O}_{\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)}$ is a direct limit of crystalline representations (see [8, Theorem D.3]). There is also a comparison isomorphism between $\pi_1^{\mathrm{crys}}(X^\circ_k, x)$ and $\pi_1^{\mathrm{dR}}(X^\circ_K, x)$.

1.3. The goal of the present paper is to explain what happens in the case when the base point $x_K \in X^\circ(K)$ specializes to a point of the boundary D in the closed fiber. In this case $\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)$ and $\pi_1^{\mathrm{dR}}(X^\circ_K, x_K)$ still make sense with no modification. We explain in this paper how to make sense of $\pi_1^{\mathrm{crys}}(X^\circ_k, x)$ in this setting, and in particular that the coordinate ring of this group scheme now carries a monodromy operator. After introducing these constructions we show the following result.

Theorem 1.4. *Let B_{st} denote Fontaine's ring of semistable periods. Then there is a canonical isomorphism of group schemes over B_{st}*

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(V) \simeq \pi_1^{\mathrm{crys}}(X^\circ_k, x) \otimes_K \mathrm{B}_{\mathrm{st}}(V), \tag{1.4.1}$$

compatible with Galois actions, Frobenius, and monodromy operators. Moreover, the coordinate ring $\mathscr{O}_{\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)}$ is a direct limit of semistable representations.

Remark 1.5. We also discuss a more general result about torsors of paths between two points.

1.6. Since $\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)$ is a pro-unipotent group scheme, we can write it canonically as a projective limit (using the derived series)

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K) = \varprojlim_N \pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)_N,$$

where $\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)_0$ is the abelianization, which is isomorphic to $H^1(X^\circ_{\overline{K}}, \mathbb{Q}_p)^\vee$, and such that the map

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)_N \to \pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)_{N-1}$$

is surjective with abelian kernel. We have a similar description on the crystalline side

$$\pi_1^{\mathrm{crys}}(X^\circ_k, x) = \varprojlim_N \pi_1^{\mathrm{crys}}(X^\circ_k, x)_N$$

and the isomorphism (6.11.1) induces isomorphisms for all N

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_K)_N \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(V) \simeq \pi_1^{\mathrm{crys}}(X^\circ_k, x)_N \otimes_{K_0} \mathrm{B}_{\mathrm{st}}(V).$$

Passing to Lie algebras this gives examples of finite dimensional semistable extensions which admit a filtration whose successive quotients are crystalline.

Remark 1.7. In this case we consider only the unramified case of varieties over the ring of Witt vectors rather than over a possibly ramified extension. We expect that similar techniques should yield analogous results in the ramified case, but this requires additional foundational work (in particular the setting of [8] is in the unramified case).

The paper is organized as follows. Sections 2, 3, and 4 are devoted to the foundational aspects of defining the monodromy operator on the crystalline fundamental group in our setting, and to explaining the Hyodo-Kato isomorphism for fundamental groups. In section 5 we discuss the comparison between de Rham and crystalline fundamental groups. Much of this material can already be extracted from Shiho's work [12]. In section 6 we review the necessary facts about semistable representations that we need, and discuss an equivalent variant of 1.4, which in fact is the result that we prove. The proof is based on various techniques using differential graded algebras and the methods of [8]. Section 7 contains some background material on differential graded algebras, and the proof of the main theorem is given in section 8. Finally the last two sections are devoted to the example of fundamental groups of punctured curves, and in particular the projective line minus three points.

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2. Unipotent isocrystals on the log point

2.1. Let k be a perfect field with ring of Witt vectors V. Let M_k be the log structure on $\mathrm{Spec}(k)$ associated to the map $\mathbb{N} \to k$ sending all nonzero elements to 0 (so $M_k \simeq \mathscr{O}^*_{\mathrm{Spec}(k)} \oplus \mathbb{N}$). Let $\mathscr{I}$ denote the category on unipotent isocrystals on $(\mathrm{Spec}(k), M_k)/K$, where K denotes the field of fractions of V.

2.2. Let $\mathrm{Mod}_K(\mathcal{N})$ denote the category of pairs (M, N), where M is a finite dimensional vector space over K, and $N : M \to M$ is a nilpotent endomorphism. We let $\mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N}) \subset \mathrm{Mod}_K(\mathcal{N})$ denote the full subcategory of pairs (M, N) for which there exists an N-stable filtration

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = M$$

such that the endomorphism of F^i/F^{i+1} induced by N is zero for all i.

2.3. There is a functor

$$\tilde{\eta}_0 : \mathscr{I} \rightarrow \mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N}) \tag{2.3.1}$$

defined as follows. Let L_V denote the log structure on $\mathrm{Spec}(V)$ induced by the map $\mathbb{N} \rightarrow V$ sending 1 to 0. The natural closed immersion

$$(\mathrm{Spec}(k), M_k) \hookrightarrow (\mathrm{Spec}(V), L_V)$$

defines an object of the crystalline topos of $(\mathrm{Spec}(k), M_k)/V$, which we denote by T.

If E is an isocrystal on $(\mathrm{Spec}(k), M_k)/V$ we can evaluate it on T to get a K-vector space, which we denote by $\mathcal{E}_0$. The crystal structure on E induces an endomorphism $N_0 : \mathcal{E}_0 \rightarrow \mathcal{E}_0$ as follows.

Consider the ring of dual numbers $V[\epsilon]$ (so we have $\epsilon^2 = 0$, but we suppress this from the notation), and let $L_{V[\epsilon]}$ denote the log structure on $\mathrm{Spec}(V[\epsilon])$ induced by pulling back L_V along the morphism

$$p : \mathrm{Spec}(V[\epsilon]) \rightarrow \mathrm{Spec}(V)$$

induced by the unique map of V-algebras

$$V \rightarrow V[\epsilon].$$

So we have

$$L_{V[\epsilon]} \simeq \mathscr{O}^*_{\mathrm{Spec}(V[\epsilon])} \oplus \mathbb{N}.$$

There is an automorphism ι of $L_{V[\epsilon]}$ defined by the map

$$\mathbb{N} \rightarrow \mathscr{O}^*_{\mathrm{Spec}(V[\epsilon])} \oplus \mathbb{N}, \quad 1 \mapsto (1 + \epsilon, 1).$$

Define $p_1^b : p^*L_V \rightarrow L_{V[\epsilon]}$ to be the natural map (by definition $p^*L_V = L_{V[\epsilon]}$ and under this identification p_1^b is the identity map), and let $p_2^b := \alpha \circ p_1^b$. Define

$$p_i : (\mathrm{Spec}(V[\epsilon]), L_{V[\epsilon]}) \rightarrow (\mathrm{Spec}(V), L_V), \quad i = 1, 2,$$

to be (p, p_i^b).

Setting ϵ to 0 defines a closed immersion of log schemes

$$j : (\mathrm{Spec}(V), L_V) \hookrightarrow (\mathrm{Spec}(V[\epsilon]), L_{V[\epsilon]}), \tag{2.3.2}$$

and we obtain a commutative diagram

$$\begin{tikzcd}
 & & (\mathrm{Spec}(k), M_k) \arrow[dll, hook'] \arrow[dl, hook'] \arrow[ddl, hook'] \\
(\mathrm{Spec}(V), L_V) \arrow[r, hook, "j"] \arrow[dr, "\mathrm{id}"] & (\mathrm{Spec}(V[\epsilon]), L_{V[\epsilon]}) \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\
 & (\mathrm{Spec}(V), L_V) &
\end{tikzcd} \tag{2.3.3}$$

The crystal structure on E therefore defines an isomorphism

$$\sigma : p_2^*\mathcal{E}_0 \rightarrow p_1^*\mathcal{E}_0,$$

which reduces to the identity modulo ϵ. Such an isomorphism is simply a map

$$\sigma : \mathcal{E}_0 \otimes_K K[\epsilon] \rightarrow \mathcal{E}_0 \otimes_K K[\epsilon] \tag{2.3.4}$$

reducing to the identity modulo ϵ. Giving such a map σ is equivalent to giving an endomorphism $N_0 : \mathcal{E}_0 \to \mathcal{E}_0$. Indeed, given σ we define N_0 by the formula

$$\sigma(x \otimes 1) = e + N_0(e) \cdot \epsilon.$$

Note also that if E is unipotent then $(\mathcal{E}_0, N_0) \in \mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N})$. We therefore get the functor $\tilde{\eta}_0$ by sending E to $(\mathcal{E}_0, N_0)$.

Remark 2.4. The category $\mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N})$ is Tannakian with fiber functor the forgetful functor to Vec_K. As discussed for example in [11, Chapitre IV, §2.5] the Tannaka dual group is isomorphic to $\mathbb{G}_a$. If (A, N) is an object of $\mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N})$ then the corresponding action of $\mathbb{G}_a$ on A is characterized by the element $1 \in \mathbb{G}_a$ acting by $\exp(N)$.

2.5. The category $\mathscr{I}$ can be described explicitly using modules with connection. Consider the surjection $V[t] \to V$ sending t to 0, and let $V\langle t\rangle$ denote the p-adically completed divided power envelope of the composite map

$$V[t] \to V \to k.$$

We write $K\langle t\rangle$ for $V\langle t\rangle[1/p]$. Let $M_{V\langle t\rangle}$ denote the log structure on $\mathrm{Spec}(V\langle t\rangle)$ induced by the map $\mathbb{N} \to V\langle t\rangle$ sending 1 to t. We then have a strict closed immersion

$$i : (\mathrm{Spec}(V), L_V) \hookrightarrow (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle})$$

obtained by setting $t = 0$. For an isocrystal E on $(\mathrm{Spec}(k), M_k)/K$ let $\mathcal{E}_{V\langle t\rangle}$ denote the value on

$$(\mathrm{Spec}(k), M_k) \hookrightarrow (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}),$$

which is a free $K\langle t\rangle$-module of finite rank. Furthermore, we have a canonical isomorphism

$$\mathcal{E}_{V\langle t\rangle} \otimes_{K\langle t\rangle} K \simeq \mathcal{E}_0,$$

induced by the closed immersion i.

2.6. There is a natural differential

$$d : K\langle t\rangle \to K\langle t\rangle \mathrm{dlog}(t)$$

sending $t^{[i]}$ to $it^{[i]}\mathrm{dlog}(t)$. If M is a $K\langle t\rangle$-module, we define a *connection on* M to be a K-linear map

$$\nabla : M \to M \cdot \mathrm{dlog}(t)$$

satisfying the Leibnitz rule

$$\nabla(fm) = (df) \cdot m + f\nabla(m).$$

Define $\mathrm{Mod}_{K\langle t\rangle}(\nabla)$ to be the category of pairs (M, ∇), where M is a finitely generated free $K\langle t\rangle$-module and ∇ is a connection on M. Define $\mathrm{Mod}_{K\langle t\rangle}^{\mathrm{un}}(\nabla) \subset \mathrm{Mod}_{K\langle t\rangle}(\nabla)$ to be the full subcategory of pairs (M, ∇) for which there exists a finite ∇-stable filtration by $K\langle t\rangle$-submodules

$$0 = F^n \subset F^{n-1} \subset \cdots \subset F^0 = M$$

such that each successive quotient F^i/F^{i+1} is isomorphic to a finite direct sum of copies of $(K\langle t\rangle, d)$.

Let $J \subset K\langle t\rangle$ denote the kernel of the surjection

$$K\langle t\rangle \to K, \quad t \mapsto 0.$$

Note that for any $K\langle t\rangle$-module M with connection ∇, the connection ∇ induces a K-linear map

$$\nabla_0 : M/JM \to M/JM,$$

characterized by the condition that for any $m \in M$ we have $\nabla_0(\bar{m}) \cdot \mathrm{dlog}(t)$ equal to the reduction of $\nabla(m)$. It follows from the construction that we get a functor

$$\Pi : \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla) \to \mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N}).$$

2.7. Now by the standard correspondence between isocrystals and modules with integrable connection (see for example [5, 6.2]), evaluation on

$$(\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle})$$

defines an equivalence of categories

$$\tilde{\eta}_{V\langle t\rangle} : \mathscr{I} \to \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla).$$

Furthermore, the composite $\Pi \circ \tilde{\eta}_{V\langle t\rangle}$ is the functor $\tilde{\eta}_0$.

There is also a functor

$$\mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N}) \to \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla)$$

defined by sending an object $(A, N) \in \mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N})$ to the object $(M, \nabla) \in \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla)$ obtained by setting $M = A \otimes_K K\langle t\rangle$, and defining ∇ to be the unique connection sending $a \otimes 1 \in V \otimes_K K\langle t\rangle$ to $(N(a) \otimes 1) \cdot \mathrm{dlog}(t)$.

2.8. If one incorporates also Frobenius then the functor Π becomes an equivalence. This is a consequence of the so-called Hyodo-Kato isomorphism [4, 4.13] (see also [9, Chapter 5]).

Let $\mathrm{Mod}^{\mathrm{un}}_K(\varphi, \mathcal{N})$ denote the category of triples (A, N, φ_A), where $(A, N) \in \mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N})$ and $\varphi_A : \sigma^* A \to A$ is an isomorphism of K-vector spaces such that

$$\varphi_A \circ N = pN \circ \varphi_A.$$

The ring $V\langle t\rangle$ has a lifting of Frobenius given by σ on V and the map $t \mapsto t^p$. We denote this map by $\sigma_{V\langle t\rangle}$, and the induced map on $K\langle t\rangle$ by $\sigma_{K\langle t\rangle}$. Let $F{-}\mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla)$ denote the category of triples (M, ∇, φ_M) consisting of an object $(M, \nabla) \in \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla)$ and an isomorphism

$$\varphi_M : \sigma^*_{K\langle t\rangle}(M, \nabla) \to (M, \nabla)$$

in $\mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla)$.

Finally let $F - \mathscr{I}$ denote the category of F-isocrystals on $(\mathrm{Spec}(k), M_k)/K$ for which the underlying isocrystal is unipotent.

The previously defined functors then extend to give functors

$$\begin{tikzcd} F - \mathscr{I} \arrow[r, "\tilde{\eta}_{V\langle t\rangle}"] \arrow[rr, bend left, "\tilde{\eta}_0"] & F - \mathrm{Mod}^{\mathrm{un}}_{K\langle t\rangle}(\nabla) \arrow[r, "\Pi"] & \mathrm{Mod}^{\mathrm{un}}_K(\varphi, \mathcal{N}). \end{tikzcd} \tag{2.8.1}$$

Proposition 2.9. *All the functors in* (2.8.1) *are equivalences.*

Proof. The statement that the functor labelled $\tilde{\eta}_{V\langle t\rangle}$ is an equivalence follows from the corresponding statement without the Frobenius structure. It therefore suffices to show that the functor Π in (2.8.1) is an equivalence, which follows from [8, 5.3.25]. $\square$

3. The monodromy operator on π_1^{crys}

3.1. Let X/V, $D \subset X$, and $X^\circ \subset X$ be as in the introduction. Let M_X denote the fine log structure on X defined by D.

Since X/V is proper, the point x_K extends uniquely to a point

$$x : \mathrm{Spec}(V) \to X,$$

and in fact uniquely to a morphism of log schemes

$$x : (\mathrm{Spec}(V), M_V) \to (X, M_X),$$

where M_V is the log structure on V associated to the chart $\mathbb{N} \to V$ sending 1 to p.

3.2. Let (X_k, M_{X_k}) denote the reduction modulo p of (X, M_X). Note that the reduction modulo p of $(\mathrm{Spec}(V), M_V)$ is the log point as discussed in section 2.

Let $\mathscr{C}^{\mathrm{crys}}$ denote the category of unipotent log isocrystals on $((X_k, M_{X_k})/K)$. As discussed in [12, 4.1.4] this is a Tannakian category over K. The point

$$y : (\mathrm{Spec}(k), M_k) \to (X_k, M_{X_k}),$$

obtained by reduction from x, defines a functor

$$y^* : \mathscr{C}^{\mathrm{crys}} \to \mathscr{I},$$

where $\mathscr{I}$ is defined as in 2.1. Composing with the functor $\tilde{\eta}_0$ (2.3.1), we get a functor

$$\tilde{\omega}_0^{\mathrm{crys}} : \mathscr{C}^{\mathrm{crys}} \to \mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N}).$$

By further composing with the forgetful functor

$$\mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N}) \to \mathrm{Vec}_K,$$

we obtain a functor

$$\omega_0^{\mathrm{crys}} : \mathscr{C}^{\mathrm{crys}} \to \mathrm{Vec}_K.$$

Proposition 3.3. *The functor ω_0^{crys} is a fiber functor.*

Proof. This follows from [8, 8.11]. □

3.4. Let $\pi_1^{\mathrm{crys}}(X_K^\circ, x)$ denote the Tannaka dual of the category $\mathscr{C}^{\mathrm{crys}}$ with respect to the fiber functor ω_0^{crys}. This is a pro-unipotent group scheme over K.

It has a Frobenius automorphism defined as follows. First note that there is a commutative diagram

$$\begin{tikzcd}
(\mathrm{Spec}(k), M_k) \arrow[r, "F_k"] \arrow[d, "y"] & (\mathrm{Spec}(k), M_k) \arrow[d, "y"] \\
(X_k, M_{X_k}) \arrow[r, "F_Y"] & (X_k, M_{X_k}),
\end{tikzcd}$$

where the horizontal arrows are the Frobenius endomorphisms. We therefore have a 2-commutative diagram

$$
\begin{tikzcd}
\mathscr{C}^{\mathrm{crys}} \arrow[r, "F^*_{X_k}"] \arrow[d, "y^*"] & \mathscr{C}^{\mathrm{crys}} \arrow[d, "y^*"] \\
\mathscr{I} \arrow[r, "F^*_k"] & \mathscr{I}.
\end{tikzcd}
\tag{3.4.1}
$$

It follows for example from [8, 4.26] that the horizontal functors are equivalences of categories. Since the log scheme $(\mathrm{Spec}(V), L_V)$ also has a lifting of Frobenius given by $\sigma : V \to V$ and multiplication by p on L_V, there is a natural isomorphism between the composite functor

$$\mathscr{I} \xrightarrow{F^*_k} \mathscr{I} \xrightarrow{\tilde{\eta}_0} \mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N}) \xrightarrow{\text{forget}} \mathrm{Mod}_K,$$

and the composite functor

$$\mathscr{I} \xrightarrow{\tilde{\eta}_0} \mathrm{Mod}^{\mathrm{un}}_K(\mathcal{N}) \xrightarrow{\text{forget}} \mathrm{Mod}_K \xrightarrow{(-)\otimes_\sigma K} \mathrm{Mod}_K.$$

We therefore obtain an isomorphism of functors

$$\omega_0^{\mathrm{crys}} \circ F^*_{X_k} \simeq \omega_0^{\mathrm{crys}} \otimes_{K,\sigma} K.$$

This defines an isomorphism of group schemes over K

$$\varphi : \pi_1^{\mathrm{crys}}(X_k^\circ, x) \otimes_{K,\sigma} K \to \pi_1^{\mathrm{crys}}(X_k^\circ, x),$$

which we refer to as the *Frobenius endomorphism of* $\pi_1^{\mathrm{crys}}(X_K^\circ, x)$.

3.5. There is also a monodromy operator on $\pi_1^{\mathrm{crys}}(X_K^\circ, x)$ defined as follows. As in 2.3 let $V[\epsilon]$ denote the ring of dual numbers over V. Then the monodromy operator will, by definition, be an isomorphism of group schemes over $V[\epsilon]$

$$\mathscr{N} : \pi_1^{\mathrm{crys}}(X_k^\circ, x) \otimes_K K[\epsilon] \to \pi_1^{\mathrm{crys}}(X_k^\circ, x) \otimes_K K[\epsilon]$$

whose reduction modulo ϵ is the identity. Note that, by the discussion in 2.3 such an isomorphism is specified by a K-linear map

$$N : \mathscr{O}_{\pi_1^{\mathrm{crys}}(X_k^\circ, x)} \to \mathscr{O}_{\pi_1^{\mathrm{crys}}(X_k^\circ, x)}. \tag{3.5.1}$$

The isomorphism $\mathscr{N}$ is constructed as follows. Let

$$\eta_{K[\epsilon]} : \mathscr{I} \to \mathrm{Mod}_{K[\epsilon]}$$

be functor evaluating an isocrystal on the object (2.3.2). We then get a fiber functor

$$\omega_{V[\epsilon]} : \mathscr{C}^{\mathrm{crys}} \to \mathrm{Mod}_{K[\epsilon]}$$

by taking the composite

$$\mathscr{C}^{\mathrm{crys}} \xrightarrow{y^*} \mathscr{I} \xrightarrow{\eta_{V[\epsilon]}} \mathrm{Mod}_{K[\epsilon]},$$

and we can consider the corresponding Tannaka dual group

$$\pi_1(\mathscr{C}^{\mathrm{crys}}, \omega_{V[\epsilon]}).$$

The diagram (2.3.3) induces two isomorphisms of functors

$$\alpha_i : \omega_0^{\mathrm{crys}} \otimes_K K[\epsilon] \to \omega_{V[\epsilon]}, \quad i = 1, 2,$$

which in turn induce an automorphism of group schemes

$$\pi_1^{\text{crys}}(X_k^\circ, x) \otimes_K K[\epsilon] \xrightarrow{\alpha_1} \pi_1(\mathscr{C}^{\text{crys}}, \omega_{V[\epsilon]}) \xrightarrow{\alpha_2^{-1}} \pi_1^{\text{crys}}(X_k^\circ, x) \otimes_K K[\epsilon]. \tag{3.5.2}$$

We define the monodromy operator $\mathscr{N}$ to be this composite.

3.6. More generally, given $x_{i,K} \in X^\circ(K)$ for $i = 1, 2$, we get two points

$$x_i : (\text{Spec}(V), M_V) \to (X, M_X),$$

and reductions y_i. Let

$$\pi^{\text{crys}}(X_k^\circ, x_1, x_2)$$

denote the functor of isomorphisms of fiber functors between the resulting two functors

$$\omega_{x_i,0}^{\text{crys}} : \mathscr{C}^{\text{crys}} \to \text{Vec}_K.$$

Then $\pi^{\text{crys}}(X_k^\circ, x_1, x_2)$ is a torsor under the group scheme $\pi_1^{\text{crys}}(X_k^\circ, x_1)$ and by a similar construction to the one in 3.4 and 3.5 comes equipped with a Frobenius automorphism and monodromy operator.

Remark 3.7. By the general theory of unipotent group schemes the functor taking Lie algebras induces an equivalence of categories between the category of unipotent group schemes over K and the category of nilpotent Lie algebras over K. The inverse functor is given by sending a Lie algebra L to the scheme $\mathbb{L}$ corresponding to L with group structure given by the Campbell-Hausdorf series. One consequence of this for our purposes is that the coordinate ring of $\pi_1^{\text{crys}}(X_k^\circ, x)$ is canonically isomorphic to the symmetric algebra on the dual of $\text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x))$. In particular, the monodromy operator is determined by its action on the Lie algebra.

Remark 3.8. A reformulation of the above construction of the monodromy operator is the following. The isomorphisms (2.3.4) define an automorphism of the fiber functor $\omega_0^{\text{crys}} \otimes_K K[\epsilon]$, and therefore an element

$$\alpha \in \text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x)) = \text{Ker}(\pi_1^{\text{crys}}(X_k^\circ, x)(K[\epsilon]) \to \pi_1^{\text{crys}}(X_k^\circ, x)(K)).$$

By the general Tannakian formalism the isomorphism (3.5.2) is given by conjugation by α. If we denote by $[-,-]$ the Lie bracket on $\text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x))$ it follows that the action on $\text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x))$ induced by (3.5.2) is the map

$$[-, \alpha] : \text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x)) \to \text{Lie}(\pi_1^{\text{crys}}(X_k^\circ, x)).$$

This implies in particular that for any surjective homomorphism of algebraic groups $\pi_1^{\text{crys}}(X_k^\circ, x) \to H$ the endomorphism N in (3.5.1) restricts to an endomorphism of $\mathscr{O}_H$.

4. The Hyodo-Kato isomorphism for the fundamental group

We proceed with the notation of the preceding section.

4.1. It will be useful to consider connections on geometric objects such as algebraic groups or Lie algebras. This can be done in the following manner.

As usual for a ring A let $A[\epsilon]$ denote the ring of dual numbers on A. There are two maps

$$p_1, p_2 : V[t] \to V[t][\epsilon]$$

over V given by sending t to t and $t+\epsilon t$ respectively. This extends naturally to a morphism of log schemes and induces a commutative diagram

$$\begin{tikzcd}
(\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}) \arrow[r, hook, "\iota"] \arrow[dr, equal] & (\mathrm{Spec}(V\langle t\rangle[\epsilon]), M_{\langle t\rangle[\epsilon]}) \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
 & (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}).
\end{tikzcd}$$

4.2. Let

$$\eta_{K\langle t\rangle} : \mathscr{I} \rightarrow \mathrm{Mod}_{K\langle t\rangle}$$

be the functor obtained by evaluating an isocrystal on the object

$$(\mathrm{Spec}(k), M_k) \hookrightarrow (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}).$$

Composing with $y^* : \mathscr{C}^{\mathrm{crys}} \rightarrow \mathscr{I}$ we get a fiber functor

$$\omega_{K\langle t\rangle} : \mathscr{C}^{\mathrm{crys}} \rightarrow \mathrm{Mod}_{K\langle t\rangle}.$$

Let

$$\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$$

denote the corresponding Tannaka dual group over $K\langle t\rangle$.

This group scheme over $K\langle t\rangle$ comes equipped with the following structure:

(i) An isomorphism

$$\varphi_{K\langle t\rangle} : \pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}) \otimes_{K\langle t\rangle, \sigma_{K\langle t\rangle}} K\langle t\rangle \rightarrow \pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}).$$

We refer to this as a *Frobenius structure* on $\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$.

(ii) An isomorphism

$$\epsilon_{K\langle t\rangle} : p_1^*\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}) \rightarrow p_2^*\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$$

over $K\langle t\rangle[\epsilon]$ reducing to the identity over $K\langle t\rangle$. This isomorphism is obtained by noting that the two functors $p_1^*\omega_{K\langle t\rangle}$ and $p_2^*\omega_{K\langle t\rangle}$ are canonically isomorphic. We refer to such an isomorphism $\epsilon_{K\langle t\rangle}$ as a *connection*.

4.3. We have a commutative diagram of rings

$$\begin{tikzcd}
K \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & K[\epsilon] \\
K\langle t\rangle \arrow[u, "t=0"'] \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & K\langle t\rangle[\epsilon] \arrow[u, "t=0"'] \\
K \arrow[u] \arrow[uu, bend left=60, "\mathrm{id}"] \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & K[\epsilon] \arrow[u] \arrow[uu, bend right=60, "\mathrm{id}"']
\end{tikzcd}$$

By construction we have an isomorphism of group schemes with Frobenius structure and monodromy operator (notation as in 3.5

$$(\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}), \varphi_{K\langle t\rangle}, \epsilon_{K\langle t\rangle}) \otimes_{K\langle t\rangle, t=0} K \simeq (\pi_1^{\mathrm{crys}}(X_k^\circ, x), \varphi, \mathcal{N}). \tag{4.3.1}$$

Conversely, we can base change along $K \to K\langle t\rangle$ to get a group scheme with Frobenius structure and connection

$$(\pi_1^{\mathrm{crys}}(X_k^\circ, x), \varphi, \mathcal{N}) \otimes_K K\langle t\rangle.$$

Lemma 4.4. *There exists a unique isomorphism of group schemes over $K\langle t\rangle$ with Frobenius structure and connection*

$$(\pi_1^{\mathrm{crys}}(X_k^\circ, x), \varphi, \mathcal{N}) \otimes_K K\langle t\rangle \simeq (\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}), \varphi_{K\langle t\rangle}, \epsilon_{K\langle t\rangle})$$

reducing to the isomorphism (4.3.1) *after setting $t = 0$.*

Proof. It suffices to prove the corresponding statement for the Lie algebras of the quotients by the derived series (see 1.6). In this case the result follows form the usual Hyodo-Kato isomorphism as discussed for example in [8, 5.3.25]. □

Remark 4.5. Likewise one can consider torsors of paths between two points. With notation as in 3.6 we can consider the two fiber functors to $\mathrm{Mod}_{K\langle t\rangle}$ obtained by evaluation as in the preceding construction to get a $\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$-torsor (where $\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$ is defined using the point x_1)

$$\pi_1^{\mathrm{crys}}(X_k^\circ, x_{1,K\langle t\rangle}, x_{2,K\langle t\rangle})$$

equipped with a Frobenius structure $\varphi_{K\langle t\rangle}$ and connection $\epsilon_{K\langle t\rangle}$ compatible with the structures on $\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$. Then by an argument similar to the above one gets an isomorphism

$$(\pi_1^{\mathrm{crys}}(X_k^\circ, x_1, x_2), \varphi, \mathcal{N}) \otimes_K K\langle t\rangle \simeq (\pi_1^{\mathrm{crys}}(X_k^\circ, x_{1,K\langle t\rangle}, x_{2,K\langle t\rangle}), \varphi_{K\langle t\rangle}, \epsilon_{K\langle t\rangle})$$

of torsors compatible with the isomorphism in 4.4.

5. Crystalline and de Rham comparison

We follow the method of [12, Chapter V] with a slight modification to take into account the specialization of the base point to the boundary.

5.1. Let $\mathscr{C}^{\mathrm{dR}}$ denote the category of unipotent modules with integrable connection on X_K°/K. This is a Tannakian category, and the point $x_K \in X^\circ(K)$ defines a fiber functor

$$\omega_{x_K}^{\mathrm{dR}} : \mathscr{C}^{\mathrm{dR}} \to \mathrm{Vec}_K.$$

We let $\pi_1^{\mathrm{dR}}(X_K^\circ, x_K)$ denote the Tannaka dual of $\mathscr{C}^{\mathrm{dR}}$ with respect to the fiber functor $\omega_{x_K}^{\mathrm{dR}}$.

There is a natural isomorphism

$$\pi_1^{\mathrm{crys}}(X_k^\circ, x) \simeq \pi_1^{\mathrm{dR}}(X_K^\circ, x_K) \tag{5.1.1}$$

defined as follows.

5.2. As before, let $\mathscr{C}^{\mathrm{crys}}$ denote the category of unipotent log isocrystals on $(X_k, M_{X_k})/K$. The correspondence between isocrystals and modules with integrable connection furnishes a natural equivalence of categories

$$\mathscr{C}^{\mathrm{crys}} \to \mathscr{C}^{\mathrm{dR}}.$$

Moreover, this equivalence identifies the functor $\omega_{x_K}^{\mathrm{dR}}$ with the fiber functor

$$\omega_x^{\mathrm{crys}} : \mathscr{C}^{\mathrm{crys}} \to \mathrm{Vec}_K$$

which evaluates an isocrystal on the p-adic enlargement

$$(\mathrm{Spec}(k), M_k) \hookrightarrow (\mathrm{Spec}(V), M_V).$$

5.3. On the other hand, we have a commutative diagram

$$\begin{tikzcd}
 & (\mathrm{Spec}(V), M_V) \arrow[d, "t\mapsto \pi"] \\
(\mathrm{Spec}(k), M_k) \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] \arrow[dd, "y"] & (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}) \\
 & (\mathrm{Spec}(V), L_V) \arrow[u, "t\mapsto 0"'] \\
(X_k, M_{X_k}). &
\end{tikzcd}$$

From this diagram we obtain an isomorphism of fiber functors on $\mathscr{C}^{\mathrm{crys}}$

$$\omega_x^{\mathrm{crys}} \simeq \omega_{K\langle t\rangle}^{\mathrm{crys}} \otimes_{K\langle t\rangle, t\mapsto p} K,$$

where the right side is the fiber functor obtained by evaluating on $(\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle})$. This defines an isomorphism of group schemes over K

$$\pi_1^{\mathrm{crys}}(X_k^\circ, \omega_{K\langle t\rangle}) \otimes_{K\langle t\rangle, t\mapsto p} K \simeq \pi_1^{\mathrm{dR}}(X_K^\circ, x_K).$$

Combining this with the isomorphism 4.4 we obtain the isomorphsim (5.1.1).

5.4. Similarly for two points $x_{i,K} \in X^\circ(K)$ we can consider the torsor of isomorphisms of fiber functors $\omega_{x_{1,K}}^{\mathrm{dR}} \simeq \omega_{x_{2,K}}^{\mathrm{dR}}$ which we denote by

$$\pi^{\mathrm{dR}}(X_K^\circ, x_{1,K}, x_{2,K}).$$

Using the preceding isomorphisms of fiber functors for each of the points x_i we get an isomorphism of torsors

$$\pi^{\mathrm{dR}}(X_K^\circ, x_{1,K}, x_{2,K}) \simeq \pi^{\mathrm{crys}}(X_k^\circ, x_1, x_2).$$

6. Review of semistable representations

For the convenience of the reader, and to establish some basic notation, we summarize in this section some of the basic definitions and results about period rings that we need in the following sections.

6.1. For a $\mathbb{Z}_p$-algebra A with $A/pA \neq 0$ and Frobenius surjective on A/pA, we write $\mathrm{A}_{\mathrm{cris}}(A)$ for the ring defined in [3, 2.2.2] (a good summary can be found in [14, §1]). Recall (see for example [6, 2.2]) that this can can be described as

$$\mathrm{A}_{\mathrm{cris}}(A) = \varprojlim_n B_n(A),$$

where

$$B_n(A) := \Gamma((\mathrm{Spec}(A/p^nA)/W_n)_{\mathrm{crys}}, \mathcal{O}).$$

As discussed in [6, 2.2] the choice of elements $\epsilon_m \in \widehat{A}$ with $\epsilon_0 = 1$, and $\epsilon_{m+1}^p = \epsilon_m$, and $\epsilon_1 \neq 1$, defines an element $t \in \mathrm{A}_{\mathrm{cris}}(A)$ and we have rings

$$\mathrm{B}_{\mathrm{cris}}(A)^+ := \mathrm{A}_{\mathrm{cris}}(A) \otimes \mathbb{Q},$$

and

$$\mathrm{B}_{\mathrm{cris}}(A) := \mathrm{B}_{\mathrm{cris}}(A)^+[1/t].$$

6.2. Next let us recall the definition of $\mathrm{B}_{\mathrm{st}}(\overline{V})$ following [6, §2]. We will only consider the unramified case, though of course these definitions can be made more generally.

Let V, k, K, σ and M_k be as in 1.2 and 2.1. Fix also the following notation:

M_V	The log structure on $\mathrm{Spec}(V)$ defined by the closed fiber.
V_n	The quotient $V/p^{n+1}V$.
M_{V_n}	The pullback of M_V to $\mathrm{Spec}(V_n)$
$\overline{K}$	An algebraic closure of K.
$\overline{V}$	The integral closure of V in $\overline{K}$.
$M_{\overline{V}}$	The log structure on $\mathrm{Spec}(\overline{V})$ defined by the closed fiber. Note that $M_{\overline{V}}$ is not fine but is a colimit of fine log structures.
$\overline{V}_n$	The quotient $\overline{V}/p^{n+1}\overline{V}$.
$M_{\overline{V}_n}$	The pullback of $M_{\overline{V}}$ to $\mathrm{Spec}(\overline{V}_n)$.

We then have a morphism of log schemes over V_n

$$(\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n}) \to (\mathrm{Spec}(V_n), M_{V_n}),$$

which induces a morphism of topoi

$$h : ((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}} \to ((\mathrm{Spec}(V_n), M_{V_n})/V_n)_{\mathrm{crys}}.$$

There is a surjection

$$V[t] \to V$$

sending t to p. Let $V_n\langle t\rangle$ denote the divided power envelope of the induced surjection

$$V_n[t] \to V_n.$$

There is a log structure $M_{V_n\langle t\rangle}$ on $\mathrm{Spec}(V_n\langle t\rangle)$ induced by the composite morphism

$$\mathbb{N} \xrightarrow{1\mapsto t} V_n[t] \longrightarrow V_n\langle t\rangle.$$

The resulting strict closed immersion

$$(\mathrm{Spec}(V_n), M_{V_n}) \hookrightarrow (\mathrm{Spec}(V_n\langle t\rangle), M_{V_n\langle t\rangle})$$

is an object of the crystalline site $\mathrm{Cris}((\mathrm{Spec}(V_n), M_{V_n})/V_n)$. Let P_n^{st} denote the value of

$$h_*\mathscr{O}_{((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}}}$$

on this object. The ring P_n^{st} is a $V_n\langle t\rangle$-algebra, and there is a natural map

$$P_n^{\mathrm{st}} \to \overline{V}_n,$$

whose kernel is a PD-ideal. This map even extends to a strict closed immersion of log schemes

$$(\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n}) \hookrightarrow (P_n^{\mathrm{st}}, M_{P_n^{\mathrm{st}}}),$$

where the log structure $M_{P_n^{\mathrm{st}}}$ is defined as in [6, 3.9].

There is a natural map (where the right side has trivial log structure)

$$(\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n}) \to \mathrm{Spec}(\overline{V}_n)$$

which induces a morphism

$$B_n(\overline{V}) \to \Gamma(((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}}, \mathscr{O}_{((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}}}).$$

In particular, the structure sheaf

$$\mathscr{O}_{((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}}}$$

has a natural structure of $B_n(\overline{V})$-algebra, and hence P_n^{st} also has a natural structure of a $B_n(\overline{V})$-algebra.

The ring P_n^{st} can be described explicitly. It is shown in [6, 3.3] that the choice of a p^{n+1}-th root of p in $\overline{V}$ induces an element $\nu_\beta \in P_n^{\mathrm{st},*}$ such that $\nu_\beta - 1$ lies in the divided power ideal of P_n^{st}, and that the resulting map

$$B_n(\overline{V}_n)\langle z\rangle \to P_n^{\mathrm{st}}, \quad z \mapsto \nu_\beta - 1 \tag{6.2.1}$$

is an isomorphism.

6.3. Passing to the limit, define

$$P^{\mathrm{st}} := \varprojlim_n P_n,$$

and let $P_{\mathbb{Q}}^{\mathrm{st}}$ denote $P^{\mathrm{st}} \otimes \mathbb{Q}$. If we fix a compatible sequence of p^n-th roots of p, then the construction in [6, 3.3] defines an isomorphism between P^{st} and the p-adically completed PD-polynomial algebra $\mathrm{A}_{\mathrm{cris}}(\overline{V})\langle z\rangle$.

In particular, the ring $P_{\mathbb{Q}}^{\mathrm{st}}$ is a $\mathrm{B}_{\mathrm{cris}}(\overline{V})^+$-algebra.

6.4. There is an endomorphism

$$\mathscr{N} : P^{\mathrm{st}} \to P^{\mathrm{st}}$$

defined as follows. Let $V_n\langle t\rangle[\epsilon]$ denote the ring of dual numbers over $V_n\langle t\rangle$ (so $\epsilon^2 = 0$). Let $(J_{V_n\langle t\rangle}, \gamma)$ be the divided power ideal of $V_n\langle t\rangle$. Then the ideal $J_{V_n\langle t\rangle} + \epsilon V_n\langle t\rangle \subset V_n\langle t\rangle[\epsilon]$ carries a canonical divided power structure compatible with that on $V_n\langle t\rangle$ (this is an immediate verification). Let $M_{V_n\langle t\rangle[\epsilon]}$ denote the log structure on $\mathrm{Spec}(V_n\langle t\rangle[\epsilon])$ obtained by pulling back the log structure $M_{V_n\langle t\rangle}$ along the retraction

$$V_n\langle t\rangle \to V_n\langle t\rangle[\epsilon].$$

Then we obtain a commutative diagram of objects in $\mathrm{Cris}((\mathrm{Spec}(V_n), M_{V_n})/V_n)$

$$\begin{tikzcd}
 & & (\mathrm{Spec}(V_n), M_{V_n}) \arrow[dll, hook'] \arrow[dl, hook'] \arrow[ddl, hook'] \\
(\mathrm{Spec}(V_n\langle t\rangle), M_{V_n\langle t\rangle}) \arrow[r, hook, "j"] \arrow[dr, "\mathrm{id}"] & (\mathrm{Spec}(V_n\langle t\rangle[\epsilon]), M_{V_n\langle t\rangle[\epsilon]}) \arrow[d, shift right, "p_2"'] \arrow[d, shift left, "p_1"] & \\
 & (\mathrm{Spec}(V_n\langle t\rangle), M_{V_n\langle t\rangle}). &
\end{tikzcd}$$

By [6, 3.1] the sheaf

$$h_*\mathscr{O}_{((\mathrm{Spec}(\overline{V}_n), M_{\overline{V}_n})/V_n)_{\mathrm{crys}}}$$

is a quasi-coherent crystal, and therefore we obtain an isomorphism

$$\gamma_{P_n^{\mathrm{st}}} : P_n^{\mathrm{st}} \otimes_{V_n\langle t\rangle} V_n\langle t\rangle[\epsilon] \to P_n^{\mathrm{st}} \otimes_{V_n\langle t\rangle} V_n\langle t\rangle[\epsilon] \tag{6.4.1}$$

reducing to the identity modulo ϵ. We define

$$\mathcal{N} : P^{\mathrm{st}}_n \to P^{\mathrm{st}}_n$$

to be the map characterized by the property that the isomorphism (6.4.1) sends $x \otimes 1$ to $x \otimes 1 + \mathcal{N}(x) \otimes \epsilon$. By passing to the inverse limit over n we then also obtain a connection $\gamma_{P^{\mathrm{st}}}$ with associated endomorphism $\mathcal{N} : P^{\mathrm{st}} \to P^{\mathrm{st}}$, and also an endomorphism of $P^{\mathrm{st}}_{\mathbb{Q}}$ (which we will again denote by $\mathcal{N}$).

Explicitly, if we fix a p^{n+1}-st root β of p in $\overline{V}$, defining an isomorphism (6.2.1), then the endomorphism $\mathcal{N}$ sends $B_n(\overline{V}_n)$ to 0, and $z^{[i]}$ to $z^{[i-1]}\nu_\beta$ by [6, 3.3].

Define $\mathrm{B}_{\mathrm{st}}(\overline{V})^+ \subset P^{\mathrm{st}}_{\mathbb{Q}}$ to be the subalgebra of elements $x \in P^{\mathrm{st}}_{\mathbb{Q}}$ for which there exists an integer $i \geq 1$ with $\mathcal{N}^i(x) = 0$. Finally define

$$\mathrm{B}_{\mathrm{st}}(\overline{V}) := \mathrm{B}_{\mathrm{st}}(\overline{V})^+[1/t] = \mathrm{B}_{\mathrm{st}}(\overline{V})^+ \otimes_{\mathrm{A}_{\mathrm{cris}}(\overline{V})^+} \mathrm{B}_{\mathrm{crys}}(\overline{V}).$$

6.5. The ring $P^{\mathrm{st}}_{\mathbb{Q}}$ comes equipped with a Frobenius automorphism

$$\varphi : P^{\mathrm{st}}_{\mathbb{Q}} \to P^{\mathrm{st}}_{\mathbb{Q}},$$

which extends the Frobenius endomorphism on $\mathrm{B}_{\mathrm{cris}}(\overline{V})^+$, and we have the relation

$$p\varphi\mathcal{N} = \mathcal{N}\varphi.$$

In particular, φ restricts to an automorphism of $\mathrm{B}_{\mathrm{st}}(\overline{V})^+$. There is also an action of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ on P^{st}, which commutes with the action of $\mathcal{N}$ and φ. This action restricts to an action of G_K on $\mathrm{B}_{\mathrm{st}}(\overline{V})^+$.

6.6. Finally for the convenience of the reader let us recall the definition of a semistable representation (for more details see [3]).

Let $\mathrm{Rep}(G_K)$ denote the category of finite dimensional $\mathbb{Q}_p$-vector spaces with continuous action of G_K.

As in 2.8 define $\mathrm{Mod}_K(\varphi, \mathcal{N})$ to be the category of triples (A, N, φ_A), where A is a finite dimensional K-vector space, $\varphi_A : A \to A$ is a semilinear automorphism, and $N : A \to A$ is a nilpotent endomorphism satisfying

$$p\varphi_A N = N\varphi_A.$$

There is a functor

$$\mathbf{D}_{\mathrm{st}} : \mathrm{Rep}(G_K) \to \mathrm{Mod}_K(\varphi, \mathcal{N})$$

defined as follows.

Let M be a finite dimensional $\mathbb{Q}_p$-vector space with continuous G_K-action. Define

$$\mathbf{D}_{\mathrm{st}}(M) := (M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}))^{G_K}.$$

This has a semilinear endomorphism φ, and a nilpotent operator N induced by the endomorphisms φ and $\mathcal{N}$ on $\mathrm{B}_{\mathrm{st}}(\overline{V})$. We therefore get an object of $\mathrm{Mod}_K(\varphi, \mathcal{N})$.

There is a natural map

$$\alpha_M : \mathbf{D}_{\mathrm{st}}(M) \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \to M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V})$$

which is always injective. The representation M is called *semistable* if α_M is an isomorphism. This is equivalent to the condition that

$$\dim_K(\mathbf{D}_{\mathrm{st}}(M)) = \dim_{\mathbb{Q}_p}(M).$$

The notion of a semistable representation can also be described in terms of the rings $P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$ instead of B_{st}:

Proposition 6.7. *Let $M \in \mathrm{Rep}(G_K)$ be a representation, let (A, N, φ_A) be an object of $\mathrm{Mod}_K(\varphi, \mathcal{N})$, and suppose given an isomorphism*

$$\lambda : A \otimes_K P^{\mathrm{st}}_{\mathbb{Q}}[1/t] \to M \otimes_{\mathbb{Q}_p} P^{\mathrm{st}}_{\mathbb{Q}}[1/t].$$

compatible with Frobenius, monodromy operators, and Galois action. Then M is a semistable representation and the isomorphism λ is induced by an isomorphism over $\mathrm{B}_{\mathrm{st}}(\overline{V})$.

Proof. The key point is that the inclusion

$$A \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \hookrightarrow A \otimes_K P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$$

identifies $A \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V})$ with the elements $A \otimes_K P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$ on which the monodromy operator is nilpotent. To verify this claim notice that A admits a finite filtration stable under the monodromy operator such that the successive quotients have trivial monodromy operator. Using this one sees that to verify the claim it suffices to show that the inclusion

$$\mathrm{B}_{\mathrm{st}}(\overline{V}) \hookrightarrow P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$$

identifies $\mathrm{B}_{\mathrm{st}}(\overline{V})$ with the elements of $P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$ on which the monodromy operator is trivial. Before inverting t this is [6, 3.7]. To get our variant statement, note that the monodromy operator on an element $x \in P^{\mathrm{st}}_{\mathbb{Q}}[1/t]$ is nilpotent if and only if the monodromy operator on $t^r x$ is nilpotent for some $r > 0$. The claim therefore follows from [6, 3.7].

To deduce the proposition from this, note that since λ is compatible with the monodromy operators it induces an isomorphism of sets of elements on which the monodromy operator is nilpotent. We conclude that λ restricts to an isomorphism

$$\sigma' : A \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \to M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V})$$

which proves the proposition. □

6.8. Proposition 6.7 can be generalized to the case of infinite dimensional representations as follows.

Let M denote a possibly infinite dimensional continuous representation of G_K over $\mathbb{Q}_p$, and let (A, N, φ_A) be a triple consisting of a K-vector space A, a semilinear automorphism φ_A, and a K-linear map $N : A \to A$ satisfying $p\varphi_A N = N\varphi_A$. Suppose further given an isomorphism

$$\lambda : A \otimes_K P^{\mathrm{st}}_{\mathbb{Q}}[1/t] \to M \otimes_{\mathbb{Q}_p} P^{\mathrm{st}}_{\mathbb{Q}}[1/t].$$

compatible with Frobenius, monodromy operators, and Galois action.

Proposition 6.9. *In the situation of 6.8 the representation M is the union of finite dimensional semistable representations.*

Proof. Since M is a continuous representation we can write M as a union $M = \cup_i M_i$ of finite dimensional representations. By the description of the Galois action on $P_{\mathbb{Q}}^{\mathrm{st}}[1/t]$ given in [6, 3.3 (4)] the Galois invariants of $P_{\mathbb{Q}}^{\mathrm{st}}[1/t]$ equal K. Let A_i denote

$$(M_i \otimes_{\mathbb{Q}_p} P_{\mathbb{Q}}^{\mathrm{st}}[1/t])^{G_K},$$

so A_i is a subspace of A stable under φ_A and N. We then have a commutative diagram

$$\begin{tikzcd} A_i \otimes_K P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \arrow[r, hook] \arrow[d] & A \otimes_K P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \arrow[d, "\simeq"] \\ M_i \otimes_{\mathbb{Q}_p} P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \arrow[r, hook] & M \otimes_{\mathbb{Q}_p} P_{\mathbb{Q}}^{\mathrm{st}}[1/t]. \end{tikzcd}$$

From this it follows that A_i is finite dimensional. As noted in [3, 4.2.2] this implies that the action of N on A_i is nilpotent. Since A is the union of the A_i this in turn implies that N acts nilpotently on any element of A, and that (A, N, φ_A) is a union of objects of $\mathrm{Mod}_K(\varphi, \mathcal{N})$. Then as in the proof of 6.7 restricting λ to the set of elements on which the monodromy operator is nilpotent we get an isomorphism

$$\lambda' : A \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \to M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}).$$

Let T_i denote the quotient M/M_i and let B_i denote $(T_i \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}))^{G_K}$. We then have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A_i \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[r] \arrow[d, hook] & A \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[r] \arrow[d, "\simeq"] & B_i \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[d, hook] & \\ 0 \arrow[r] & M_i \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[r] & M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[r] & T_i \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}) \arrow[r] & 0. \end{tikzcd}$$

From this and a diagram chase it follows that the map

$$A_i \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V}) \to M_i \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V})$$

is an isomorphism, and that M_i is a semistable representation. □

6.10. The above enables us to reformulate 1.4 as follows. Let the notation be as in 1.4. In section 8 we will give a proof of the following theorem:

Theorem 6.11. *There is an isomorphism of group schemes over* $P_{\mathbb{Q}}^{\mathrm{st}}[1/t]$

$$\pi_1^{\text{ét}}(X_{\overline{K}}^{\circ}, x_K) \otimes_{\mathbb{Q}_p} P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \simeq \pi_1^{\mathrm{crys}}(X_k, \omega_{K\langle t\rangle}) \otimes_{K\langle t\rangle} P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \tag{6.11.1}$$

compatible with Galois actions, Frobenius morphisms, and connections.

6.12. Let us explain how theorem 6.11 implies 1.4.

By 4.4 the right side of (6.11.1) is isomorphic to

$$\pi_1^{\mathrm{crys}}(X_k, x) \otimes_K P_{\mathbb{Q}}^{\mathrm{st}}[1/t]$$

in a manner compatible with Frobenius and connections. Thus giving the isomorphism (6.11.1) is equivalent to giving an isomorphism

$$\pi_1^{\text{ét}}(X_{\overline{K}}^{\circ}, x_K) \otimes_{\mathbb{Q}_p} P_{\mathbb{Q}}^{\mathrm{st}}[1/t] \simeq \pi_1^{\mathrm{crys}}(X_k, x) \otimes_K P_{\mathbb{Q}}^{\mathrm{st}}[1/t]$$

compatible with Frobenius and Galois. Furthermore, looking at the Lie algebras using 6.8 and 6.9 we get from such an isomorphism the desired isomorphism in 1.4.

7. Differential graded algebras and connections

We can describe the monodromy operator on $\pi_1^{\text{crys}}(X_k^\circ, x)$ using differential graded algebras as follows.

7.1. If E is a differential graded $K\langle t\rangle$-algebra we can talk about a connection on E using the method of 4.2. Such a connection is simply an isomorphism of differential graded algebras

$$\gamma_E : p_1^*E \to p_2^*E$$

over $K\langle t\rangle[\epsilon]$ which reduces to the identity.

7.2. Let E be a differential graded $K\langle t\rangle$-algebra equipped with a connection and such that $K\langle t\rangle \to E$ is an equivalence. Let A be a differential graded K-algebra and let $f : A \to E$ be a map of differential graded algebras sending A to the horizontal elements of E. We can then consider the fundamental group

$$\pi_1(A, f)$$

defined to be the fundamental group scheme of $X := \mathbb{R}\mathrm{Spec}(D(A \otimes_K K\langle t\rangle))$ (notation as in [8]), viewed as an object of $\mathrm{Ho}(\mathrm{SPr}_*(K\langle t\rangle))$ using [8, B.4]. The connection on E defines a connection on X, and therefore also a connection

$$p_1^*\pi_1(A, f) \to p_2^*\pi_1(A, f)$$

on $\pi_1(A, f)$.

7.3. The monodromy operator on $\pi_1^{\text{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$ can be described using differential graded algebras as follows. Let

$$\mathscr{K}_{(X_k, M_{X_k})/K} \to \mathbb{R}^\bullet$$

be the standard resolution of the structure sheaf on the crystalline site of (X_k, M_{X_k}), defined by the lifting (X, M_X) (see for example [8, 4.33]). Likewise we have a resolution

$$\mathcal{K}_{(k, M_k)/W} \to \mathbb{S}^\bullet$$

of the structure sheaf in the convergent tops of $(\mathrm{Spec}(k), M_k)/K$, provided by the embedding of $(\mathrm{Spec}(k), M_k)$ into $(\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle})$. Since (X, M_X) is smooth over V we can find an extension

$$\rho : (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}) \to (X, M_X)$$

of the given map $(\mathrm{Spec}(k), M_k) \to (X_k, M_{X_k})$. By functoriality of the construction of the resolution there is a natural map

$$\rho^*\mathbb{R}^\bullet \to \mathbb{S}^\bullet.$$

7.4. Let A denote the differential graded algebra $\Gamma((X_k, M_{X_k})/K, \mathbb{R}^\bullet)$ and let E denote $\mathbb{S}^\bullet(\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle})$. Since the crystals $\mathbb{R}^i$ are acyclic (see [8, 4.33]) the differential graded algebra A is an explicit model for $R\Gamma((X_k, M_{X_k})/K, \mathscr{K}_{(X_k, M_{X_k})/K})$ and the choice of ρ induces a morphism $f : A \to E$. Furthermore, E comes equipped with a connection being the value of a differential graded algebra of crystals. Thus the preceding discussion applies and we get a connection on the group scheme $\pi_1(A, f)$ over $K\langle t\rangle$. It follows from the constructions of [10] that this gives a model for $\pi_1^{\text{crys}}(X_k^\circ, \omega_{K\langle t\rangle})$ with its connection.

8. Proof of theorem 6.11

The goal of this section is to give a proof of 6.11, and therefore also 1.4.

The approach here is to prove a comparison result for augmented differential graded algebras and then pass to fundamental groups to get 6.11.

8.1. Fix a hypercovering $U_. \to X$ with each U_n very small in the sense of [8, 6.1] and such that each U_n is a disjoint union of open subsets of X, and furthermore assume that each connected component of U_n meets the closed fiber of X. Write $U_n = \mathrm{Spec}(S_n)$, with S_n a geometrically integral V-algebra. Let $M_{U_.}$ denote the log structure on $U_.$ obtained by pullback from M_X, and let $(U_.^\wedge, M_{U_.^\wedge})$ be the simplicial formal log scheme obtained by p-adically completing $(U_., M_{U_.})$.

Since U_0 is a disjoint union of affine open subsets of X, there exists a lifting

$$u : (\mathrm{Spec}(V), M_V) \to (U_., M_{U_.})$$

of the map

$$x : (\mathrm{Spec}(V), M_V) \to (X, M_X).$$

We fix one such u.

Fix also a geometric generic point

$$\bar{\eta} : \mathrm{Spec}(\Omega) \to X$$

over $K \hookrightarrow \overline{K}$. Since each connected component of U_n maps isomorphically to an open subset of X, the point $\bar{\eta}$ also defines a geometric generic point of each connected component of U_n.

We can do this more canonically as follows. Let $\Upsilon_.$ be the simplicial set with Υ_n equal to the set of connected components of U_n, with the natural transition maps. Then we have a canonical morphism

$$\bar{\eta} : \Upsilon_. \times \mathrm{Spec}(\Omega) \to U_..$$

8.2. Let $\eta_0 \in X$ be the generic point of the closed fiber. Then $\mathscr{O}_{X,\eta_0}$ is a discrete valuation ring with uniformizer p, and fraction field the function field $k(X)$. Let $k(X)^\wedge$ be the completion of $k(X)$ with respect to the discrete valuation defined by $\mathscr{O}_{X,\eta_0}$. Fix an algebraic closure $\Omega^\wedge$ of $k(X)^\wedge$, and a commutative diagram of inclusions

$$\begin{tikzcd} k(X) \arrow[r, hook, "\bar{\eta}"] \arrow[d, hook] & \Omega \arrow[d, hook] \\ k(X)^\wedge \arrow[r, hook, "\bar{\eta}^\wedge"] & \Omega^\wedge. \end{tikzcd}$$

We then get a morphism of simplicial schemes

$$\bar{\eta}^\wedge : \Upsilon_. \times \mathrm{Spec}(\Omega^\wedge) \to \mathrm{Spec}(S_.^\wedge),$$

over $\bar{\eta}$.

Let $\mathrm{A}_{\mathrm{cris}}(U_.^\wedge)$ be the cosimplicial algebra obtained by applying the functor $\mathrm{A}_{\mathrm{cris}}(-)$ to each S_n with respect to the algebraic closure on each connected component $e \in \Upsilon_n$ given by the map

$$\mathrm{Spec}(\Omega^\wedge) = \{e\} \times \mathrm{Spec}(\Omega^\wedge) \hookrightarrow \Upsilon_n \times \mathrm{Spec}(\Omega^\wedge) \xrightarrow{\bar{\eta}^\wedge} \mathrm{Spec}(S_n).$$

Let $GC(U_\cdot^\wedge, \mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge))$ be the Galois cohomology of this cosimplicial Galois module, as defined in [8, 5.21 and 5.40].

There is a natural map

$$R\Gamma(X^\circ_{\overline{K},\mathrm{et}}, \mathbb{Q}_p) = GC(U_\cdot, \mathbb{Q}_p) \to GC(U_\cdot^\wedge, \mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge))$$

induced by the natural map $\mathbb{Q}_p \to \mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge)$.

8.3. Next we need to relate the base points. For $e \in \Upsilon_n$, write $S_n^{(e)}$ for the coordinate ring of the connected component of U_n corresponding to e. Define $E'_n \subset E_n$ to be the subset of $e \in E_n$ such that $\mathrm{Spec}(S_n^{(e)}) \subset X$ contains the point x. The E'_n are preserved under the simplicial structure maps, and therefore define a sub-simplicial set $E'_\cdot \subset E_\cdot$.

Let $y \in X(k)$ be the intersection of $x : \mathrm{Spec}(V) \hookrightarrow X$ with the closed fiber, and consider the local ring $\mathscr{O}_{X,y}$. Let $\mathscr{O}_{X,y}^\wedge$ be the p-adic completion of this ring. There is a natural map

$$\mathscr{O}_{X,y}^\wedge \to V$$

induced by the map $\mathscr{O}_{X,y} \to V$. There is also a natural map

$$\mathscr{O}_{X,y}^\wedge \to \mathscr{O}_{X,\eta_0}^\wedge$$

and hence an inclusion $\mathscr{O}_{X,y}^\wedge \hookrightarrow \Omega^\wedge$. Let $(\mathscr{O}_{X,y}^\wedge)^\dagger$ be the p-adic completion of the integral closure of $\mathscr{O}_{X,y}$ in $\Omega^\wedge$. Fix a morphism

$$(\mathscr{O}_{X,y}^\wedge)^\dagger \to \overline{V}^\wedge \tag{8.3.1}$$

extending the map $\mathscr{O}_{X,y}^\wedge \to V$. Here $\overline{V}^\wedge$ denotes the p-adic completion of $\overline{V}$.

We then get a map

$$E'_\cdot \times \mathrm{Spec}((\mathscr{O}_{X,y}^\wedge)^\dagger) \to U_\cdot^\wedge,$$

and hence also a map

$$E'_\cdot \times \mathrm{Spec}(\overline{V}^\wedge) \to U_\cdot^\wedge,$$

over the natural map

$$E'_\cdot \times \mathrm{Spec}(V) \to U_\cdot.$$

8.4. As before let $V\langle t\rangle$ denote the p-adically completed divided power envelope of the surjection $V[t] \to V$ sending t to p. Since (X, M_X) is log smooth, we can find a dotted arrow filling in the following diagram

$$\begin{tikzcd} (\mathrm{Spec}(V), M_V) \arrow[r, hook] \arrow[d, "x"] & (\mathrm{Spec}(V\langle t\rangle), M_{V\langle t\rangle}) \arrow[dl, dashed] \\ (X, M_X). & \end{tikzcd}$$

Fix one such dotted arrow

$$\lambda : (\mathrm{Spec}(R), M_R) \to (X, M_X).$$

For $e \in E_n$ let $\overline{U}_n^{(e),\wedge}$ denote the spectrum of the integral closure of $S_n^\wedge$ in the maximal subextension of $\Omega^\wedge$ which is unramified over $\mathrm{Spec}(S_n^\wedge) \times_X X_K^\circ$. For $e \in E'_n$ the map 8.3.1 induces a morphism

$$\mathrm{Spec}(\overline{V}^\wedge) \to \overline{U}_n^{(e),\wedge}.$$

For every n, let $\overline{U}_n^\wedge$ denote the coproduct

$$\coprod_{e\in E_n} \overline{U}_n^{(e),\wedge}.$$

These schemes form in a natural way a simplicial scheme $\overline{U}_n^\wedge$, and we obtain a commutative diagram

$$\begin{tikzcd}
E'_\cdot \times \mathrm{Spec}(\overline{V}^\wedge) \arrow[rr] \arrow[d] & & \overline{U}_\cdot^\wedge \arrow[d] \\
E'_\cdot \times \mathrm{Spec}(V) \arrow[r] & E'_\cdot \times \mathrm{Spec}(R) \arrow[r] & U_\cdot .
\end{tikzcd}$$

By the universal property of $\mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge)$ (it is obtained by taking global section of the structure sheaf in the crystalline topos of $\overline{U}_\cdot^\wedge/V$), there is a natural map

$$E'_\cdot \times \mathrm{Spec}(P^{\mathrm{st}}) \to \mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge)).$$

We can therefore extend the above diagram to a commutative diagram

$$\begin{tikzcd}
E'_\cdot \times \mathrm{Spec}(\overline{V}^\wedge) \arrow[rr] \arrow[dr, hook] \arrow[dd] & & \overline{U}_\cdot^\wedge \arrow[dd] \arrow[dr, hook] & \\
& E'_\cdot \times \mathrm{Spec}(P^{\mathrm{st}}) \arrow[rr] & & \mathrm{Spec}(\mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge)) \\
E'_\cdot \times \mathrm{Spec}(V) \arrow[r] & E'_\cdot \times \mathrm{Spec}(R) \arrow[r] & U_\cdot . &
\end{tikzcd}$$

In particular, for any isocrystal F on $(X_k, M_{X_k})/K$ we obtain a natural map of cosimplicial K-spaces

$$F(\mathrm{A}_{\mathrm{cris}}(U_\cdot^\wedge)) \to x^*F(P^{\mathrm{st}}) \otimes \mathbb{Z}^{E'_\cdot}.$$

Observe also that the natural map $\mathbb{Z} \to \mathbb{Z}^{E'_\cdot}$ induces a quasi-isomorphism

$$x^*F(P^{\mathrm{st}}) \to x^*F(P^{\mathrm{st}}) \otimes \mathbb{Z}^{E'_\cdot}.$$

8.5. As in 7.3, let

$$\mathscr{K}_{(X_k, M_{X_k})/K} \to \mathbb{R}^\bullet$$

be the standard resolution of the structure sheaf, defined by the lifting (X, M_X), and let

$$\mathscr{K}_{(\mathrm{Spec}(k), M_k)/K} \to \mathbb{S}^\bullet$$

be the resolution of the structure sheaf defined by the surjection $V\langle t\rangle \to k$.

By functoriality of the construction of these resolutions there is a natural map $x^*\mathbb{R}^\bullet \to \mathbb{S}^\bullet$. Putting all of this together we obtain the following commutative diagrams of cosimplicial differential graded algebras:

$$\begin{tikzcd}
GC(U^\circ_{\cdot,\overline{K}}, \mathbb{Q}_p) \otimes P^{\mathrm{st}} \arrow[r, "\tilde{a}"] \arrow[d] & GC(U_\cdot^\wedge, \mathrm{A}_{\mathrm{cris}}(\overline{U}_\cdot^\wedge)) \otimes_{\mathrm{A}_{\mathrm{cris}}(V)} P^{\mathrm{st}} \arrow[d] \\
P^{\mathrm{st}} \arrow[r, "a"] & P^{\mathrm{st}} \otimes \mathbb{Z}^{E'_\cdot}
\end{tikzcd} \tag{8.5.1}$$

$$
\begin{tikzcd}
GC(U_\cdot^\wedge, \mathrm{A}_{\mathrm{cris}}(\overline{U}_\cdot^\wedge)) \otimes_{\mathrm{A}_{\mathrm{cris}}(V)} P^{\mathrm{st}} \arrow[r, "\tilde{b}"] \arrow[d] & GC(U_\cdot^\wedge, \mathbb{R}^\bullet(\mathrm{A}_{\mathrm{cris}}(\overline{U}_\cdot^\wedge))) \otimes_{\mathrm{A}_{\mathrm{cris}}(V)} P^{\mathrm{st}} \arrow[d] \\
P^{\mathrm{st}} \otimes \mathbb{Z}^{E'_\cdot} \arrow[r, "b"] & \mathbb{S}^\bullet(P^{\mathrm{st}}) \otimes \mathbb{Z}^{E'_\cdot},
\end{tikzcd}
\tag{8.5.2}
$$

$$
\begin{tikzcd}
GC(U_\cdot^\wedge, \mathbb{R}^\bullet(\mathrm{A}_{\mathrm{cris}}(\overline{U}_\cdot^\wedge))) \otimes_{\mathrm{A}_{\mathrm{cris}}(V)} P^{\mathrm{st}} \arrow[d] & \Gamma((X_{kk}, M_{X_k})/K, \mathbb{R}^\bullet) \otimes_K P^{\mathrm{st}} \arrow[l, "\tilde{c}"'] \arrow[d] \\
\mathbb{S}^\bullet(P^{\mathrm{st}}) \otimes \mathbb{Z}^{E'_\cdot} & \Gamma((\mathrm{Spec}(k), M_k)/K, \mathbb{S}^\bullet) \otimes P^{\mathrm{st}}, \arrow[l, "c"']
\end{tikzcd}
\tag{8.5.3}
$$

$$
\begin{tikzcd}
\Gamma((X_{kk}, M_{X_k})/K, \mathbb{R}^\bullet) \otimes_K P^{\mathrm{st}} \arrow[d] & R\Gamma((X_k, M_{X_k})/K, \mathscr{K}) \otimes_K P^{\mathrm{st}} \arrow[l, "\tilde{d}"'] \arrow[d] \\
\Gamma((\mathrm{Spec}(k), M_k)/K, \mathbb{S}^\bullet) \otimes P^{\mathrm{st}} & P^{\mathrm{st}}. \arrow[l, "d"']
\end{tikzcd}
\tag{8.5.4}
$$

Since $E'_\cdot$ has geometric realization a point, the map a is an equivalence. Furthermore, the map $\tilde{a}$ induces an equivalence after inverting $t \in \mathrm{A}_{\mathrm{cris}}(V)$. The map $\tilde{b}$ (resp. b) is an equivalence since

$$\mathscr{K}_{(X_k, M_{X_k})/K} \to \mathbb{R}^\bullet, \quad (\mathscr{K}_{(k, M_k)/K} \to \mathbb{S}^\bullet)$$

is an equivalence. Let $\mathrm{dga}_{P^{\mathrm{st}}[1/t],/P^{\mathrm{st}}}$ denote the category of commutative differential graded $P^{\mathrm{st}}[1/t]$-algebras with an augmentation to P^{st}. Applying the functor of Thom-Sullivan cochains we then obtain a morphism

$$(R\Gamma((X_k, M_{X_k})/K, \mathscr{K}) \otimes_K P^{\mathrm{st}}[1/t] \xrightarrow{y^*} P^{\mathrm{st}}[1/t]) \to (GC(U^\circ_{\cdot,\overline{K}}, \mathbb{Q}_p) \otimes P^{\mathrm{st}}[1/t] \xrightarrow{x^*} P^{\mathrm{st}}[1/t])$$

in $\mathrm{Ho}(\mathrm{dga}_{P^{\mathrm{st}}[1/t],/P^{\mathrm{st}}})$. This morphism is an equivalence by Faltings' theory of almost étale extensions (see [1, 2.33]). Applying the π_1-functor, as described in [8, Chapters 4 and 5], we obtain an isomorphism

$$\pi_1^{\mathrm{crys}}(X_K^\circ, x) \otimes_K P^{\mathrm{st}}[1/t] \simeq \pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x) \otimes_{\mathbb{Q}_p} P^{\mathrm{st}}[1/t].$$

It follows from the construction that this isomorphism is compatible with the Frobenius operators, connections, and the G_K-action.

This completes the proof of 6.11. □

Remark 8.6. By the same argument one gets a comparison isomorphism for torsors of paths. Given two points $x_{1,K}, x_{2,K} \in X^\circ(K)$ we can then consider the torsors of paths

$$\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_{1,\overline{K}}, x_{2,\overline{K}}) \tag{8.6.1}$$

of isomorphisms between the fiber functors on the category of unipotent $\mathbb{Q}_p$-local systems on $X^\circ_{\overline{K}}$ defined by the points, and similarly we have the torsor $\pi_1^{\mathrm{crys}}(X_k^\circ, x_1, x_2)$ defined in 5.4.

As discussed in [8, 8.27-8.32] the torsor (8.6.1) is described by the differential graded algebra $GC(U^\circ_{\cdot,\overline{K}}, \mathbb{Q}_p)$ equipped with the two augmentations defined by the points. Similarly the torsor $\pi_1^{\mathrm{crys}}(X_k^\circ, x_1, x_2)$ is desribed by the differential graded algebra $R\Gamma((X_k, M_{X_k})/K, \mathscr{K})$ equipped with its two augmentations. Chasing through the above proof one obtains an isomorphism

$$\mathscr{O}_{\pi_1^{\mathrm{et}}(X^\circ_{\overline{K}}, x_{1,\overline{K}}, x_{2,\overline{K}})} \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(\overline{V}) \simeq \mathscr{O}_{\pi_1^{\mathrm{crys}}(X_k^\circ, x_1, x_2)} \otimes_K \mathrm{B}_{\mathrm{st}}(\overline{V})$$

compatible with Frobenius, monodromy operators, and Galois action. Furthermore, proposition 6.9 implies that $\mathscr{O}_{\pi_1^{\mathrm{crys}}(X_k^\circ, x_1, x_2)}$ is a colimit of semistable representations.

9. The case of curves

9.1. Let C/V be a smooth proper curve, and let $s_i : \mathrm{Spec}(V) \to C$ $(i = 1, \dots, r)$ be a finite number of distinct sections. Let $C^\circ \subset C$ be the complement of the sections, and let D denote the union of the sections. Let M_C be the log structure on C defined by D. Let L_V be the hollow log structure on $\mathrm{Spec}(V)$ given by the map $\mathbb{N} \to V$ sending all nonzero elements to 0. The choice of a uniformizer for each section defines morphisms

$$s_i : (\mathrm{Spec}(V), L_V) \to (C, M_C).$$

Also let L_K denote the hollow log structure on $\mathrm{Spec}(K)$.

9.2. If $(\mathcal{E}, \nabla)$ is a module with integrable connection, we can pull $\mathcal{E}$ back along s_i to get a K-vector space $\mathcal{E}(s_i)$ together with an endomorphism, called the *residue at* s_i,

$$R_{s_i} : \mathcal{E}(s_i) \to \mathcal{E}(s_i)$$

induced by the connection. This map can be described as follows.

There is a natural inclusion

$$\Omega^1_{C_K/K} \hookrightarrow \Omega^1_{(C_K, M_{C_K})/K}$$

with cokernel canonically isomorphic to $\oplus_i K_{s_i}$. The composite map

$$\mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{(C_K, M_{C_K})/K} \longrightarrow \mathcal{E} \otimes K_{s_i} = \mathcal{E}(s_i)$$

is $\mathscr{O}_{C_K}$-linear, and therefore induces a map $\mathcal{E}(s_i) \to \mathcal{E}(s_i)$, which by definition is the map R_{s_i}.

Lemma 9.3. *Let $MIC(C_K/K)$ (resp. $MIC((C_K, M_{C_K})/K)$) denote the category of modules with integrable connection on C_K/K (resp. $(C_K, M_{C_K})/K$). Then the natural functor*

$$MIC(C_K/K) \to MIC((C_K, M_{C_K})/K)$$

is fully faithful with essentially image those objects $(\mathcal{E}, \nabla)$ for which the residue mappings R_{s_i} are all zero.

Proof. Note that the residues of a module with logarithmic integrable connection $(\mathcal{E}, \nabla)$ are all zero, if and only if

$$\nabla(\mathcal{E}) \subset \mathcal{E} \otimes \Omega^1_{C_K/K} \subset \mathcal{E} \otimes \Omega^1_{(C_K, M_{C_K})/K}.$$

From this observation the lemma follows. □

9.4. Let $(C_k, M_{C_k})/k$ be the reduction of (C, M_C). If E is an isocrystal on $(C_k, M_{C_k})/K$, we can evaluate E on the enlargement discussed in 2.3

$$\begin{array}{ccc} (\mathrm{Spec}(k), M_k) & \hookrightarrow & (\mathrm{Spec}(V), L_V) \\ {\scriptstyle s_i}\downarrow & & \\ (C_V, M_{C_V}) & & \end{array}$$

to get a K-vector space $E(s_i)$ with an endomorphism $N_i : E(s_i) \to E(s_i)$.

9.5. Let $(\mathcal{E}, \nabla)$ be the module with integrable connection on (C_K, M_{C_K}) associated to E. From the commutative diagram

$$\begin{tikzcd}
(\mathrm{Spec}(k), M_k) \arrow[r, hook] \arrow[d, "s_i"] & (\mathrm{Spec}(V), L_V) \arrow[d, "s_i"] \\
(C_k, M_{C_k}) \arrow[r, hook] & (C, M_C),
\end{tikzcd}$$

we obtain a canonical isomorphism

$$E(s_i) \simeq \mathcal{E}(s_i).$$

It follows from the construction that this isomorphism identifies N_i with R_{s_i}.

Lemma 9.6. *The natural functor*

$$(\text{unip. isocrystals on } C_k/K) \to (\text{unip. isocrystals on } (C_k, M_{C_k})/K)$$

is fully faithful, with essential image the full subcategory of unipotent isocrystals E for which the maps $N_i : E(s_i) \to E(s_i)$ are are all zero.

Proof. This follows from the fact that there is an equivalence of categories

$$(\text{unip. isocrystals on } (C_k, M_{C_k})/K) \simeq (\text{unip. modules with connection on } (C_K, M_{C_K})/K)$$

compatible with residues, and the corresponding result for modules with integrable connections. □

9.7. Fix now a point

$$x : \mathrm{Spec}(V) \to C$$

sending the closed fiber to D and the generic point to C°. Let $s \in D(V)$ be the section whose closed fiber is the closed fiber of x.

As before let $\mathscr{C}^{\mathrm{crys}}$ (resp. $\mathscr{C}^{\mathrm{dR}}$) denote the category of unipotent isocrystals (resp. modules with integrable connection) on (C_k, M_{C_k}) (resp. (C_K, M_{C_K})). Let $\mathscr{H}^{\mathrm{crys}} \subset \mathscr{C}^{\mathrm{crys}}$ be a Tannakian subcategory corresponding to a surjection of affine K-group schemes

$$\pi_1^{\mathrm{crys}}(C_k^\circ, x) \twoheadrightarrow H^{\mathrm{crys}}.$$

Denote by H^{dR} the quotient of $\pi_1^{\mathrm{dR}}(C_K^\circ, x)$ obtained from H^{crys} and the isomorphism

$$\pi_1^{\mathrm{crys}}(C_k^\circ, x) \otimes_{K_0} K \simeq \pi_1^{\mathrm{dR}}(C_K^\circ, x).$$

By Tannaka duality, the group H^{dR} corresponds to a Tannakian subcategory $\mathscr{H}^{\mathrm{dR}} \subset \mathscr{C}^{\mathrm{dR}}$.

It follows from the discussion in 3.8 that the monodromy operator on $\mathscr{O}_{\pi_1^{\mathrm{crys}}(C_k^\circ, x)}$ restricts to a monodromy operator on $\mathscr{O}_{H^{\mathrm{crys}}}$. In fact, the discussion in 3.8 implies the following. Taking residues at s defines a tensor functor from the category $\mathscr{H}^{\mathrm{dR}}$ to the category of K-vector spaces equipped with a nilpotent endomorphism. Giving such a functor is equivalent to giving a homomorphism

$$\rho_s : \mathbb{G}_{a,K} \to H^{\mathrm{dR}}.$$

The monodromy operator on $\mathrm{Lie}(H^{\mathrm{crys}}) \simeq \mathrm{Lie}(H^{\mathrm{dR}})$ is given $[-, \mathrm{Lie}(\rho_s)(1)]$, where

$$\mathrm{Lie}(\rho_s) : \mathbb{G}_{a,K} \to \mathrm{Lie}(H^{\mathrm{dR}})$$

is the map obtained from ρ_s by passing to Lie algebras.

Corollary 9.8. *The monodromy operator on H^{crys} is trivial if and only if the image of ρ_s is in the center of H^{dR}.*

Proof. This follows from the preceding discussion. $\square$

10. Example: $\mathbb{P}^1 - \{0,1,\infty\}$

To give a very explicit example, we discuss in this section the Kummer torsor following Deligne in [2, §16].

10.1. Let $X = \mathbb{P}^1$, and let $D = \{0,1,\infty\} \subset X$. For any point $x \in X^\circ(K)$, define the *Kummer torsor* to be the following torsor under $\mathbb{Q}_p(1)$

$$K(x) := \{(y_n \in \overline{K})_{n\geq 0} | y_n^p = y_{n-1}, \quad y_0 = x\}.$$

Equivalently, we can think of $K(x)$ as a class in

$$K(x) \in \mathrm{Ext}^1_{\mathrm{Rep}_{G_K}(\mathbb{Q}_p)}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

Let us write

$$0 \to \mathbb{Q}_p(1) \to \mathcal{K}_x \to \mathbb{Q}_p \to 0$$

for this extension of $G_{\mathbb{Q}_p}$-representations.

10.2. The Kummer torsor has the following description in terms of $\pi_1^{\mathrm{et}}(X_K^\circ, x)$ (see [2, §16]). There is a natural map

$$X^\circ \hookrightarrow \mathbb{G}_m$$

which induces a morphism

$$T : \pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x) \to \pi_1^{\mathrm{et}}(\mathbb{G}_m, x).$$

Let $U_1(x)$ be the abelianization of $\mathrm{Ker}(T)$. Pushing out the exact sequence

$$1 \longrightarrow \mathrm{Ker}(T) \longrightarrow \pi_1^{\mathrm{et}}(X_K^\circ, x) \xrightarrow{T} \pi_1^{\mathrm{et}}(\mathbb{G}_m, x) \longrightarrow 1$$

along $\mathrm{Ker}(T) \to U_1(x)$ and taking Lie algebras, we obtain an exact sequence of G_K-representations

$$0 \to U_1(x) \to U(x) \to \mathbb{Q}_p(1) \to 0,$$

where we use the canonical isomorphism $\mathrm{Lie}(\pi_1^{\mathrm{et}}(\mathbb{G}_m, x)) \simeq \mathbb{Q}_p(1)$. Since $U_1(x)$ is abelian, the Lie bracket on $U(x)$ defines an action of $\mathbb{Q}_p(1)$ on $U_1(x)$. Set

$$U_1^n(x) := \mathrm{ad}^n(U_1(x)).$$

We then have a natural map

$$\mathbb{Q}_p(1)^{\otimes n} \otimes U_1(x)/U_1^1(x) \to U_1^n(x)/U_1^{n+1}(x). \tag{10.2.1}$$

Proposition 10.3. *(a) The projection map*

$$\pi_1^{\mathrm{et}}(X_K^\circ, x) \to \pi_1(\mathbb{A}_K^1 - \{1\}, x) \simeq \mathbb{Q}_p(1)$$

induces an isomorphism

$$U_1(x)/U_1^1(x) \simeq \mathbb{Q}_p(1).$$

(b) For every $n \geq 1$ the map

$$\mathbb{Q}_p(n+1) \xrightarrow{(a)} \mathbb{Q}_p(n) \otimes U_1(x)/U_1^1(x) \xrightarrow{(10.2.1)} U_1^n(x)/U_1^{n+1}(x)$$

is an isomorphism.

(c) The class of the extension

$$\begin{tikzcd} \mathbb{E}_x: \quad 0 \arrow[r] & U_1^1(x)/U_1^2(x) \arrow[r] \arrow[d, "\simeq"] & U_1(x)/U_1^2(x) \arrow[r] & U_1(x)/U_1^1(x) \arrow[r] \arrow[d, "\simeq"] & 0 \\ & \mathbb{Q}_p(2) & & \mathbb{Q}_p(1) & \end{tikzcd}$$

in

$$\mathrm{Ext}^1_{G_K}(\mathbb{Q}_p(1), \mathbb{Q}_p(2)) \simeq \mathrm{Ext}^1_{G_K}(\mathbb{Q}_p, \mathbb{Q}_p(1))$$

is the class of the Kummer torsor $K(x)$.

Proof. Statements (a) and (b) follow from the proof of [2, 16.3].

Statement (c) essentially follows from [2, 14.2 and 16.3]. Let us explain this. First of all, let us describe the sum

$$[\mathbb{E}(x)(-1)] + [-K(x)] \in \mathrm{Ext}^1_{G_K}(\mathbb{Q}_p, \mathbb{Q}_p(1)).$$

In general, if L is a Lie algebra, and H is a group scheme acting on L by Lie algebra homomorphism, then for any H-torsor P we can form a new Lie algebra

$$L \wedge^H P := (L \times P)/((l,p) \sim (hl, hp))_{h \in H}.$$

The Lie algebra structure is obtained as follows. If we fix a trivialization $s \in P$, then we obtain an isomorphism

$$L \xrightarrow{l \mapsto (l,s)} L \wedge^H P,$$

and we define the Lie algebra structure using this isomorphism. This Lie algebra structure on $L \wedge^H P$ is independent of the choice of s. Indeed if s' is another trivialization, say $s' = hs$ for $h \in H$, then the diagram

$$\begin{tikzcd} L \arrow[dd, "h"'] \arrow[dr, "l \mapsto (l,hs)"] & \\ & L \wedge^H P \\ L \arrow[ur, "l \mapsto (l,s)"'] & \end{tikzcd}$$

commutes, and the left vertical arrow is an isomorphism of Lie algebras. It follows that the Lie algebra structure on $L \wedge^H P$ is well-defined even when P is not trivial.

In particular, starting with the extension

$$\mathbb{E}(x)(-1): \quad 0 \to \mathbb{Q}_p(1) \to U_1(x)/U_1^2(x) \to \mathbb{Q}_p \to 0,$$

we can view $U_1(x)/U_1^2(x)$ as having a $\mathbb{Q}_p(1)$-action given by conjugation. In particular, we can push this sequence out along any $\mathbb{Q}_p(1)$-torsor P to get a new extension of Lie algebras

$$0 \to \mathbb{Q}_p(1) \to U_1(x)/U_1^2(x) \wedge^{\mathbb{Q}_p(1)} P \to \mathbb{Q}_p \to 0.$$

It follows from the definition of Baer sum of extensions, that the sum

$$[\mathbb{E}(x)(-1)] + [-\mathcal{K}_x]$$

is the pushout of $\mathbb{E}(x)(-1)$ along $-K(x)$. On the other hand, by [2, 14.2 and 15.51], the torsor $K(x)$ is isomorphic to the space of paths $P_{(0,1),x}$, of isomorphism of fiber functors between the fiber functor given by tangential base point at 0 in the direction of 1 (see [2, §15]). By [2, 16.11.3] we have

$$[\mathbb{E}(0,1)(-1)] = [\mathbb{E}(x)(-1)] + [P_{(0,1),x}],$$

and $[\mathbb{E}(0,1)(-1)]$ is the zero class by [2, 16.13]. We conclude that

$$[\mathbb{E}(x)(-1)] = -[\mathcal{K}_x].$$

□

Remark 10.4. As discussed in [2, 16.12] the choice of a section $a : U_1(x)/U_1^1(x) \to U_1(x)$ induces an isomorphism

$$(\prod_{n\geq 1} \mathbb{Q}_p(n)) \rtimes \mathbb{Q}_p(1) \simeq U(x)$$

with trivial Lie bracket on $\prod_{n\geq 1} \mathbb{Q}_p(n)$ and action of $\mathbb{Q}_p(1)$ on $\prod_{n\geq 1} \mathbb{Q}_p(n)$ induced by the maps (10.2.1).

10.5. Suppose now that x reduces modulo the maximal ideal $\mathfrak{m}_K$ of $\mathscr{O}_K$ to 0. Let X_k be the reduction of X modulo $\mathfrak{m}_K$, and let

$$y : (\mathrm{Spec}(k), M_k) \to (X_k, M_{X_k})$$

be the reduction of x. In our case, $X_k = \mathbb{P}^1_k$ with log structure defined by the divisor $\{0, 1, \infty\}$ and y is the inclusion of $0 \in \mathbb{P}^1_k$. Let $(\mathcal{G}_m, M_{\mathcal{G}_m})$ denote the scheme $\mathbb{P}^1_k$ with log structure defined by the divisor $\{0, \infty\}$. We then have a natural map of log schemes

$$t : (X_k, M_{X_k}) \to (\mathcal{G}_m, M_{\mathcal{G}_m}).$$

This map induces a morphism of group schemes

$$T^{\mathrm{crys}} : \pi_1^{\mathrm{crys}}(X^\circ, x) \to \pi_1^{\mathrm{crys}}(\mathbb{G}_m, t_0),$$

where t_0 denotes the tangential base point at 0 (see for example [8, Chapter 9]). This map is the crystalline realization of the map T in 10.2. On the other hand, it follows from a basic calculation of cohomology that the composite functor

$$\begin{array}{c}
(\text{unip. isocrystals on } (\mathcal{G}_m, M_{\mathcal{G}_m})) \\
\downarrow{\scriptstyle t^*} \\
(\text{unip. isocrystals on } (X_k, M_{X_k})) \\
\downarrow{\scriptstyle y^*} \\
(\text{unip. isocrystals on } (\mathrm{Spec}(k), M_k)) \\
\downarrow \\
\mathrm{Mod}_K^{\mathrm{un}}(\mathcal{N})
\end{array}$$

is an equivalence of categories. We therefore obtain a section

$$s : \pi_1^{\mathrm{crys}}(\mathbb{G}_m, t_0) \to \pi_1^{\mathrm{crys}}(X^\circ, x)$$

compatible with Frobenius and the monodromy operator.

10.6. Repeating the previous discussion in the crystalline realization as opposed to the étale realization, we obtain an extension (φ, N)-modules

$$\mathbb{E}_x^{\mathrm{crys}}: \quad 0 \to K_0(2) \to U_1^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x) \to K_0(1) \to 0,$$

where $K_0(i)$ has underlying K_0-vector space K_0, trivial monodromy operator, and Frobenius given by multiplication by $1/p^i$. Moreover, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(T^{\mathrm{crys}}) \arrow[r] \arrow[d] & \pi_1^{\mathrm{crys}}(X_k, M_{X_k}) \arrow[r, "T^{\mathrm{crys}}"] \arrow[d, "\kappa"] & \pi_1^{\mathrm{crys}}(\mathbb{G}_m, t_0) \simeq K_0(1) \arrow[r] \arrow[d, "="'] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & U_1^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x) \arrow[r] & U^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x) \arrow[r] & K_0(1) \arrow[r] & 0.
\end{tikzcd}$$

By 9.7 if the monodromy operator on $U^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x)$ is given by the adjoint action of the image of s, which in particular is nonzero (for example by the crystalline analogue of the explicit description in 10.4). Since the section s identifies $K_0(1)$ with a direct summand of $U^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x)$ we conclude that the monodromy operator on $U_1^{\mathrm{crys}}(x)/U_1^{\mathrm{crys},2}(x)$ is also nontrivial. In particular, the G_K-representation $\mathcal{K}_x$ is semistable, but not crystalline.

10.7. Of course the extension $\mathcal{K}_x$, and its trivialization over $\mathrm{B_{st}}(V)$ can be described explicitly. For the convenience of the reader, let us write out this exercise.

Fix a sequence $\underline{\beta} = (\beta_n)_{n\geq 0}$ of elements $\beta_n \in \overline{V}$, with $\beta_0 = \pi$ and $\beta_{n+1}^p = \beta_n$. As discussed in [6, 3.3 and 3.5] this sequence defines an element $u_\beta \in \mathrm{B_{st}}(V)$ such that the induced map

$$\mathrm{B_{cris}}(V)[u_\beta] \to \mathrm{B_{st}}(V)$$

is an isomorphism. For $g \in G_K$, define

$$\lambda_g = (\lambda_{g,n})_{n\geq 0} \in \mathbb{Z}_p(1)$$

to be the system of roots of unity characterized by the equalities

$$g(\beta_n) = \lambda_{g,n}\beta_n.$$

Now recall (see for example [6, 2.2], where the map is called ϵ) that there is a map

$$\alpha : \mathbb{Z}_p(1) \to \mathrm{Ker}(\mathrm{A_{cris}}(V)^* \to \overline{V}^{\wedge *}) \subset \mathrm{A_{cris}}(V)^*.$$

Since the kernel of the map $\mathrm{A_{cris}}(V) \to \overline{V}^\wedge$ has a divided power structure, we can take the logarithm of α to get an additive map

$$\log(\alpha(-)) : \mathbb{Z}_p(1) \to \mathrm{A_{cris}}(V).$$

It follows from [6, 3.3] that the action of $g \in G_K$ on $u_\beta \in \mathrm{B_{st}}(V)$ is given by

$$u_\beta^g = \log(\alpha(\lambda_g)) + u_\beta.$$

10.8. Consider now our torsor $K(x)$ with associated 2-dimensional G_K-representation $\mathcal{K}_x$.

Write $x = u\pi^z$ with $u \in \mathscr{O}_K^*$ and $z \geq 1$. Note that we may assume that $u \equiv 1 \pmod{\pi}$. Indeed multiplying x by an element of K_0 gives an isomorphic torsor. Therefore by multiplying x by the inverse of the Teichmuller lifting of $u \pmod{\pi}$ we may assume that $u \equiv 1 \pmod{\pi}$.

Fix a sequence of roots $\underline{x} = (x_n)_{n\geq 0}$, with $x_0 = x$ and $x_{n+1}^p = x_n$. Then we can write $x_n = u_n\beta_n^z$, where $u_n \in \overline{V}^*$, $u_0 = u$, and $u_{n+1}^p = u_n$.

Let $b \in \mathcal{K}_x$ be the lifting of $1 \in \mathbb{Q}_p$ given by $\underline{x}$, so we have a direct sum decomposition

$$\mathcal{K}_x \simeq \mathbb{Q}_p(1) \oplus \mathbb{Q}_p \cdot b.$$

The action of an element $g \in G_K$ is given in terms of this decomposition by sending

$$(s, t \cdot b) \in \mathbb{Q}_p(1) \oplus \mathbb{Q}_p \cdot b$$

to

$$(s^g + t\epsilon_g, t \cdot b),$$

where $\epsilon_g \in \mathbb{Z}_p(1)$ is the element characterized by

$$x_n^g = \epsilon_{g,n} x_n.$$

10.9. The map $\log(\alpha(-))$ induces an isomorphism

$$\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{st}}(V) \simeq \mathrm{B}_{\mathrm{st}}(V).$$

It follows that the base change of $\mathcal{K}_x$ to $\mathrm{B}_{\mathrm{st}}(V)$ is isomorphic to the free module on two generators

$$\mathcal{K}_x \otimes \mathrm{B}_{\mathrm{st}}(V) \simeq \mathrm{B}_{\mathrm{st}}(V) \cdot b_1 \oplus \mathrm{B}_{\mathrm{st}}(V) \cdot b_2,$$

where b_1 is the element $1 \in K_0 = \mathrm{B}_{\mathrm{st}}(V)^{G_K}$. An element $g \in G_K$ acts by

$$g(\gamma_1 \cdot b_1 + \gamma_2 \cdot b_2) = (\gamma_1^g + \log(\alpha(\epsilon_g)))b_1 + \gamma_2^g \cdot b_2.$$

From this we see that the G_K-invariant sections of $\mathcal{K}_x \otimes \mathrm{B}_{\mathrm{st}}(V)$ are spanned by b_1 and an element

$$w = \rho b_1 + b_2,$$

where $\rho \in \mathrm{B}_{\mathrm{st}}(V)$ is an element such that

$$\log(\alpha(\epsilon_g)) = \rho - \rho^g,$$

for all $g \in G_K$. Thus $\mathcal{K}_x$ is semistable if and only there exists such a ρ, which we now write down explicitly.

10.10. Let S_V be the perfection of $\overline{V}/p\overline{V}$ and let $\underline{\bar{u}} \in S_V^*$ be the element defined by the reductions of the u_n. We can then consider the image $[\underline{u}] \in \mathrm{A}_{\mathrm{cris}}(V)$ of the Teichmuller lifting of $\underline{\bar{u}}$ under the natural map

$$W(S_V) \to \mathrm{A}_{\mathrm{cris}}(V).$$

Then $[\underline{u}] - 1$ is in the divided power ideal of $\mathrm{A}_{\mathrm{cris}}(V)$ since $u \equiv 1 \pmod{\pi}$, so we can define the logarithm $\log([\underline{u}])$. Moreover, by the definition of ϵ_g and λ_g we have

$$u_n \cdot \epsilon_{g,n} = u_n^g \lambda_{g,n}^z.$$

This relation implies that in $\mathrm{A}_{\mathrm{cris}}(V)$ we have

$$\log(\alpha(\epsilon_g)) = (\log[\underline{u}])^g - \log([\underline{u}]) + z\log(\alpha(\lambda_g)).$$

It follows that we can take

$$\rho = -(\log([\underline{u}]) + zu_\beta) \in \mathrm{B}_{\mathrm{st}}(V).$$

Remark 10.11. Note that this description of $(\mathcal{K}_x \otimes \mathrm{B}_{\mathrm{st}}(V))^{G_K}$ also shows that the monodromy operator is nontrivial.

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