ℓ-DERIVED SCHEMES AND LOCAL TERMS

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Contents

1. Introduction

This paper is a continuation of our study of local terms and independence of ℓ begun in [16, 17] and continuing in subsequent papers.

Let k be an algebraically closed field and let ℓ be a prime invertible in k. Let E denote either a finite extension of $\mathbb{Q}_\ell$ or an algebraic closure $\overline{\mathbb{Q}}_\ell$. Recall that if $c : C \to X \times X$ is a correspondence of quasi-projective schemes over k, and if $F \in D^b_c(X, E)$ is a complex then an *action* of c on F is a morphism $u : c_1^*F \to c_2^!F$. For such an action one gets classes $\mathrm{Tr}_c(u) \in H^0(\mathrm{Fix}(c), \Omega_{\mathrm{Fix}(c)})$, where the fixed point scheme $\mathrm{Fix}(c)$ is defined to be the fiber product

$$\begin{array}{ccc} \mathrm{Fix}(c) & \xrightarrow{\delta} & C \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ X & \xrightarrow{\Delta_X} & X \times X. \end{array}$$

For a proper connected component $Z \subset \mathrm{Fix}(c)$ the local term $\mathrm{lt}_Z(F, u) \in E$ is defined as the proper pushforward of $\mathrm{Tr}_c(u)|_Z$ from Z to the point, commonly written as $\int_Z \mathrm{Tr}_c(u)$. These local terms are difficult to compute in general.

In the case when C and X are smooth of the same dimension d and F is a lisse sheaf, the class $\mathrm{Tr}_c(u)$ can be described in terms of cycles as follows. There is a canonical isomorphism $v : E \simeq c_2^! E$ which induces an isomorphism $c_2^* F \to c_2^! F$. Under this identification the map u defines a map $a : c_1^* F \to c_2^* F$ such that u is given by the tensor product

$$a \otimes v : c_1^* F \to c_2^* F \otimes c_2^! F \simeq c_2^! F.$$

Pulling back a along δ we get a map

$$c'^* F \to c'^* F,$$

whose trace is a global section $t_a \in H^0(\mathrm{Fix}(c), E)$. It is not hard to show that one then has

$$\mathrm{Tr}_c(u) = t_a \cdot \mathrm{Tr}_c(v).$$

This factorization has several advantages. The local term $\mathrm{Tr}_c(v)$ is given by the cohomology class of an algebraic cycle (see [16, 1.7]) so this factor is in some sense understood. The term t_a has the advantage that it is functorial in C and X, contrary to the class $\mathrm{Tr}_c(u)$, and therefore lends itself to various devissage arguments.

The purpose of the present paper is to generalize this picture to a setting that arises as follows. Consider a correspondence

$$c : C \to X \times X$$

over k, where C and X are smooth separated schemes of finite type of the same dimension d and c_1 and c_2 are dominant (and hence generically quasi-finite). Let $j : U \hookrightarrow X$ be a dense open subset with complement D a divisor with normal crossings in X such that $c_1^{-1}(D) = c_2^{-1}(D)$. Let V be a lisse sheaf on U with an action $u : c_{U1}^* V \to c_{U2}^* V$ over $C_U := c_1^{-1}(U) = c_2^{-1}(U)$. Since the boundary D is invariant, the pushforward $j_* V$ on X also has an action $j_* u$ of c.

Theorem 1.1. *Assume that the sheaf has unipotent local monodromy along D and that the action u is filtered (see 9.2). Then there exists a global section $t_{j_* u} \in H^0(\mathrm{Fix}(c), E)$, which generalizes the section t_a discussed above, with the following properties:*

- (i) *(Proposition 7.4) It is functorial.*
- (ii) *(Corollary 8.9) If $v : c_1^* \Lambda \to c_2^! \Lambda$ denotes the standard action, then*

$$\mathrm{Tr}_c(j_* u) = t_{j_* u} \cdot \mathrm{Tr}_c(v).$$

The basic idea behind the definition of $t_{j_* u}$ is in the spirit of derived algebraic geometry. The pushforward $M := Rj_* V$ is not just an object of the derived category of sheaves of E-modules, but an object of the derived category of sheaves of A_X-modules, where A_X denotes the sheaf of differential graded algebras $Rj_* E$. Our assumption that V has unipotent local monodromy along D implies that this A_X-module M behaves in many ways like a projective A_X-module. Furthermore, if $\tilde{j} : c_1^{-1}(U) \hookrightarrow C$ is the inclusion and A_C denotes $R\tilde{j}_* E$, then we can form the derived pullback

$$\mathrm{L}c_i^* M := c_i^{-1} M \otimes^{\mathbb{L}}_{c_i^{-1} \mathscr{A}_X} A_C.$$

The map $Rj_* u$ then defines a map $\mathrm{L}c_1^* M \to \mathrm{L}c_2^* M$ in the derived category of A_C-modules, and the element $t_{j_* u}$ is defined as a kind of trace of this map.

The organization of the paper is as follows.

To properly work with sheaves of differential graded algebras and modules over them we need several foundational results on the model category structures on such sheaves. Because we will be needing a good theory of tensor products we use the so-called flat model category structure studied in [8, 9, 10]. We develop the necessary results on this model category structure in section 2. This model category structure has also been considered by Liu and Zheng [15].

In section 3 we develop the theory of commutative differential graded algebras in a ringed topos (T, Λ), where Λ is a sheaf of $\mathbb{Q}$-algebras, as well as the model category of modules over such a differential graded algebra.

In section 4 we use this general machinery to introduce the notion of an ℓ-derived scheme. This is a pair (X, A_X), where X is a scheme and A_X is a sheaf of differential graded algebras on the pro-étale site of X as defined in [3]. The reason for working with the pro-étale site here is that we consider only commutative differential graded algebras over a $\mathbb{Q}$-algebra and using the usual étale site would involve first passing to torsion coefficients where the theory of commutative differential graded algebras is not well-behaved.

In section 5 we consider correspondences of ℓ-derived schemes and naive local terms of actions. The element t_{j_*u} mentioned above will be constructed as the naive local term of an action of an ℓ-derived correspondence.

The main example where the general theory applies is for pushforwards of lisse sheaves with unipotent local monodromy. We discuss this in section 6.

In section 7 we discuss the basic properties of these ℓ-derived naive local terms, and in particular establish their functoriality. In section 8 we elucidate their relationship with true local terms. Finally in section 9 we discuss a key property of actions of correspondences arising in the case of Weil sheaves over finite fields.

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2. The flat model category structure

2.1. Following [8], if $\mathscr{D}$ is an abelian category and $\mathscr{A} \subset \mathscr{D}$ is a subcategory, we write $\mathscr{A}^\perp \subset \mathscr{D}$ for the full subcategory whose objects are $M \in \mathscr{D}$ for which

$$\mathrm{Ext}^1_{\mathscr{D}}(A, M) = 0$$

for all $A \in \mathscr{A}$. We also define ${}^\perp\mathscr{A}$ to be the full subcategory of objects $M \in \mathscr{D}$ for which

$$\mathrm{Ext}^1_{\mathscr{D}}(M, A) = 0$$

for all $A \in \mathscr{A}$.

Definition 2.2 ([8, §2])**.** A cotorsion pair in $\mathscr{D}$ is a pair of subcategories $(\mathscr{A}, \mathscr{B})$ such that $\mathscr{B} = \mathscr{A}^\perp$ and $\mathscr{A} = {}^\perp\mathscr{B}$.

2.3. Let C be a site with associated topos T, and let Λ be a sheaf of rings on C. Assume that the topos T has enough points.

Let Mod_Λ denote the category of Λ-modules in T, and let $\mathscr{A} \subset \mathrm{Mod}_\Lambda$ be the subcategory of flat Λ-modules.

Remark 2.4. We expect that the assumption that T has enough points can be omitted, but the arguments below use points in various places and the case when T has enough points suffices for the applications we have in mind.

Definition 2.5. A Λ-module $M \in \mathrm{Mod}_\Lambda$ is *cotorsion* if

$$\mathrm{Ext}^1_\Lambda(A, M) = 0$$

for all $A \in \mathscr{A}$. Let $\mathscr{B} \subset \mathrm{Mod}_\Lambda$ denote the subcategory of cotorsion sheaves of Λ-modules, so $\mathscr{B} = \mathscr{A}^\perp$ by definition.

Lemma 2.6. *The pair $(\mathscr{A}, \mathscr{B})$ is a cotorsion pair in Mod_Λ. Equivalently, we have*

$$\mathscr{A} = \{S \in \mathrm{Mod}_\Lambda | \ \mathrm{Ext}^1_\Lambda(S, M) = 0 \text{ for all } M \in \mathscr{B}\}.$$

Proof. Let $S \in \mathrm{Mod}_\Lambda$ be an object with $\mathrm{Ext}^1_\Lambda(S, M) = 0$ for all $M \in \mathscr{B}$. We must show that S is a flat sheaf of Λ-modules. Since T has enough points, it suffices to show that for any point

$$j : \{\star\} \to T$$

the stalk $j^{-1}S$ is a flat $j^{-1}\Lambda$-module. The pushforward functor

$$j_* : \mathrm{Mod}_{j^{-1}\Lambda} \to \mathrm{Mod}_\Lambda$$

is exact, since for any $U \in T$ and $M \in \mathrm{Mod}_{j^{-1}\Lambda}$ we have

$$(j_*M)(U) = \mathrm{Hom}(j^{-1}h_U, M),$$

where we write h_U for the representable presheaf on C defined by U. We therefore have

$$\mathrm{Ext}^1_\Lambda(S, j_*M) = \mathrm{Ext}^1_{j^{-1}\Lambda}(j^{-1}S, M).$$

This implies that if M is a $j^{-1}\Lambda$-module with

$$\mathrm{Ext}^1_{j^{-1}\Lambda}(A, M) = 0$$

for all flat $j^{-1}\Lambda$-modules A, then $j_*M \in \mathscr{B}$.

Therefore we have

$$\mathrm{Ext}^1_\Lambda(S, j_*M) = \mathrm{Ext}^1_{j^{-1}\Lambda}(j^{-1}S, M) = 0$$

for all cotorsion $j^{-1}\Lambda$-modules M. This reduces the proof to the punctual case where the result is [7, 7.1.4]. □

2.7. Let $C(\Lambda)$ denote the category of complexes (possibly unbounded in both directions) of Λ-modules in T. For $X \in C(\Lambda)$, set

$$Z^nX := \mathrm{Ker}(X^n \to X^{n+1}).$$

Let $\widetilde{\mathscr{A}} \subset C(\Lambda)$ (resp. $\widetilde{\mathscr{B}} \subset C(\Lambda)$) denote the full subcategory of exact complexes X with Z^nX in $\mathscr{A}$ (resp. $\mathscr{B}$) for all n. Let $dg\widetilde{\mathscr{A}} \subset C(\Lambda)$ denote the full subcategory of complexes X with $X^n \in \mathscr{A}$ for all n, and such that any morphism

$$X \to B$$

with $B \in \widetilde{\mathscr{B}}$ is homotopic to 0. Similarly define $dg\widetilde{\mathscr{B}}$ to be the full subcategory of complexes X with $X^n \in \mathscr{B}$ for all n and such that any morphism

$$A \to X$$

with $A \in \widetilde{\mathscr{A}}$ is homotopic to 0.

Theorem 2.8. *There exists a cofibrantly generated model category structure on $C(\Lambda)$ in which a morphism $f : X \to Y$ is an*

(i) *equivalence if f is a quasi-isomorphism;*
(ii) *cofibration if f is a monomorphism with cokernel in $dg\widetilde{\mathscr{A}}$.*
(iii) *fibration if f is an epimorphism with kernel in $dg\widetilde{\mathscr{B}}$.*

Proof. See [15, 2.1.3]. □

Definition 2.9. The model category structure described in the theorem is called the *flat model category structure* on $C(\Lambda)$.

Remark 2.10. In the case when T is the category of sheaves on a topological space theorem 2.8 reduces to [8, 4.12].

Example 2.11. The constant sheaf Λ viewed as a complex concentrated in degree 0 is cofibrant. Certainly Λ is a complex of flat modules. If $B \in \widetilde{\mathscr{B}}$ then giving a map

$$\Lambda \to B$$

is equivalent to giving a map of Λ-modules $\Lambda \to Z^0 B$. Since B is assumed exact, there is an exact sequence

$$0 \to Z^{-1}B \to B^{-1} \to Z^0 B \to 0.$$

Since $Z^{-1}B$ is assumed cotorsion, there exists a lifting $\Lambda \to B^{-1}$ of the map to $Z^0 B$. This map defines a homotopy between our original map $\Lambda \to B$ and 0.

Remark 2.12. A bounded below complex of injective Λ-modules is a fibrant object of $C(\Lambda)$.

Remark 2.13. In the case when T is the punctual topos and Λ is a field, fibrations in $C(\Lambda)$ are epimorphisms, cofibrations are monomorphisms, and equivalences are quasi-isomorphisms.

2.14. **Another characterization of cofibrant objects.** The main result of this section is the following proposition 2.15. In the punctual case this result is [8, 5.6], and the general result is stated without proof in [15, 2.1.8]. For the convenience of the reader we include a proof here.

Proposition 2.15. *A complex $A \in C(\Lambda)$ is a cofibrant object if and only if A^n is flat for all n, and for every exact complex E the complex $A \otimes E$ is exact.*

Before beginning the proof let us introduce the following terminology:

Definition 2.16. Let $E \in C(\Lambda)$ be a complex of Λ-modules on C. We say that E is

(1) *injective*, if E is an exact complex and $Z^n E$ is an injective Λ-module for all n;
(2) *dg-injective*, if E^n is an injective Λ-module for all n and any morphism $X \to E$ with $X \in C(\Lambda)$ is homotopic to 0;

(3) *cotorsion*, if $E \in \widetilde{\mathscr{B}}$;
(4) *dg-cotorsion*, if $E \in dg\widetilde{\mathscr{B}}$.

Proof of 2.15. Let $\{x_i\}_{i\in I}$ be a conservative set of points, and set

$$\mathscr{Q} := \prod_{i\in I} x_{i*}(\mathbb{Q}/\mathbb{Z}),$$

a sheaf of abelian groups on C. For a complex E of Λ-modules on C, let $E^\vee$ denote the complex of Λ-modules

$$\mathscr{H}om^{\cdot}_{\mathrm{Ab}}(E, \mathscr{Q}).$$

Note that this complex is isomorphic to

$$\prod_{i\in I} x_{i*}\mathscr{H}om^{\cdot}_{\mathrm{Ab}}(E_x, \mathbb{Q}/\mathbb{Z}).$$

Lemma 2.17. *A sequence of sheaves of* Λ*-modules*

$$0 \to F \to G \to H \to 0$$

is exact if and only if the sequence

$$0 \to \mathscr{H}om(H, \mathscr{Q}) \to \mathscr{H}om(G, \mathscr{Q}) \to \mathscr{H}om(F, \mathscr{Q}) \to 0$$

is exact.

Proof. Since the second sequence is a product of sequences of the form

$$0 \to \mathrm{Hom}(H_{x_i}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(G_{x_i}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(F_{x_i}, \mathbb{Q}/\mathbb{Z}) \to 0 \tag{2.17.1}$$

and the first sequence is exact if and only if it is exact on stalks, it suffices to verify that for a point x_i the sequence

$$0 \to F_{x_i} \to G_{x_i} \to H_{x_i} \to 0$$

is exact if and only if (2.17.1) is exact. This reduces the proof to the punctual case which follows from [8, 5.2]. □

Lemma 2.18. *A sheaf of* Λ*-modules* F *is flat if and only if the* Λ*-module* $\mathscr{H}om(F, \mathscr{Q})$ *is injective.*

Proof. If F is flat, then F_{x_i} is a flat Λ_{x_i}-module for every i, which implies that

$$\mathscr{H}om(F, x_{i*}\mathbb{Q}/\mathbb{Z}) \simeq x_{i*}\mathscr{H}om(F_{x_i}, \mathbb{Q}/\mathbb{Z})$$

is an injective Λ-module, by the punctual case which follows from [8, 5.3]. Since the product of injective Λ-modules is injective this implies that $\mathscr{H}om(F, \mathscr{Q})$ is injective.

Conversely, suppose $\mathscr{H}om(F, \mathscr{Q})$ is injective, and consider an exact sequence of Λ-modules

$$0 \to G' \to G \to G'' \to 0.$$

To verify that the sequence

$$0 \to G' \otimes F \to G \otimes F \to G'' \otimes F \to 0$$

is exact, it suffices by 2.17 to show that the sequence

$$0 \to \mathscr{H}om(G'' \otimes F, \mathscr{Q}) \to \mathscr{H}om(G \otimes F, \mathscr{Q}) \to \mathscr{H}om(G' \otimes F, \mathscr{Q}) \to 0$$

is exact. By adjunction, this sequence is isomorphic to the sequence

$$0 \to \mathscr{H}om_\Lambda(G'', \mathscr{H}om(F, \mathscr{Q})) \to \mathscr{H}om_\Lambda(G, \mathscr{H}om(F, \mathscr{Q})) \to \mathscr{H}om_\Lambda(G', \mathscr{H}om(F, \mathscr{Q})) \to 0,$$

which is exact since $\mathscr{H}om(F, \mathscr{Q})$ is injective. □

Lemma 2.19. *For any sheaf of Λ-modules S, the sheaf $\mathscr{H}om_{\mathrm{Ab}}(S, \mathscr{Q})$ is a cotorsion sheaf.*

Proof. Let $F \to S$ be a flat resolution of S. Then by 2.18 each $\mathscr{H}om(F^i, \mathscr{Q})$ is an injective Λ-module, and by 2.17 the map

$$\mathscr{H}om(S, \mathscr{Q}) \to \mathscr{H}om(F, \mathscr{Q})$$

is an injective resolution of $\mathscr{H}om(S, \mathscr{Q})$. Therefore for any flat sheaf of Λ-modules M we have

$$\mathrm{Ext}^1(M, \mathscr{H}om(S, \mathscr{Q})) \simeq H^1(\mathrm{Hom}_\Lambda(M, \mathscr{H}om(F, \mathscr{Q})),$$

which by adjunction is isomorphic to

$$H^1(\mathrm{Hom}_{\mathrm{Ab}}(M \otimes_\Lambda F, \mathscr{Q})).$$

Since $\mathrm{Hom}_{\mathrm{Ab}}(-, \mathscr{Q})$ is an exact functor by 2.17, this implies that this group is zero. □

Lemma 2.20. *Let $F \in C(\Lambda)$ be an object.*

(i) *F is in $\widetilde{\mathscr{A}}$ if and only if $F^\vee$ is injective.*
(ii) *F is injective if and only if F is exact and dg-injective.*

Proof. For (i), note that by 2.17, the complex F is exact if and only if $F^\vee$ is exact, and moreover if this holds then

$$Z^n F^\vee = \mathscr{H}om(F^{-n}/Z^{-n}F, \mathscr{Q}) \simeq \mathscr{H}om(Z^{-n+1}F, \mathscr{Q}) \tag{2.20.1}$$

which by 2.18 is injective if and only if $Z^{-n+1}F$ is flat.

For (ii) let us first show that if F is injective then F is dg-injective (note that an injective F is exact by definition). Since Z^nF is injective, the inclusion $Z^nF \hookrightarrow F^n$ is split for each n, and so since F is exact we have noncanonical isomorphisms

$$F^n \simeq Z^nF \oplus Z^{n+1}F$$

for each n. This implies that F^n is an injective Λ-module. The proof that any map $X \to F$ is homotopic to 0 is the same as in [9, Lemma 3.9].

Conversely, if F is exact and dg-injective, then we show that each Z^nF is injective as follows. Let $i : N \hookrightarrow M$ be an inclusion of Λ-modules, and let $\varphi : N \to Z^nF$ be a morphism. Since F^n is injective, this morphism can be extended to a morphism $\psi : M \to F^n$. Let X be the complex which is equal to M in degree n and M/N in degree $n+1$ with differential the projection π. The maps φ and ψ induce a morphism of complexes

$$f : X \to F,$$

which since F is dg-injective is homotopic to zero. Let $h : M \to F^{n-1}$ and $h' : M/N \to F^n$ be maps defining a homotopy to 0. Then we have

$$\psi = dh + h'\pi.$$

In particular, we have $\varphi = dh$, so dh defines a morphism $M \to Z^nF$ extending φ. □

Lemma 2.21. *A complex $F \in C(\Lambda)$ is exact if and only if $F^\vee$ is in $\widetilde{\mathscr{B}}$.*

Proof. Note that F is exact if and only if $F^\vee$ is exact. It therefore suffices to show that if F is exact then the cycles of $F^\vee$ are cotorsion. This follows from 2.19 and the isomorphism (2.20.1). □

We can now complete the proof of 2.15.

First let us show that if F is a cofibrant object and E is an exact complex in $C(\Lambda)$, then $F \otimes_\Lambda E$ is exact. By 2.17 it suffices to show that $(F \otimes_\Lambda E)^\vee$ is exact. We have

$$(F \otimes E)^\vee = \mathscr{H}om^\cdot(F \otimes E, \mathscr{Q}) \simeq \mathscr{H}om^\cdot_\Lambda(F, \mathscr{H}om_{\mathrm{Ab}}(E, \mathscr{Q})) \simeq \mathscr{H}om^\cdot_\Lambda(F, E^\vee).$$

Now by 2.21 we have $E^\vee \in \widetilde{\mathscr{B}}$, and since $(dg\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$ is a cotorsion pair this implies that the complex

$$\mathscr{H}om^\cdot(F, E^\vee)(T)$$

is exact for any $T \in C$. Therefore the complex $(F \otimes E)^\vee$ is exact.

Conversely, suppose $A \in C(\Lambda)$ is a complex with A^n flat for all n and such that for any exact complex E the complex $A \otimes E$ is exact. We show that then $\mathrm{Hom}^\cdot(A, T)$ is exact for any $T \in \widetilde{\mathscr{B}}$. Since $(dg\widetilde{\mathscr{A}}, \widetilde{\mathscr{B}})$ is a cotorsion pair, this implies that $A \in dg\widetilde{\mathscr{A}}$.

By [8, 3.7], there exists an exact sequence

$$0 \to A \to C \to D \to 0,$$

where $C \in \widetilde{\mathscr{B}}$ and $D \in dg\widetilde{\mathscr{A}}$. We claim that in fact $C \in \widetilde{\mathscr{A}}$. For this it suffices by 2.20 to show that $C^\vee$ is exact and dg-injective. Now since A^n and D^n are flat for all n, the Λ-modules C^n are also flat and therefore the terms $C^{\vee n}$ are injective for all n. Furthermore $C^\vee$ is exact since C is exact. It remains to show that if E is an exact complex, then any map $E \to C^\vee$ is homotopic to zero. For this it suffices to show that for any $T \in C$ the complex

$$\mathscr{H}om^\cdot(E, C^\vee)(T)$$

is exact. Now just as in the first part of the proof, we have

$$\mathscr{H}om^\cdot(E, C^\vee) \simeq (E \otimes C)^\vee.$$

Furthermore, by the first part of the proof the complex $D^\cdot \otimes E$ is exact as is the complex $A \otimes E$ by assumption. It follows that $E \otimes C$ is exact, which implies that $(E \otimes C)^\vee$ is presheaf exact. This completes the proof that $C \in \widetilde{\mathscr{A}}$.

Now let $T \in \widetilde{\mathscr{B}}$. Notice that for any n, the module T^n is an extension of cotorsion modules, and therefore is also cotorsion. For any integers n and m we have an exact sequence

$$0 \to \mathrm{Hom}(D^n, T^m) \to \mathrm{Hom}(C^n, T^m) \to \mathrm{Hom}(A^n, T^m) \to \mathrm{Ext}^1(D^n, T^m),$$

and since T^m is cotorsion and D^n is flat we have $\mathrm{Ext}^1(D^n, T^m) = 0$. Therefore the sequence

$$0 \to \mathrm{Hom}^\cdot(D, T) \to \mathrm{Hom}^\cdot(C, T) \to \mathrm{Hom}^\cdot(A, T) \to 0$$

is an exact sequence of complexes. Now since C and D are in $dg\widetilde{\mathscr{A}}$, the complexes $\mathrm{Hom}^\cdot(D, T)$ and $\mathrm{Hom}^\cdot(C, T)$ are exact, whence $\mathrm{Hom}^\cdot(A, T)$ is also exact. □

2.22. **Relation with the derived category.**

2.23. Let $D(\Lambda)$ denote the derived category of sheaves of Λ-modules in T, so we have a functor $C(\Lambda) \to D(\Lambda)$. This functor takes equivalences in $C(\Lambda)$ to isomorphisms in $D(\Lambda)$ so induces a functor

$$q : \mathrm{Ho}(C(\Lambda)) \to D(\Lambda), \tag{2.23.1}$$

where $\mathrm{Ho}(C(\Lambda))$ denotes the homotopy category of the model category $C(\Lambda)$.

Remark 2.24. I L (resp. I) is a cofibrant (resp. fibrant) object of $C(\Lambda)$ then

$$\mathrm{Hom}_{\mathrm{Ho}(C(\Lambda))}(L, I)$$

is given by homotopy classes of morphisms of complexes $L \to I$. This follows, as in the classical case of bounded below chain complexes with the usual model category structure, from noting that the cone L' of the map

$$L \to L \oplus L, \quad a \mapsto (a, -a)$$

together with the map $L \to L$ given by the sum map $L \oplus L \to L$ is a cylinder object for L in $C(\Lambda)$ in the sense of [13, 1.2.4].

Theorem 2.25. *The functor (2.23.1) is an equivalence of categories.*

The proof occupies (2.26)–(2.32) For the proof we need some basic facts and definitions about unbounded derived categories. The basic references for this is [21] in the case of ringed spaces and [14] in the case of sites.

2.26. Let $\mathfrak{B} \subset C(\Lambda)$ be the full subcategory whose objects are bounded above complexes of sheaves of the form $\oplus_{i \in I} j_{i!}\Lambda_{U_i}[m]$, where m is an integer, I is a set of objects $\{U_i\}$ of C, and $j_{i!}$ is the left adjoint to the restriction functor from sheaves of Λ modules in T to sheaves of Λ-modules in the localized category T/U_i. Explicitly $j_{i!}\Lambda_{U_i}$ is the sheaf associated to the presheaf sending $V \in C$ to

$$\oplus_{\varphi : V \to U} \Lambda(V).$$

Recall from [21, 2.6] that a *special direct system in* $\mathfrak{B}$ is a system $(P_n)_{n \in E}$ of complexes with E a totally ordered set such that the following conditions hold:

(i) If $n \in E$ has no predecessor then $P_n = \varinjlim_{m<n} P_m$.

(ii) If n has a predecessor $n-1$ then the map $P_{n-1} \to P_n$ is injective, its cokernel C_n belongs to $\mathfrak{B}$, and for every integer r the sequence

$$0 \to P^r_{n-1} \to P^r_n \to C^r_n \to 0$$

is split.

Let $\widehat{\mathfrak{B}} \subset C(\Lambda)$ be the smallest class of complexes in $C(\Lambda)$ which contains $\mathfrak{B}$ and is closed under special direct limits in the sense [21, 2.6 (b)]. Note that since the category of cofibrant objects in $C(\Lambda)$ is closed under filtering direct limits the category of cofibrant objects in $C(\Lambda)$ is closed under special direct limits, and therefore every object of $\widehat{\mathfrak{B}}$ is cofibrant.

Definition 2.27. A complex $A \in C(\Lambda)$ is *K-limp* if for every acyclic object $L \in \mathfrak{B}$ the complex

$$\mathrm{Hom}^{\cdot}(L, A)$$

is acyclic.

Lemma 2.28. *If $I \in C(\Lambda)$ is fibrant, then I is K-limp.*

Proof. If $L \in \mathfrak{B}$ is an acyclic object then

$$H^n(\mathrm{Hom}^{\cdot}(L, I)) \simeq H^0(\mathrm{Hom}^{\cdot}(L[-n], I)) = \mathrm{Hom}_{\mathrm{Ho}(C(\Lambda))}(L[-n], I).$$

It therefore suffices to show that if I is fibrant and $L \in C(\Lambda)$ is trivially cofibrant then any morphism $L \to I$ is homotopic to 0, which is immediate. □

Lemma 2.29. *If $A \in C(\Lambda)$ is acyclic and K-limp, then for every $U \in C$ the complex $\Gamma(U, A)$ is exact.*

Proof. Let $z \in \Gamma(U, A^m)$ be a section with $dz = 0$. Define a subcomplex $R \subset A$ as follows. For $n > m$ set $R^n = 0$. Let $R^m \subset A^m$ be the image of the morphism $j_!\Lambda_U \to A^m$ corresponding to z, where $j : T/U \to T$ is the localization morphism, and define R^n for $n < m$ inductively by letting R^n be the preimage of $\mathrm{Ker}(R^{n+1} \to R^{n+2}) \subset A^{n+1}$ under the map $A^n \to A^{n+1}$. Then by construction R is an acyclic subcomplex of A. Let $L \to R$ be a quasi-isomorphism with $L^n = 0$ for $n > m$, $L^m = R^m$, and $L \in \mathfrak{B}$. Since A is K-limp the map $L \to A$ is homotopic to 0. In particular, looking in degree m we obtain a morphism $h_m : j_!\Lambda_U \to A^{m-1}$ whose composite with the differential $A^{m-1} \to A^m$ is the map corresponding to z. By adjunction the map h_m corresponds to a section $w \in \Gamma(U, A^{m-1})$ with $dw = z$. □

Lemma 2.30. *If $K \in C(\Lambda)$ is acylic and K-limp, then for any $L \in \widehat{\mathfrak{B}}$ the complex $\mathrm{Hom}^{\cdot}(L, K)$ is acylic.*

Proof. Let $\mathfrak{A} \subset \mathfrak{B}$ denote the subcategory of complexes P with $P^n = 0$ for all but one index n. Then every object of $\mathfrak{B}$ can be written as a special direct limit in $\mathfrak{A}$, and therefore if $\widehat{\mathfrak{A}}$ denotes the smallest class of complexes in $C(\Lambda)$ which contains $\mathfrak{A}$ and is closed under special direct limits, then $\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}}$. By [21, 2.5] the class of complexes $L \in C(\Lambda)$ for which $\mathrm{Hom}^{\cdot}(L, K)$ is acylic is closed under special direct limits. It therefore suffices to show that $\mathrm{Hom}^{\cdot}(L, K)$ is acylic for $L \in \mathfrak{A}$. This follows from 2.29. □

2.31. To prove 2.25 it suffices to show that the functor (2.23.1) is fully faithful. By [21, 3.4], every object of $\mathrm{Ho}(C(\Lambda))$ can be represented by a complex in $\widehat{\mathfrak{B}}$, so by [13, 1.2.10] it suffices to prove the following:

Proposition 2.32. *Let $K \in \widehat{\mathfrak{B}}$ and $L \in C(\Lambda)$ be complexes and assume that L is K-limp. Then the map*

$$\mathrm{Hom}_{\mathrm{Ho}(C(\Lambda))}(K, L) \to \mathrm{Hom}_{D(\Lambda)}(K, L)$$

is bijective.

Proof. Let $L \to I$ be a quasi-isomorphism with I a K-injective complex, and let C be the cone of the morphism $L \to I$. Then as in [21, 5.15] the complex C is K-limp and acyclic. Since we also have a distinguished triangle

$$\mathrm{Hom}^{\cdot}(K, L) \to \mathrm{Hom}^{\cdot}(K, I) \to \mathrm{Hom}^{\cdot}(K, C) \to \mathrm{Hom}^{\cdot}(K, L)[1]$$

it follows that it suffices to prove that $\mathrm{Hom}^{\cdot}(K, C)$ is acylic, which follows from 2.30. □

Remark 2.33. In the case of ringed spaces this follows from [21, 6.1].

This completes the proof of 2.25. □

2.34. Tensor product of complexes defines a symmetric monoidal structure on the category $C(\Lambda)$, and it is shown in [15, 2.1.3] that this monoidal structure is compatible with the model category structure in the sense of [13, Chapter 4].

In fact a stronger compatibility of the monoidal structure with the model category structure holds. Namely the monoid axiom of [19, 3.3] holds, as we now explain. Let $\mathscr{R}$ be the class of morphisms $f : X \to Y$ in $C(\Lambda)$ obtained by (possibly transfinite) compositions of pushouts of maps

$$A \otimes S \xrightarrow{i\otimes \mathrm{id}} B \otimes S, \tag{2.34.1}$$

where $S \in C(\Lambda)$ is an arbitrary complex and $i : A \to B$ is a trivial cofibration in $C(\Lambda)$. Then the monoid axiom in this case is the statement that every morphism in $\mathscr{R}$ is an equivalence.

To see that this stronger condition holds, note that a composition of equivalences is an equivalence so it suffices to show that a pushout of a map (2.34.1), with $i : A \to B$ a trivial cofibration, is an equivalence. Let C be the cokernel of i. Then by definition i is injective and C is an acyclic object of $dg\widetilde{\mathscr{A}}$. In particular the terms C^n are flat so the sequence

$$0 \to A \otimes S \to B \otimes S \to C \otimes S \to 0$$

is exact, and if $g : A \otimes S \to D$ is any morphism with resulting pushout $g_*i : D \to E$ then we have an exact sequence

$$0 \longrightarrow D \xrightarrow{g_*i} E \longrightarrow C \otimes S \longrightarrow 0.$$

This reduces the verification of the monoid axiom to the following:

Lemma 2.35. *Let $C \in dg\widetilde{\mathscr{A}}$ be an acyclic object. Then for any complex S the complex $C \otimes S$ is acyclic.*

Proof. Choose a surjective quasi-isomorphism $F \to S$ with F cofibrant, and let K be the kernel. Since C is trivially cofibrant each C^n is flat and the sequence

$$0 \to C \otimes K \to C \otimes F \to C \otimes S \to 0$$

is exact. By 2.15 both the complexes $C \otimes K$ and $C \otimes F$ are exact, whence $C \otimes S$ is exact also. □

2.36. Let (T, Λ) and (T', Λ') be ringed topoi as in 2.3, and let $f : (T', \Lambda') \to (T, \Lambda)$ be a morphism of ringed topoi. So there is no confusion, we write $C(T, \Lambda)$ (resp. $C(T', \Lambda')$) for the category of complexes of sheaves of Λ-modules in T (resp. sheaves of Λ'-modules in T').

Consider the functor

$$f^{-1} : C(T, \Lambda) \to C(T', f^{-1}\Lambda),$$

left adjoint to the functor

$$f_* : C(T', f^{-1}\Lambda) \to C(T, \Lambda),$$

and

$$(-) \otimes_{f^{-1}\Lambda} \Lambda' : C(T', f^{-1}\Lambda) \to C(T', \Lambda'),$$

left adjoint to the forgetful functor

$$C(T', \Lambda') \to C(T', f^{-1}\Lambda).$$

Lemma 2.37. *The pair (f^{-1}, f_*) defines a Quillen adjunction between the model categories $C(T, \Lambda)$ and $C(T', f^{-1}\Lambda)$.*

Proof. Consider first the case when T' is the punctual topos. In this case the functor f_* is exact, and therefore takes equivalences to equivalences and epimorphisms to epimorphisms. If M is a cotorsion module over $f^{-1}\Lambda$, then f_*M is a cotorsion Λ-module in T. Indeed for any flat Λ-module A we have

$$\mathrm{Ext}^1_\Lambda(A, f_*M) = \mathrm{Ext}^1_{f^{-1}\Lambda}(f^{-1}A, M) = 0,$$

where the second equality follows from the fact that $f^{-1}A$ is a flat $f^{-1}\Lambda$-module. To prove that f_* takes fibrations to fibrations it therefore suffices to show that if X is a fibrant complex of $f^{-1}\Lambda$-modules, and A is an exact complex of Λ-modules in T with Z^nA flat over Λ for all n, then any map $A \to f_*X$ is homotopic to 0. This follows from the corresponding statement from the adjoint morphism $f^{-1}A \to X$. This proves the lemma in the case when T' is the punctual topos.

For the general case, note that f^{-1} takes equivalences to equivalences and flat modules to flat modules. It therefore suffices to show that if X is a cofibrant complex of Λ-modules in T then $f^{-1}X$ is a cofibrant complex of Λ-modules in T', and for this in turn it suffices by 2.15 to show that for an exact complex E of $f^{-1}\Lambda$-modules the complex $f^{-1}X \otimes E$ is exact. This can be verified on stalks where it follows from the case when T' is the punctual topos. □

2.38. By 2.15 and 2.35 the functor

$$C(T', f^{-1}\Lambda) \to C(T', \Lambda'), \quad M \mapsto M \otimes_{f^{-1}\Lambda} \Lambda'$$

preserves cofibrations and trivial cofibrations, and it has a right adjoint given by the natural forgetful functor. Extension of scalars along $f^{-1}\Lambda \to \Lambda'$ is therefore part of a Quillen adjunction and we get derived functors

$$\mathrm{L}f^* : \mathrm{Ho}(C(T, \Lambda)) \to \mathrm{Ho}(C(T', \Lambda')), Rf_* : \mathrm{Ho}(C(T', \Lambda')) \to \mathrm{Ho}(C(T, \Lambda)),$$

where for a cofibrant $M \in C(T, \Lambda)$ we have

$$\mathrm{L}f^*M := f^{-1}M \otimes_{f^{-1}\Lambda} \Lambda'.$$

Furthermore, this implies that $f_* : C(T', \Lambda') \to C(T, \Lambda)$ takes fibrations (resp. trivial fibrations) to fibrations (resp. trivial fibrations).

3. Differential graded algebras

3.1. **Preliminaries on symmetric powers.** In this subsection we gather together various facts about symmetric power functors for modules. In the following we will also consider symmetric powers for sheaves, but this will rely on the results in the punctual case discussed here.

Throughout this subsection Λ denotes a $\mathbb{Q}$-algebra.

3.2. For an integer n and $N \in C(\Lambda)$ let $T^n(N)$ denote the n-fold tensor product of N with itself over Λ, so in degree r we have

$$T^n(N)^r = \oplus_{i_1+\cdots i_n=r} N^{i_1} \otimes N^{i_2} \otimes \cdots \otimes N^{i_n}.$$

There is an action of the symmetric group S_n on $T^n(N)$ in which the transposition interchanging s and $s+1$ sends

$$x_1 \otimes \cdots \otimes x_n \in N^{i_1} \otimes N^{i_2} \otimes \cdots \otimes N^{i_n}$$

to

$$(-1)^{i_s i_{s+1}} x_1 \otimes \cdots \otimes x_{i_{s+1}} \otimes x_{i_s} \otimes \cdots \otimes x_n.$$

Note that this is compatible with the differentials. The coinvariants of this action is denoted $\mathrm{Sym}^n(N)$, the *n-th symmetric power of N*.

3.3. Let $\mathrm{Mod}_\Lambda^\dagger$ denote the category of $\mathbb{Z}/(2)$-graded Λ-modules. For an object $V \in \mathrm{Mod}_\Lambda^\dagger$ we write $V = V_+ \oplus V_-$ to indicate the grading. For two objects $V, W \in \mathrm{Mod}_\Lambda^\dagger$ the tensor product

$$V \otimes W \in \mathrm{Mod}_\Lambda^\dagger$$

is the usual tensor product with $\mathbb{Z}/(2)$-grading given by

$$(V \otimes W)_+ = (V_+ \otimes W_+) \oplus (V_- \otimes W_-), \quad (V \otimes W)_- = (V_- \otimes W_+) \oplus (V_+ \otimes W_-).$$

Using this tensor product we can define

$$T^n : \mathrm{Mod}_\Lambda^\dagger \to \mathrm{Mod}_\Lambda^\dagger$$

sending V to $V^{\otimes n}$. Once again there is an action of the symmetric group S_n on $T^n(V)$ defined by the same formula as in 3.2, noting that the action of a transposition depends only on the parity of indices i_s and i_{s+1}. We define

$$\mathrm{Sym}^n : \mathrm{Mod}_\Lambda^\dagger \to \mathrm{Mod}_\Lambda^\dagger$$

by taking the S_n-invariants of T^n.

3.4. There is a functor

$$\epsilon : C(\Lambda) \to \mathrm{Mod}_\Lambda^\dagger$$

sending a complex N to the Λ-module $\epsilon(N) := \oplus_i N^i$ with $\mathbb{Z}/(2)$-grading

$$\epsilon(N)_+ = \oplus_{i \text{ even}} N^i, \quad \epsilon(N)_- = \oplus_{i \text{ odd}} N^i.$$

This functor commutes with tensor products, and therefore for $N \in C(\Lambda)$ we have a natural isomorphism

$$\epsilon(T^n(N)) \simeq T^n(\epsilon(N)).$$

This functor is compatible with the S_n-actions and induces an isomorphism

$$\epsilon(\mathrm{Sym}^n(N)) \simeq \mathrm{Sym}^n(\epsilon(N)).$$

3.5. Let $j : N \hookrightarrow M$ be a monomorphism in $C(\Lambda)$ such that the cokernel Q has Q^i flat over Λ for all i (the case we will be interested in is when $N \to M$ is a cofibration in $C(\Lambda)$).

In this case there is a filtration $F^j_\cdot$

$$\mathrm{Sym}^n(N) = F_0^j \subset F_1^j \subset \cdots F_n^j = \mathrm{Sym}^n(M)$$

on $\mathrm{Sym}^n(M)$, where F_i^j is defined to be the image of the natural map

$$\mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(M) \to \mathrm{Sym}^n(M).$$

Similarly for a monomorphism $j : N \hookrightarrow M$ in $\mathrm{Mod}_\Lambda^\dagger$ with cokernel flat over Λ we have a filtration $F^j_\cdot$ on $\mathrm{Sym}^n(M)$. The forgetful functor ϵ is compatible with these filtrations (since it commutes with the formation of images) in the sense that for a monomorphism $N \hookrightarrow M$ in $C(\Lambda)$ the filtration $F_\cdot^{\epsilon(j)}$ on $\mathrm{Sym}^n(\epsilon(M)) \simeq \epsilon(\mathrm{Sym}^n(M))$ is equal to $\epsilon(F_\cdot^j)$.

Lemma 3.6. *(i) Let $j : N \hookrightarrow M$ be a monomorphism in $\mathrm{Mod}_\Lambda^\dagger$ with cokernel Q flat over Λ. Then for every i the projection $\mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(M) \to \mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(Q)$ factors through F_i^j, which by definition is a quotient of $\mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(M)$ and induces an isomorphism*

$$F_i^j/F_{i-1}^j \simeq \mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(Q).$$

(ii) Let $j : N \hookrightarrow M$ be a monomorphism in $C(\Lambda)$ with cokernel Q term-wise flat over Λ. Then for every i the projection $\mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(M) \to \mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(Q)$ factors through F_i^j and induces an isomorphism

$$F_i^j/F_{i-1}^j \simeq \mathrm{Sym}^{n-i}(N) \otimes \mathrm{Sym}^i(Q).$$

Proof. To prove (i), note that giving a $\mathbb{Z}/(2)$-grading on a Λ-module K is equivalent to specifying a $\widetilde{\Lambda} := \Lambda[x]/(x^2-1)$-module structure on K compatible with the given Λ-module structure. In this dictionary K_+ (resp. K_0) is the $+1$ (resp. -1) eigenspace of the operator $\cdot x : K \to K$. Since Q is flat over Λ it is also flat over $\widetilde{\Lambda}$, and therefore we can write Q is a filtered direct limit

$$Q = \varinjlim_s Q_s,$$

of projective $\widetilde{\Lambda}$-modules Q_s. For an index s let M_s denote $M \times_Q Q_s$ so we have an inclusion $N \hookrightarrow M_s$ with cokernel Q_s. Then $M = \varinjlim_s M_s$ and since the functors Sym^n commute with filtered colimits it suffices to prove (i) for the M_s. This reduces the proof of (i) to the case when Q is a projective $\widetilde{\Lambda}$-module in which case the projection $M \to Q$ has a section in $\mathrm{Mod}_\Lambda^\dagger$. Thus to prove (i) it suffices to consider the case when $M = N \oplus Q$, with Q a projective $\widetilde{\Lambda}$-module. In this case (see for example [6, Appendix 2, A2.2]) we have

$$\mathrm{Sym}^n(M) = \oplus_{s+t=n} \mathrm{Sym}^s(N) \otimes_\Lambda \mathrm{Sym}^t(Q).$$

The filtration $F_\cdot^j$ is given by

$$F_i^j = \oplus_{s+t=n, t\leq i} \mathrm{Sym}^s(N) \otimes_\Lambda \mathrm{Sym}^t(Q)$$

from which (i) immediately follows.

Statement (ii) follows from (i) and the observation that all the statement in (ii) can be verified after applying the functor ϵ. □

3.7. **Symmetric algebras and sheaves.**

3.8. Let C be a site with associated topos T as in 2.3, and let Λ be a sheaf of commutative $\mathbb{Q}$-algebras on C.

3.9. As in the punctual case 3.2 for an integer n and object $N \in C(\Lambda)$ let $T^n(N)$ denote the n-fold tensor product of N with itself over Λ, so in degree r we have

$$T^n(N)^r = \oplus_{i_1+\cdots i_n = r} N^{i_1} \otimes N^{i_2} \otimes \cdots \otimes N^{i_n}.$$

This has an action of the symmetric group S_n and the coinvariants of this action are denoted $\mathrm{Sym}^n(N)$, the *n-th symmetric power of N*.

Lemma 3.10. *Let $u : A \to B$ be a trivial cofibration in $C(\Lambda)$. Then any pushout of $T^n(u)$ (resp. $\mathrm{Sym}^n(u)$) is an equivalence.*

Proof. The statement for $T^n(u)$ is a formal consequence of the monoid axiom. Indeed note that the map $T^n(u)$ can be factored as the composition of finitely many maps of the form

$$A^{\otimes s} \otimes A \otimes B^{\otimes(n-s-1)} \xrightarrow{\mathrm{id}\otimes u \otimes \mathrm{id}} A^{\otimes s} \otimes B \otimes B^{\otimes(n-s-1)}, \tag{3.10.1}$$

whence is an equivalence by the monoid axiom 2.34.

As for the statement about $\mathrm{Sym}^n(u)$, note that since Λ is a $\mathbb{Q}$-algebra for any N the complex $\mathrm{Sym}^n(N)$ is a direct summand of $T^n(N)$, functorialy in N (average over the group). □

Lemma 3.11. *Let $N \in C(\Lambda)$ be a cofibrant object. Then $T^n(N)$ and $\mathrm{Sym}^n(N)$ are cofibrant objects of $C(\Lambda)$.*

Proof. The statement for $T^n(N)$ is immediate, and the statement for $\mathrm{Sym}^n(N)$ follows by noting that $\mathrm{Sym}^n(N)$ is a direct summand of $T^n(N)$. □

3.12. **Basic definitions.**

Definition 3.13. A *sheaf of commutative differential graded Λ-algebras* (abbreviated dga) on C is a triple

$$(A, e, \boldsymbol{\mu}),$$

where $A \in C(\Lambda)$,

$$\boldsymbol{\mu} : A \otimes_\Lambda A \to A, \quad e : \Lambda \to A,$$

are morphisms of complexes, such that the following conditions hold:

(i) $\boldsymbol{\mu}$ and e make A^0 and $\oplus_p A^p$ into associative Λ-algebras with unit.
(ii) For local sections $a \in A^p$ and $b \in A^q$ we have
$$d(\boldsymbol{\mu}(a, b)) = \boldsymbol{\mu}(d(a), b) + (-1)^p \boldsymbol{\mu}(a, d(b)).$$
(iii) For local sections $a \in A^p$ and $b \in A^q$ we have
$$\boldsymbol{\mu}(a, b) = (-1)^{pq} \boldsymbol{\mu}(b, a).$$

We denote the category of sheaves of differential graded Λ-algebras by dga_Λ.

Remark 3.14. In this paper we consider only commutative differential graded algebras, and often omit the adjective "commutative".

Remark 3.15. In what follows we often write simply $a \cdot b$ for $\boldsymbol{\mu}(a, b)$ and $1 \in A^0$ for the unit.

3.16. There is a functor

$$\mathrm{dga}_\Lambda \to C(\Lambda), \quad A \mapsto A^\sharp$$

sending a differential graded algebra A to its underlying complex of Λ modules.

Definition 3.17. Let $A \in \mathrm{dga}_\Lambda$ be a sheaf of differential graded Λ-algebras. A *left A-module*, is a complex $M \in C(\Lambda)$ with a map of complexes

$$\rho : A \otimes_\Lambda M \to M$$

making $\oplus_p M^p$ a graded left-module over $\oplus_p A^p$.

For a left A-module M we denote the underlying object of $C(\Lambda)$ by $M^\sharp$. The usual shift in grading extends to the category of left A-modules.

Remark 3.18. Similarly one can define a notion of *right A-module*. Since our differential graded algebras are always assumed commutative there is an equivalence of categories between the categories of left and right A-modules, and in what follows we call them simply *A-modules*. We denote the category of A-modules by Mod_A.

3.19. **Model category structure on Mod_A.**

Theorem 3.20. *Let $A \in \mathrm{dga}_\Lambda$ be an algebra. Then there exists a cofibrantly generated model category structure on Mod_A in which a morphism $M \to N$ is a fibration (resp. equivalence) if and only if it is a fibration (resp. equivalence) in $C(\Lambda)$.*

Proof. This follows from [19, 4.1]. □

Remark 3.21. We could also consider two objects $A, B \in \mathrm{dga}_\Lambda$ and the resulting category $\mathrm{Mod}_{(A,B)}$ of differential graded (A, B)-bimodules. This category is equivalent to the category of left modules over $A^{\mathrm{op}} \otimes_\Lambda B$, so there exists a model category structure on $\mathrm{Mod}_{(A,B)}$ in which a morphism $M \to N$ is a fibration (resp. trivial fibration) if and only if the underlying map in $C(\Lambda)$ is a fibration (resp. trivial fibration).

3.22. Generating cofibrations in Mod_A can be given as follows (see [19, §6, proof of Theorem 4.1]). Let

$$\{N_i \to M_i\}_{i\in I}, \quad (\text{resp. } \{S_j \to T_j\}_{j\in J})$$

be sets of generating cofibrations (resp. generating trivial cofibrations) in $C(\Lambda)$. Then the sets of morphisms in Mod_A given by

$$\{N_i \otimes_\Lambda A \to M_i \otimes_\Lambda A\}_{i\in I}, \quad (\text{resp.} \{S_j \otimes_\Lambda A \to T_j \otimes_\Lambda A\}_{j\in J})$$

are generating cofibrations (resp. generating trivial cofibrations) in Mod_A. We denote these generating sets by I_A and J_A.

3.23. A variant construction of cofibrations in Mod_A is the following. Let $M \in \mathrm{Mod}_A$ be an A-module, let $N \in C(\Lambda)$ be a cofibrant object, and let $\beta : N \to M^\sharp$ be a morphism in $C(\Lambda)$. By adjunction the morphism β defines a morphism $\tilde\beta : A \otimes_\Lambda N \to M$ in Mod_A. Following [11, 3.1] we write $M\{N, \beta\}$ for the cone of the morphism $\tilde\beta$ (see for example [1, Tag 09K9] for the construction of the cone). For any $Q \in \mathrm{Mod}_A$ giving a morphism $M\{N, \beta\} \to Q$ in Mod_A is equivalent to giving a morphism $f : M \to Q$ and a homotopy between the induced map $N \to Q^\sharp$ and the zero map in $C(\Lambda)$. By the following lemma it follows that the map $M \to M\{N, \beta\}$ is a cofibration.

Lemma 3.24. *Let $N \in C(\Lambda)$ be a cofibrant object, let $g : S' \to S$ be a trivial fibration in $C(\Lambda)$ and let $f : N \to S'$ be a morphism. Then any homotopy t between gf and 0 lifts to a homotopy between f and 0.*

Proof. For a complex $Q \in C(\Lambda)$ let Q_I denote $\mathrm{Cone}(-\mathrm{id}_Q)[-1]$. Then for any complex $N \in C(\Lambda)$ the set of morphisms $N \to Q_I$ are in bijection with the set of pairs (f, σ), where $f : N \to Q$ is a morphism in $C(\Lambda)$ and σ is a homotopy between f and 0.

We apply this to prove the lemma as follows. Let K denote the kernel of $S' \to S$. The preceding construction is functorial and we get a surjective morphism $S'_I \to S_I \times_S S'$ with kernel $K[-1]$. It follows that $S'_I \to S_I \times_S S'$ is a trivial fibration. Since N is cofibrant we can lift the morphism $N \to S_I \times_S S'$ defined by f and t to a morphism $N \to S'_I$, which defines the required lifting of t. □

Remark 3.25. In the case of the punctual topos, every cofibrant object $N \in \mathrm{Mod}_A$ is a retract of a cofibration obtained as a direct limit of a sequence

$$0 \to M_1 \to M_2 \to \cdots ,$$

where each $M_i \to M_{i+1}$ is of the form $M_i \to M_i\{N, \beta\}$ for a complex N of Λ-modules with zero differentials. This is shown in [11, 2.2.5].

Lemma 3.26. *Let $i : N \to M$ be a monomorphism in Mod_A with cofibrant cokernel. Then i is a cofibration.*

Proof. Write Q for the cokernel of i.

Let $L \to M$ be a trivial fibration with L cofibrant, and let K denote $\mathrm{Ker}(L \to Q)$. Then the diagram

$$\begin{array}{ccc} K & \longrightarrow & L \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

is a pushout square, so it suffices to prove that $K \to L$ is a cofibration. This reduces the proof to the case when M is in addition cofibrant, which we assume for the rest of the proof.

Let $S' \to S$ be a trivial fibration with kernel J and consider a commutative diagram of solid arrows

$$\begin{array}{ccc} N & \xrightarrow{a} & S' \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ M & \xrightarrow{b} & S. \end{array}$$

Since M is cofibrant we can find a morphism $b' : M \to S'$ lifting b. Let $\epsilon : N \to J$ denote the difference $a - b'i$. To prove the lemma it suffices to show that we can extend ϵ to a morphism $M \to J$. Equivalently, if E denotes the pushout of the diagram

$$\begin{array}{ccc} N & \longrightarrow & J \\ \downarrow & & \\ M, & & \end{array}$$

then the inclusion $J \hookrightarrow E$ is split. This follows from noting that $E/J \simeq Q$ and that the quotient map $E \to Q$ is a trivial fibration. □

3.27. For $A \in \mathrm{dga}_\Lambda$ the homotopy category $\mathrm{Ho}(\mathrm{Mod}_A)$ is a triangulated category. Indeed it is obtained by localization along the null system of acyclic modules from the homotopy category $K(\mathrm{Mod}_A)$, which is triangulated in the usual way (see for example [1, Tag 09P5]).

3.28. Let $A \in \mathrm{dga}_\Lambda$ be an algebra, and let $M \in \mathrm{Mod}_A$ and $N \in \mathrm{Mod}_A$ be A-modules. We can then form the complex

$$M \otimes_A N \in \mathrm{Mod}_A.$$

The functor

$$(-) \otimes_A N : \mathrm{Mod}_A \to \mathrm{Mod}_A$$

is left adjoint to the functor

$$\mathscr{H}om_A(N, -) : \mathrm{Mod}_A \to \mathrm{Mod}_A.$$

Lemma 3.29. *If N is cofibrant in Mod_A, then the adjoint pair $((-) \otimes_A N, \mathscr{H}om_A(N, -))$ is a Quillen adjunction.*

Proof. It suffices to show that $(-) \otimes_A N$ takes cofibrations (resp. trivial cofibrations) to cofibrations (resp. trivial cofibrations).

In the case when $A = \Lambda$ this follows from the discussion in 2.34. This special case has the following consequence. Let $p : S' \to S$ be a trivial fibration (resp. fibration) and let $i : M \to M'$ be a cofibration (resp. trivial cofibration) in $C(\Lambda)$. Let $F \in C(\Lambda)$ be the fiber product of the diagram

$$\begin{array}{ccc} & & \mathscr{H}om_\Lambda(M, S') \\ & & \downarrow \\ \mathscr{H}om_\Lambda(M', S) & \longrightarrow & \mathscr{H}om_\Lambda(M, S), \end{array}$$

so there is a map

$$q : \mathscr{H}om_\Lambda(M', S') \to F.$$

Then q is a trivial fibration. Indeed if $N \to N'$ is a cofibration in $C(\Lambda)$ then by adjunction the problem of finding a dotted arrow filling in a commutative square

$$\begin{array}{ccc} N & \longrightarrow & \mathscr{H}om_\Lambda(M', S') \\ \downarrow & & \downarrow{\scriptstyle q} \\ N' & \longrightarrow & F \end{array} \tag{3.29.1}$$

is equivalent to finding a dotted arrow filling in any commutative diagram of solid arrows

$$\xymatrix{ N \otimes M \ar[r] \ar[d] & N \otimes M' \ar[r] \ar[d] & S' \ar[d] \\ N' \otimes M \ar[r] \ar[urr] & N' \otimes M' \ar[r] \ar@{-->}[ur] & S. }$$

Let E denote the pushout of

$$\begin{array}{ccc} N \otimes M & \longrightarrow & N \otimes M' \\ \downarrow & & \\ N' \otimes M, & & \end{array}$$

and let $e : E \to N' \otimes M'$ be the natural map. Then the problem of filling in 3.29.1 is in turn equivalent to finding a dotted arrow filling in any commutative diagram of solid arrows

$$\begin{array}{ccc} E & \longrightarrow & S' \\ {\scriptstyle e}\downarrow & \nearrow & \downarrow \\ N' \otimes M' & \longrightarrow & S. \end{array}$$

If Q (resp. R) denotes the cokernel of $N \to N'$ (resp. $M \to M'$) then e is a monomorphism with cokernel $Q \otimes R$. It follows that e is a cofibration, and if i is a trivial cofibration so is e. Thus we can find the desired dotted arrow in both our two cases:

(1) p is a trivial fibration and i is a cofibration.
(2) p is a fibration and i is a trivial cofibration.

For the case of general $A \in \mathrm{dga}_\Lambda$, it suffices by [13, 2.1.20] and the description of cofibrant generators in Mod_A given in 3.22 to show that if $M \to M'$ is a cofibration (resp. trivial cofibration) in $C(\Lambda)$, then the map

$$M \otimes_\Lambda N \to M' \otimes_\Lambda N \tag{3.29.2}$$

is a cofibration (resp. trivial cofibration) in Mod_A as well. To verify this let $S' \to S$ be a trivial fibration (resp. fibration) in Mod_A, and let F be the fiber product of the diagram Mod_A (where the A-module structure on the terms is induced by the A-module structure on S' and S)

$$\begin{array}{ccc} & & \mathscr{H}om_\Lambda(M, S') \\ & & \downarrow \\ \mathscr{H}om_\Lambda(M', S) & \longrightarrow & \mathscr{H}om_\Lambda(M, S). \end{array}$$

Then the lifting problem for verifying that 3.29.2 is a cofibration (resp. trivial cofibration) is equivalent to the problem of finding a dotted arrow as follows:

$$\begin{array}{ccc} & & \mathscr{H}om_\Lambda(M', S') \\ & \nearrow & \downarrow{\scriptstyle q} \\ N & \longrightarrow & F, \end{array}$$

where q is the natural map. By the above discussion the map q is a trivial fibration, as this can be verified in $C(\Lambda)$, and since N is cofibrant this lifting problem can be solved. □

3.30. In particular, we get an induced derived functor

$$(-) \otimes^{\mathbb{L}}_A (-) : \mathrm{Ho}(\mathrm{Mod}_A) \times \mathrm{Ho}(\mathrm{Mod}_A) \to \mathrm{Ho}(\mathrm{Mod}_A).$$

Lemma 3.29 implies that if $N \in \mathrm{Mod}_A$ is cofibrant then $(-) \otimes_A N$ preserves trivial cofibrations. In fact more is true:

Lemma 3.31. *Let $N \to N'$ be a cofibration in Mod_A and let $i : M \to M'$ be an equivalence. If $M \otimes_A N \to M' \otimes_A N$ is an equivalence then $M \otimes_A N' \to M \otimes_A N'$ is also an equivalence.*

In particular, if $N \in \mathrm{Mod}_A$ is cofibrant and $M \to M'$ is an equivalence then the map

$$M \otimes_A N \to M' \otimes_A N$$

is an equivalence.

Proof. Let $x : \{*\} \to T$ be a point of the topos. Then we get an adjunction (x^*, x_*) between Mod_A and Mod_{A_x}, where Mod_{A_x} is the category of differential graded modules over A_x in the puntual topos. This is a Quillen adjunction as x_* preserves fibrations and trivial fibrations by 2.37. It follows that it suffices to prove the lemma in the case of the punctual topos.

In this case $N \to N'$ is a retract of cofibration $N \to N''$ obtained as a transfinite composition of pushouts of the generating cofibrations described in 3.22. It therefore suffices to consider the case when $N \to N'$ is a pushout of a generating cofibration. In other words, we may assume given a cofibration $N_0 \hookrightarrow N_0'$ in $C(\Lambda)$ and a morphism $A \otimes_\Lambda N_0 \to N$ such that $N \to N'$ is the pushout of the resulting diagram

$$\begin{array}{ccc} A \otimes_\Lambda N_0 & \longrightarrow & A \otimes_\Lambda N_0' \\ \downarrow & & \\ N. & & \end{array}$$

Let Q denote the cokernel of $N_0 \to N_0'$ so Q is a cofibrant object of $C(\Lambda)$; in particular, Q is termwise flat. Then for any $S \in \mathrm{Mod}_A$ the sequence

$$0 \to S \otimes_A (A \otimes_\Lambda N_0) \to S \otimes_A (A \otimes_\Lambda N_0') \to S \otimes_A (A \otimes_\Lambda Q) \to 0$$

is exact. By pushing out we find that the sequence

$$0 \to S \otimes_A N \to S \otimes_A N' \to S \otimes_A (A \otimes_\Lambda Q) \to 0$$

is also exact.

We therefore get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M \otimes_A N & \longrightarrow & M \otimes_A N' & \longrightarrow & M \otimes_\Lambda Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' \otimes_A N & \longrightarrow & M' \otimes_A N' & \longrightarrow & M' \otimes_\Lambda Q & \longrightarrow & 0. \end{array}$$

This reduces us to showing that the map $M \otimes_\Lambda Q \to M' \otimes_\Lambda Q$ is an equivalence. Let C be the cone of $M \to M'$. Since Q is cofibrant the tensor product $C \otimes_\Lambda Q$ is equivalent to the cone of $M \otimes_\Lambda Q \to M' \otimes_\Lambda Q$. Since C is acyclic we then get from 2.15 that $C \otimes_\Lambda Q$ is acyclic completing the proof. □

3.32. If $A \to B$ is a morphism in dga_Λ, then we have adjoint functors

$$(-) \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B,$$

$$\varphi : \mathrm{Mod}_B \to \mathrm{Mod}_A,$$

where φ is the forgetful functor. The functor φ clearly preserves fibrations and trivial fibrations, and therefore the pair

$$((-) \otimes_A B, \varphi)$$

is a Quillen adjunction. We write

$$(-) \otimes_A^{\mathbb{L}} B : \mathrm{Ho}(\mathrm{Mod}_A) \to \mathrm{Ho}(\mathrm{Mod}_B)$$

for the resulting derived functor.

Proposition 3.33. *If $A \to B$ is a weak equivalence, then the Quillen adjunction $((-)\otimes_A B, \varphi)$ is a Quillen equivalence.*

Proof. This follows from [19, 4.3] and 3.31, which implies that if $N \in \mathrm{Mod}_A$ is cofibrant then the functor $(-) \otimes_A N$ preserves weak equivalences. □

3.34. **Model category structure on** dga_Λ**.**

3.35. The forgetful functor $\sharp : \mathrm{dga}_\Lambda \to C(\Lambda)$ has a left adjoint $\mathrm{Sym} : C(\Lambda) \to \mathrm{dga}_\Lambda$ given by

$$\mathrm{Sym}(N) = \oplus_{n\geq 0}\mathrm{Sym}^n(N),$$

with the natural algebra structure.

3.36. Let $i : N \to M$ be a cofibration in $C(\Lambda)$ with cokernel Q so we get a morphism

$$\mathrm{Sym}(N) \to \mathrm{Sym}(M)$$

in dga_Λ. Let $\mathrm{Sym}(M)^a \in \mathrm{Mod}_{\mathrm{Sym}(N)}$ denote the differential graded $\mathrm{Sym}(N)$-module underlying $\mathrm{Sym}(M)$. We have a filtration

$$\mathrm{Sym}(N) = F_0 \subset F_1 \subset F_2 \subset \cdots \subset \bigcup_{i=1}^{\infty} F_i = \mathrm{Sym}(M)^a,$$

where F_i is defined to be the image the map

$$\oplus_{j=0}^{i}\mathrm{Sym}(N) \otimes_\Lambda \mathrm{Sym}^j(M) \to \mathrm{Sym}(M)^a.$$

Lemma 3.37. *We have*

$$F_i/F_{i-1} \simeq \mathrm{Sym}(N)[-i] \otimes_\Lambda \mathrm{Sym}^i(Q).$$

Proof. This follows from 3.6. □

Lemma 3.38. *Let $i : N \to M$ be a cofibration (resp. trivial cofibration) in $C(\Lambda)$. Let $\mathrm{Sym}(M)^a \in \mathrm{Mod}_{\mathrm{Sym}(N)}$ denote the underlying $\mathrm{Sym}(N)$-module of $\mathrm{Sym}(M)$. Then the map $\mathrm{Sym}(N) \to \mathrm{Sym}(M)^a$ is a cofibration (resp. trivial cofibration) in $\mathrm{Mod}_{\mathrm{Sym}(N)}$.*

Proof. It suffices to show that the map $\mathrm{Sym}(N) \to F_i$ is a cofibration (resp. trivial cofibration) for all i. By 3.26 for this it suffices in turn to show that each quotient F_i/F_{i-1} is cofibrant (resp. trivially cofibrant). This follows from 3.37. □

Theorem 3.39. *There exists a model category structure dga_Λ in which a morphism $f : A \to B$ is a an equivalence (resp. fibration) if the morphism $A^\sharp \to B^\sharp$ in $C(\Lambda)$ is an equivalence (resp. fibration). Cofibrations in dga_Λ are characterized by the left lifting property with respect to trivial fibrations.*

Proof. The functor $\sharp : \mathrm{dga}_\Lambda \to C(\Lambda)$ has a left adjoint $\mathrm{Sym} : C(\Lambda) \to \mathrm{dga}_\Lambda$ given by

$$\mathrm{Sym}(N) = \oplus_{n\geq 0}\mathrm{Sym}^n(N),$$

with the natural algebra structure.

By [4, 3.3] it suffices to verify the following conditions:

(i) The functor Sym preserves λ-smallness for an infinite regular cardinal λ.

(ii) If d is a transfinite composition of pushouts of coproducts of morphisms $\mathrm{Sym}(c)$ for c a trivial cofibration in $C(\Lambda)$, then d is an equivalence.

Statement (i) follows from the fact that the right adjoint $\sharp$ commutes with colimits.

For (ii) note that a transfinite composition of equivalences is an equivalence. It therefore suffices to show that a pushout e of a coproduct of morphisms $\mathrm{Sym}(c)$, for c a trivial cofibration in $C(\Lambda)$, is an equivalence. Since the functor Sym commutes with coproducts in fact it suffices to show that the pushout of a single $\mathrm{Sym}(c)$, with c a trivial cofibration in $C(\Lambda)$, is an equivalence.

Let $c : N \to M$ be such a trivial cofibration. By 3.38 the map $\mathrm{Sym}(N) \to \mathrm{Sym}(M)^a$ is a trivial cofibration in $\mathrm{Mod}_{\mathrm{Sym}(N)}$. If $\mathrm{Sym}(N) \to A$ is a morphism in dga_Λ then since

$$(-) \otimes_{\mathrm{Sym}(N)} A : \mathrm{Mod}_{\mathrm{Sym}(N)} \to \mathrm{Mod}_A$$

is a left Quillen functor it follows that the map

$$A \to A \otimes_{\mathrm{Sym}(N)} \mathrm{Sym}(M)$$

is an equivalence as this can be verified in Mod_A. □

Remark 3.40. The model category dga_Λ is cofibrantly generated by [19, 2.3]. Let I (resp. J) be a set of generating cofibrations (resp. trivial cofibrations) in $C(\Lambda)$. Then the set I_{dga} (resp. J_{dga}) of cofibrations (resp. trivial cofibrations) in dga_Λ obtained by applying the Sym functor to the elements of I (resp. J) is a generating set of cofibrations (resp. trivial cofibrations) in dga_Λ.

3.41. For $A \in \mathrm{dga}_\Lambda$, let $\mathrm{dga}_{A\backslash}$ denote the category of morphisms $A \to B$ in dga_Λ. As in [12, 2.7], this has a model category structure in which a morphism $f : B \to C$ under A is a fibration (resp. weak equivalence, cofibration) if and only if the underlying morphism in dga_Λ is a fibration (resp. weak equivalence, cofibration).

It is also shown in loc. cit. that the model category $\mathrm{dga}_{A\backslash}$ is cofibrantly generated. If I (resp. J) is a generating set of cofibrations (resp. trivial cofibrations) in dga_Λ then the set of maps

$$\{A \otimes_\Lambda X \to A \otimes_\Lambda Y | (X \to Y) \in I\}$$

$$(\text{resp. } \{A \otimes_\Lambda X \to A \otimes_\Lambda Y | (X \to Y) \in J\})$$

is a generating set of cofibrations (resp. trivial cofibrations) in $\mathrm{dga}_{A\backslash}$.

3.42. If $g : A \to B$ is a morphism, then there is an induced functor

$$\mathrm{dga}_{B\backslash} \to \mathrm{dga}_{A\backslash} \tag{3.42.1}$$

which has a left adjoint given by tensor product (note that for this adjoint to coincide with module tensor product it is essential that we work with commutative differential graded algebras). The functor (3.42.1) preserves fibrations and trivial fibrations, and therefore we have a Quillen adjunction. For $C \in \mathrm{Ho}(\mathrm{dga}_{A\backslash})$, we write

$$C \otimes^{\mathbb{L}}_A B \in \mathrm{Ho}(\mathrm{dga}_{B\backslash})$$

for the output of the resulting derived functor applied to C, so there is a commutative square

$$\begin{array}{ccc} C & \longrightarrow & C\otimes^{\mathbb{L}}_A B \\ \uparrow & & \uparrow \\ A & \longrightarrow & B \end{array}$$

in $\mathrm{Ho}(\mathrm{dga}_\Lambda)$.

3.43. Fix $A \in \mathrm{dga}_\Lambda$. For $B \in \mathrm{dga}_{A\backslash}$ let $B^a \in \mathrm{Mod}_A$ denote the underlying A-module.

Proposition 3.44. *Let $B \to C$ be a cofibration in $\mathrm{dga}_{A\backslash}$ with $B^a \in \mathrm{Mod}_A$ cofibrant. Then C^a is also cofibrant in Mod_A. In particular, if B is a cofibrant object of $\mathrm{dga}_{A\backslash}$ then $B^a \in \mathrm{Mod}_A$ is cofibrant.*

Proof. Let I be a generating set of cofibrations in $C(\Lambda)$ and let $I_{A\backslash}$ be the resulting generating set of cofibrations in dga_Λ (see 3.40 and 3.41). Then every cofibration in $\mathrm{dga}_{A\backslash}$ is a retract of a colimit of a (possibly transfinite) sequence of pushouts of morphisms in $I_{A\backslash}$.

It therefore suffices to show that if $c : N \to M$ is an element of I and

$$\rho : A \otimes_\Lambda \mathrm{Sym}(N) \to B$$

is a morphism in $\mathrm{dga}_{A\backslash}$ with B^a cofibrant, then the underlying A-module of

$$B \otimes_{A\otimes_\Lambda \mathrm{Sym}(N)} (A \otimes_\Lambda \mathrm{Sym}(M))$$

is cofibrant in Mod_A. Note that this A-module can also be described as

$$B \otimes_{\mathrm{Sym}(N)} \mathrm{Sym}(M)^a.$$

By 3.37 this module can be written as a colimit

$$B = F_0 \to F_1 \to F_2 \to \cdots \cup_{i=0}^{\infty} F_i = B \otimes_{\mathrm{Sym}(N)} \mathrm{Sym}(M)^a,$$

with each successive quotient of the form $B[-i] \otimes_\Lambda \mathrm{Sym}^i(Q)$ with $Q \in C(\Lambda)$ cofibrant. By 3.26 and induction we conclude that each F_i is cofibrant in Mod_A and therefore the colimit is cofibrant as well. □

Corollary 3.45. *Let $B, C \in \mathrm{dga}_{A\backslash}$ be two objects, and let $B^\dagger, C^\dagger \in \mathrm{Mod}_A$ be the underlying A-modules. Then the natural map*

$$B^\dagger \otimes^{\mathbb{L}}_A C^\dagger \to (B \otimes^{\mathbb{L}}_A C)^\dagger$$

is an equivalence in Mod_A, where on the left the tensor product is formed in Mod_A and on the right in dga_Λ.

Proof. This follows immediately from 3.44. □

Proposition 3.46. *For a commutative diagram in* dga_Λ

$$\begin{array}{ccccc} & & B_0 & \xrightarrow{\ b\ } & B \\ & \nearrow^{\beta_0} & & \nearrow^{\beta} & \\ A_0 & \xrightarrow{\quad a \quad} & & A & \\ & \searrow_{\gamma_0} & & \searrow_{\gamma} & \\ & & C_0 & \xrightarrow{\ c\ } & C \end{array}$$

with a, b, and c equivalences, the induced map in $\mathrm{Ho}(\mathrm{dga}_{A_0\backslash})$

$$B_0 \otimes^{\mathbb{L}}_{A_0} C_0 \to B \otimes^{\mathbb{L}}_A C$$

is an isomorphism.

Proof. It suffices to consider the case of the punctual topos, and the corresponding statement for modules (using 3.45). In this case it follows from 3.33. $\square$

3.47. **Another adjunction.**

3.48. Let $A \in \mathrm{dga}_\Lambda$ be an object. For $M \in C(\Lambda)$ the complex $\mathscr{H}om_\Lambda(A, M)$ has a natural A-module structure induced by the A-algebra structure on A. The induced functor

$$\mathscr{H}om_\Lambda(A, -) : C(\Lambda) \to \mathrm{Mod}_A$$

is right adjoint to the forgetful functor

$$\mathrm{Mod}_A \to C(\Lambda).$$

Proposition 3.49. *If A is cofibrant in* dga_Λ *then* $\mathscr{H}om_\Lambda(A, -)$ *is a right Quillen functor so we obtain a Quillen adjunction* $(\varphi, \mathscr{H}om_\Lambda(A, -))$*, where* $\varphi : \mathrm{Mod}_A \to C(\Lambda)$ *is the forgetful functor.*

Proof. It suffices to verify that if $S' \to S$ is a fibration (resp. trivial fibration) in $C(\Lambda)$ then the induced morphism in $C(\Lambda)$

$$\mathscr{H}om_\Lambda(A, S') \to \mathscr{H}om_\Lambda(A, S)$$

is a fibration (resp. trivial fibration) in $C(\Lambda)$, since a morphism in Mod_A is a fibration (resp. trivial fibration) if and only if its image in $C(\Lambda)$ is a fibration (resp. trivial fibration). This follows from 3.44 which implies that A is cofibrant in $C(\Lambda)$. $\square$

3.50. **Functoriality.**

3.51. Let $f : (T', \Lambda') \to (T, \Lambda)$ be a morphism of ringed topoi satisfying the conditions in 2.3 with Λ and Λ' algebras over $\mathbb{Q}$.

Since the functor $f_* : C(T', \Lambda') \to C(T, \Lambda)$ takes fibrations (resp. trivial fibrations) to fibrations (resp. trivial fibrations) by 2.37, the induced functor

$$f_* : \mathrm{dga}_{T',\Lambda'} \to \mathrm{dga}_{T,\Lambda}$$

also takes fibrations (resp. trivial fibrations) to fibrations (resp. trivial fibrations). Here we write $\mathrm{dga}_{T,\Lambda}$ instead of dga_Λ to make clear the relevant topos. We therefore also get a Quillen adjunction

$$\mathrm{dga}_{T',\Lambda'} \underset{f^*}{\overset{f_*}{\rightleftarrows}} \mathrm{dga}_{T,\Lambda},$$

and derived functors

$$\mathrm{Ho}(\mathrm{dga}_{T',\Lambda'}) \underset{\mathrm{L}f^*}{\overset{Rf_*}{\rightleftarrows}} \mathrm{Ho}(\mathrm{dga}_{T,\Lambda}).$$

Combining this with the tensor product construction, if $A \in \mathrm{dga}_{T,\Lambda}$ and $B \in \mathrm{dga}_{T',\Lambda'}$, and $\epsilon : f^*A \to B$ is a morphism in $\mathrm{dga}_{T',\Lambda'}$, then we can define a functor

$$f^* : \mathrm{Mod}_{T,A} \to \mathrm{Mod}_{T',B}, \quad M \mapsto f^*M \otimes_{f^*A} B.$$

This functor has right adjoint

$$f_* : \mathrm{Mod}_{T',B} \to \mathrm{Mod}_{T,A}$$

where for $N \in \mathrm{Mod}_{T',B}$ the complex f_*N is viewed as a A-module via the map $A \to f_*B$ adjoint to ϵ. As above the functor f^* takes cofibrations to cofibrations so we have a Quillen adjunction, giving rise to derived functors

$$\mathrm{Ho}(\mathrm{Mod}_{T',B}) \underset{\mathrm{L}f^*}{\overset{Rf_*}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Mod}_{T,A}).$$

3.52. **The subcategory** $\mathrm{dga}_\Lambda^{\geq 0,*}$**.**

3.53. Let $\mathrm{dga}_\Lambda^{\geq 0,*} \subset \mathrm{dga}_\Lambda$ denote the full subcategory of objects $A \in \mathrm{dga}_\Lambda$ with $\mathscr{H}^i(A) = 0$ for $i < 0$ and the natural map $\Lambda \to \mathscr{H}^0(A)$ an isomorphism. Note that these conditions depend only on the image of A in $\mathrm{Ho}(\mathrm{dga}_\Lambda)$, and we write $\mathrm{Ho}(\mathrm{dga}_\Lambda^{\geq 0,*}) \subset \mathrm{Ho}(\mathrm{dga}_\Lambda)$ for the image of $\mathrm{dga}_\Lambda^{\geq 0,*}$.

Proposition 3.54. *Assume that for every point x of the topos T the ring $\Lambda_x := x^{-1}\Lambda$ is reduced. Then for a diagram in* $\mathrm{dga}_\Lambda^{\geq 0,*}$

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

the natural map $\Lambda \to \mathscr{H}^0(B \otimes^{\mathbb{L}}_A C)$ is injective.

Proof. It suffices to consider the case of the punctual topos. Furthermore, we may assume that $A \to B$ is a cofibration in which case the derived tensor product is represented by the usual tensor product $B \otimes_A C$.

For a ring homomorphism $\Lambda \to \Lambda'$ let $A_{\Lambda'}$ (resp. $B_{\Lambda'}$, $C_{\Lambda'}$) denote the base change of A (resp. B, C) to Λ'. Then $A_{\Lambda'} \to B_{\Lambda'}$ is still a cofibration and we have a commutative diagram

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda' \\ \downarrow & & \downarrow \\ H^0(B \otimes_A C) & \longrightarrow & H^0(B_{\Lambda'} \otimes_{A_{\Lambda'}} C_{\Lambda'}). \end{array}$$

Since Λ is reduced, to prove the proposition it suffices to show that it holds after base change $\Lambda \to \Lambda'$ with Λ' a field. This therefore reduces us further to the case when Λ is a field.

In this case fibrations are epimorphisms and the model category structure reduces to the one considered in [18].

In particular, there exists an equivalence $A_0 \to A$ with $A_0^i = 0$ for $i < 0$ and $A_0^i = \Lambda$ (see for example the proof of [18, 7.2]).

Then by 3.46 the map

$$B \otimes^{\mathbb{L}}_{A_0} C \to B \otimes^{\mathbb{L}}_A C$$

is an equivalence, so it suffices to prove the result in the case when $\Lambda = A^0$. Similarly we can arrange that $B^i = 0$ and $C^i = 0$ for $i < 0$.

By [18, 7.2.2] we may further assume that B is cofibrant in $\mathrm{dga}_{A\backslash}$ in which case $B \otimes_A C$ represents $B \otimes^{\mathbb{L}}_A C$. Since $B \otimes_A C$ is concentrated in degrees ≥ 0 this implies that $\mathscr{H}^i(B \otimes^{\mathbb{L}}_A C) = 0$ for $i < 0$. Furthermore, the degree 0 part of $B \otimes_A C$ is $B^0 \otimes_\Lambda C^0$ whence the map $\Lambda \to (B \otimes_A C)^0$ is injective. It follows that $\Lambda \to \mathscr{H}^0(B \otimes_A C)$ is also injective. $\square$

4. ℓ-derived schemes: Basic definitions and properties

Fix a prime ℓ, and let E denote either a finite extension of $\mathbb{Q}_\ell$ or a separable closure $\overline{\mathbb{Q}}_\ell$.

Throughout this section we consider only quasi-compact and quasi-separated schemes over $\mathbb{Z}[1/\ell]$. For such a scheme X we write also E for the sheaf on the pro-étale site of X defined by E; see [3, 4.2.12].

Definition 4.1. An *ℓ-derived scheme* is a pair (X, A_X), where X is a quasi-compact and quasi-separated $\mathbb{Z}[1/\ell]$-scheme and A_X is a sheaf of differential graded E-algebras on the pro-étale site of X. A morphism $(Y, A_Y) \to (X, A_X)$ of ℓ-derived schemes is a pair (f, f^b), where $f : Y \to X$ is a morphism of schemes and $f^b : f^{-1}A_X \to A_Y$ is a morphism of sheaves of differential graded E-algebras on Y.

Remark 4.2. We often write simply f for a morphism (f, f^b).

Remark 4.3. We often write f^b also for the map $A_X \to f_*A_Y$ corresponding to f^b by adjunction.

4.4. For an ℓ-derived scheme (X, A_X) we denote the category of sheaves of differential graded A_X-modules on the pro-étale site of X by $\mathrm{Mod}_{(X,A_X)}$. By 3.20 this is a model category and we write $D(X, A_X)$ for the resulting homotopy category. It is a triangulated category by 3.27.

If $(f, f^b) : (Y, A_Y) \to (X, A_X)$ is a morphism of ℓ-derived schemes we have by 3.50 induced derived functors

$$Rf_* : D(Y, A_Y) \to D(X, A_X), \quad Lf^* : D(X, A_X) \to D(Y, A_Y).$$

4.5. A commutative diagram of ℓ-derived schemes

$$\begin{array}{ccc} (F, A_F) & \xrightarrow{g'} & (Z, A_Z) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ (Y, A_Y) & \xrightarrow{g} & (X, A_X) \end{array}$$

is *homotopy cartesian* if the underlying diagram of schemes is cartesian, and the induced diagram of rings on the pro-étale site of F

$$\begin{array}{ccc} h^{-1}A_X & \longrightarrow & f'^{-1}A_Z \\ \downarrow & & \downarrow \\ g'^{-1}A_Y & \longrightarrow & A_F \end{array}$$

is homotopy cocartesian, where $h : F \to X$ denotes the composition $fg' = f'g$.

4.6. **Constructible modules.** Let (X, A_X) be an ℓ-derived scheme with X a topologically noetherian quasi-compact and quasi-separated $\mathbb{Z}[1/\ell]$-scheme.

Definition 4.7. An object $M \in D(X, A_X)$ is *constructible* if its image in $D(X, E)$ is constructible in the sense of [3, 6.8.8].

An object of $\mathrm{Mod}_{(X,A_X)}$ is called *constructible* if its image in $D(X, A_X)$ is constructible.

4.8. Since the forgetful functor

$$D(X, A_X) \to D(X, E)$$

is triangulated it follows from [3, 6.8.9] that the subcategory $D_c(X, A_X) \subset D(X, A_X)$ of constructible objects is a sub-triangulated category.

If $f : (Y, A_Y) \to (X, A_X)$ is a morphism of ℓ-derived schemes then the induced functor

$$Lf^* : D(X, A_X) \to D(Y, A_Y)$$

preserves constructibility by [3, 6.5.9 (5) and 6.8.15] and therefore restricts to a triangulated functor

$$Lf^* : D_c(X, A_X) \to D_c(Y, A_Y).$$

Similarly, if f is finitely presented and Y is quasi-excellent then by [3, 6.7.2 and 6.8.15] the functor

$$Rf_* : D(Y, A_Y) \to D(X, A_X)$$

preserves constructibility and induces a functor

$$Rf_* : D_c(Y, A_Y) \to D_c(X, A_X).$$

By the general machinery in 3.29 we also have internal tensor and Hom functors

$$(-) \otimes^{\mathbb{L}}_{A_X} (-) : D(X, A_X) \times D(X, A_X) \to D(X, A_X)$$

$$RHom(-, -) : D(X, A_X)^{\mathrm{op}} \times D(X, A_X) \to D(X, A_X).$$

By [3, 6.7.12] tensor product preserves constructibility and by [3, 6.7.13] so does $RHom$ if X is quasi-excellent. With these assumptions we therefore get functors

$$(-) \otimes^{\mathbb{L}}_{A_X} (-) : D_c(X, A_X) \times D_c(X, A_X) \to D_c(X, A_X)$$

$$RHom(-, -) : D_c(X, A_X)^{\mathrm{op}} \times D_c(X, A_X) \to D_c(X, A_X).$$

4.9. Let (X, A_X) be an ℓ-derived scheme with X excellent equipped with a dimension function δ. Let $\Omega_X \in D(X, E)$ be the image under the natural map

$$D(X, \mathcal{O}_E) \to D(X, E)$$

of the dualizing complex defined in [3, 6.7.20].

Let $\pi : A'_X \to A_X$ be a cofibrant replacement for A_X in the model category of differential graded Λ-algebras (so π is a trivial fibration and A'_X is a cofibrant Λ-algebra). By 3.49 we then have a derived functor

$$RHom_\Lambda(A'_X, -) : D(X, E) \to D(X, A'_X),$$

with left adjoint the forgetful functor. By 3.33 the extension of scalars functor

$$D(X, A'_X) \to D(X, A_X)$$

is an equivalence of triangulated categories so we obtain a triangulated functor

$$RHom_\Lambda(A_X, -) : D(X, E) \to D(X, A_X)$$

right adjoint to the forgetful functor. Since this functor is determined by its left adjoint it is independent of the choice of cofibrant replacement of A_X.

Definition 4.10. The *dualizing complex of* (X, A_X) is the object

$$\Omega_{(X,A_X)} := RHom_\Lambda(A_X, \Omega_X) \in D(X, A_X).$$

Remark 4.11. Note that for an object $M \in D(X, A_X)$ we have by 3.49

$$RHom_{A_X}(M, \Omega_{(X,A_X)}) \simeq RHom_\Lambda(M^\sharp, \Omega_X),$$

where $M^\sharp \in D(X, E)$ is the underlying complex of M.

Proposition 4.12. *Assume that the image of A_X in $D(X, E)$ is in $D_c(X, E)$. Then $\Omega_{(X,A_X)} \in D_c(X, A_X)$, the functor $RHom_A(-, \Omega_{(X,A_X)})$ induces a functor*

$$D_{(X,A_X)} : D_c(X, A_X)^{\rm op} \to D_c(X, A_X)$$

such that the natural map $\mathrm{id} \to D^2_{(X,A_X)}$ *is an isomorphism of functors.*

Proof. This follows immediately from [3, 6.7.20], its generalization to E-coefficients using [3, 6.8.14], and the observation that the following diagram commutes

$$\begin{array}{ccc} D(X, A_X)^{\rm op} & \longrightarrow & D(X, E) \\ \big\downarrow {\scriptstyle D_{(X,A_X)}} & & \big\downarrow {\scriptstyle D_X} \\ D(X, A_X) & \longrightarrow & D(X, E), \end{array}$$

where the horizontal arrows are the forgetful morphisms. $\square$

5. Actions of correspondences and naive local terms

Throughout this section we work over a base field k and ℓ denotes a prime invertible in k.

5.1. A *correspondence* of ℓ-derived k-schemes is a diagram of ℓ-derived k-schemes

$$\begin{array}{ccccc} & & (C, A_C) & & \\ & \overset{c_1}{\swarrow} & & \overset{c_2}{\searrow} & \\ (X, A_X) & & & & (X, A_X), \end{array}$$

with C and X quasi-excellent. We often denote such a correspondence simply by c.

Let $A_{X\times X}$ denote the sheaf of E-algebras on $X \times X$ given by $\mathrm{pr}_1^{-1}A_X \otimes_E \mathrm{pr}_2^{-1}A_X$. Then a correspondence can also be written as a morphism

$$c = (c_1, c_2) : (C, A_C) \to (X \times X, A_{X\times X}).$$

Let $M \in D(X, A_X)$ be an object. A *naive action* of c on M is a morphism

$$u : Lc_1^*M \to Lc_2^*M$$

in $D(C, A_C)$.

Definition 5.2. Let $M \in D_c(X, A_X)$ be an object, and let $M^\vee$ denote $RHom_{A_X}(M, A_X) \in D_c(X, A_X)$. We say that M is *c-quasi-perfect* if the natural map

$$(Lc_1^*M^\vee) \otimes^{\mathbb{L}}_{A_C} (Lc_2^*M) \to RHom_{A_C}(Lc_1^*M, Lc_2^*M) \tag{5.2.1}$$

is an isomorphism.

5.3. If u is a naive action of c on a c-quasi-perfect object M, we can define the naive local terms of u as follows. Let $\Delta_X : X \to X \times X$ be the diagonal morphism and fix a factorization

$$\Delta_X^{-1}A_{X\times X} \xrightarrow{a} \widetilde{A}_X \xrightarrow{b} A_X$$

in the category of sheaves of differential graded E-algebras on X, with a a cofibration and b a trivial fibration. Define (F, A_F) by setting

$$F := C \times_{c, X\times X, \Delta_X} X$$

with sheaf of algebras

$$A_F := (\widetilde{A}_X)|_F \otimes_{A_{X\times X}|_F} (A_C|_F),$$

so we have a homotopy cartesian diagram

$$\begin{array}{ccc} (F, A_F) & \xrightarrow{\delta'} & (C, A_C) \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ (X, \widetilde{A}_X) & \xrightarrow{\delta} & (X \times X, A_{X\times X}). \end{array}$$

Let $h : (F, A_F) \to (X \times X, A_{X\times X})$ denote $c \circ \delta' = \delta \circ c'$.

Let $M^\vee \boxtimes M \in D(X \times X, A_{X\times X})$ denote $L\mathrm{pr}_1^*M^\vee \otimes_{A_{X\times X}} L\mathrm{pr}_2^*M$. The isomorphism 5.2.1 enables us to view u as a morphism

$$A_C \xrightarrow{u} Lc^*(M^\vee \boxtimes M). \tag{5.3.1}$$

There is a natural evaluation map in $D(X, \widetilde{A}_X)$

$$\text{ev} : L\delta^*(M^\vee \boxtimes M) \to \widetilde{A}_X. \tag{5.3.2}$$

Indeed giving such a map is equivalent to giving a map

$$L\Delta_X^*(M^\vee \boxtimes M) = M^\vee \otimes^{\mathbb{L}}_{A_X} M \to A_X,$$

and we take the usual evaluation map. Pulling (5.3.1) and (5.3.2) back to (F, A_F) we get a diagram

$$A_F \xrightarrow{\delta'^*u} Lh^*(M^\vee \boxtimes M) \xrightarrow{c'^*\text{ev}} A_F. \tag{5.3.3}$$

Definition 5.4. The *naive local term of* u, denoted $t_u \in H^0(F, A_F)$, is the global section of A_F defined by the composite 5.3.3.

Remark 5.5. Note that the underlying $h^{-1}A_{X\times X}$-module of A_F is the derived tensor product $c'^{-1}A_X \otimes^{\mathbb{L}}_{h^{-1}A_{X\times X}} \delta'^{-1}A_C$. Therefore the naive local term is independent of the cofibrant replacement $\widetilde{A}_X$.

Remark 5.6. More generally we can define naive local terms of elements $u \in H^0(C, Lc^*(M^\vee \boxtimes M))$, without assuming that M is c-quasi-perfect.

5.7. **Addendum on A_F.** The special properties of the pro-étale site and the fact that we work with E-coefficients enable us to say a bit more than 3.54 about the derived tensor product occurring in the definition of A_F.

Remark 5.8. It is tempting to try to prove the results of this subsection, and in particular 5.14, by passing to points. This approach runs into difficulties, however, as the stalks of the sheaf E are not equal to the field E. The pro-étale topos has points not coming from geometric points of the underlying scheme (see [1, Tag 0991]).

5.9. Let X be a quasi-compact and quasi-separated quasi-excellent scheme. Following [3, 6.8.3] let Loc_X denote the category of locally free sheaves of E-modules of finite rank. Let $\widetilde{\text{Loc}}_X$ denote the category of sheaves of E-modules on the pro-étale site of X which can be written as a filtered colimit of objects of Loc_X.

By [3, 6.8.5] the category Loc_X is an abelian subcategory of the category of sheaves of E-modules.

Lemma 5.10. *(i) Every object* $M \in \widetilde{\text{Loc}}_X$ *is isomorphic to a colimit* $\text{colim}_{i\in I} V_i$ *of objects* $V_i \in \text{Loc}_X$*, with* I *a filtering category and each morphism* $V_i \to V_j$ *injective.*

(ii) Let $M = \text{colim}_{i\in I} V_i$ *be an object of* $\widetilde{\text{Loc}}_X$ *and assume that all transition maps* $V_i \to V_j$ *are injective. Then* $\Gamma(X, M) = \text{colim}_{i\in I}\Gamma(X, V_i)$.

(iii) Let $M = \text{colim}_{i\in I} V_i$ *and* $N = \text{colim}_{j\in J} W_j$ *be objects of* $\widetilde{\text{Loc}}_X$ *and assume all transition maps* $W_j \to W_{j'}$ *are injective. Then*

$$\text{Hom}_{\widetilde{\text{Loc}}_X}(M, N) = \text{lim}_{i\in I}\text{colim}_{j\in J}\text{Hom}_{\text{Loc}_X}(V_i, W_j).$$

(iv) Let $f : M \to N$ *be a morphism in* $\widetilde{\text{Loc}}_X$*. Then the kernel of* f*, the image of* f*, and the cokernel of* f *is also in* $\widetilde{\text{Loc}}_X$*.*

(v) If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of sheaves of E-modules and $M_1, M_3 \in \widetilde{\mathrm{Loc}}_X$ then $M_2 \in \widetilde{\mathrm{Loc}}_X$.

Proof. For (i) define for every $\varphi : i \to j$ the subsheaf $K_\varphi \subset V_i$ to be the kernel of the map $V_i \to V_j$. Then K_φ is a lisse subsheaf of V_i, being the kernel of a map of lisse sheaves, and since the category I is filtering the collection of subobjects $\{K_\varphi \subset V_i\}$ is connected in the sense that for any two $K_\varphi, K_{\varphi'} \subset V_i$ there exists a third $K_{\varphi''}$ containing both. Since V_i has finite rank it follows that there exists a lisse subsheaf $K_i \subset V_i$ such that

$$K_i = \cup_{\varphi: i \to j} K_\varphi.$$

Let W_i denote V_i/K_i. For $\varphi : i \to j$ in I we then get induced maps $W_i \to W_j$ and $\mathrm{colim}_I V_i \simeq \mathrm{colim}_I W_i$.

Statement (ii) is shown in [1, Tag 09A0].

For (iii) note that by the universal property of the colimit we have

$$\mathrm{Hom}_{\widetilde{\mathrm{Loc}}_X}(M, N) = \lim_{i \in I} \mathrm{Hom}_{\widetilde{\mathrm{Loc}}_X}(V_i, N).$$

Since V_i is locally free of finite rank we have

$$\mathrm{Hom}_{\widetilde{\mathrm{Loc}}_X}(V_i, N) \simeq \Gamma(X, N \otimes_E V_i^\vee),$$

and by (ii) we have

$$\Gamma(X, N \otimes_E V_i^\vee) \simeq \mathrm{colim}_{j \in J} \Gamma(X, W_j \otimes V_i^\vee) \simeq \mathrm{colim}_{j \in J} \mathrm{Hom}_{\mathrm{Loc}_X}(V_i, W_j).$$

For (iv) it suffices to prove that $\widetilde{\mathrm{Loc}}_X$ is closed under the formation of cokernels and that if $f : M \to N$ is a surjective morphism in $\widetilde{\mathrm{Loc}}_X$ then the kernel of f is in $\widetilde{\mathrm{Loc}}_X$ (indeed the image of f is the kernel of $N \to \mathrm{Coker}(f)$ and $\mathrm{Ker}(f)$ is also the kernel if the map from M to the image).

Write $M = \mathrm{colim}_{j \in J} W_j$ and $N = \mathrm{colim}_{i \in I} V_i$, with all transitions maps injective (using (i)). Let $V_i \to \overline{V}_i$ be the quotient of V_i by the subsheaf generated by images of W_j with $f(W_j) \subset V_i$. Then $\overline{V}_i$ is a lisse sheaf, for $\varphi : i \to j$ in I we have induced maps $\overline{V}_i \to \overline{V}_j$ and $\mathrm{Coker}(f) = \mathrm{colim}_{i \in I} \overline{V}_i$ (using (iii)).

For the statement about kernels, assume furthermore that f is surjective. For every $j \in J$ let K_j denote the kernel of $W_j \to N$. Then K_j is a lisse subsheaf of W_j. Indeed by (iii) there exists an index i such that $W_j \to N$ factors through the monomorphism $V_i \hookrightarrow N$ and therefore K_j can be realized as the kernel of a morphism of lisse sheaves. Since f is an epimorphism we have $N = \mathrm{colim}_{j \in J}(W_j/K_j)$ and since filtered colimits are exact it follows that $\mathrm{Ker}(f)$ is given by $\mathrm{colim}_{j \in J} K_j$.

Finally to prove (v), write $M_3 = \mathrm{colim}_{i \in I} V_i$. It then suffices to show that for each of the pulled back sequences

$$0 \to M_1 \to E_2 \to V_i \to 0,$$

where $E_2 = M_2 \times_{M_3} V_i$ the sheaf E_2 is in $\widetilde{\mathrm{Loc}}_X$. This reduces the proof to the case when $M_3 \in \mathrm{Loc}_X$ in which case the class of the extension is given by an element of $H^1(X, M_1 \otimes_E M_3^\vee)$.

Writing $M_1 = \mathrm{colim}_{j\in J} W_j$ the natural map

$$\mathrm{colim}_{j\in J} H^1(X, W_j \otimes_E M_3^\vee) \to H^1(X, M_1 \otimes_E M_3^\vee)$$

is an isomorphism, by [1, Tag 09A0]. This implies (v). □

Corollary 5.11. *Let*

$$M_1 \to M_2 \to M_3 \to M_1[1]$$

be a distinguished triangle in $D(X,E)$ and assume that for all i the sheaves $\mathscr{H}^i(M_1)$ and $\mathscr{H}^i(M_2)$ are in $\widetilde{\mathrm{Loc}}_X$. Then all cohomology sheaves of M_3 are also in $\widetilde{\mathrm{Loc}}_X$.

Proof. This follows immediately from 5.10. □

5.12. Let A be a sheaf of differential graded E-algebras on X and assume that the following conditions hold:

(i) $\mathscr{H}^i(A) = 0$ for $i < 0$.
(ii) The natural map $E \to \mathscr{H}^0(A)$ is an isomorphism.
(iii) We have $\mathscr{H}^i(A) \in \widetilde{\mathrm{Loc}}_X$ for all i.

5.13. Let $\mathcal{D}^{\geq 0}$ denote the subcategory of $D(X,A)$ consisting of modules M for which we have

(i) $\mathscr{H}^i(M) = 0$ for $i < 0$.
(ii) The sheaf $\mathscr{H}^0(M)$ is a lisse sheaf of E-modules of rank 1.
(iii) We have $\mathscr{H}^i(M) \in \widetilde{\mathrm{Loc}}_X$ for all i.

We say that an object $P \in D(X,A)$ is $\mathcal{D}^{\geq 0}$-*acyclic* of for all $C \in \mathcal{D}^{\geq 0}$ and $j < 0$ we have

$$\mathscr{H}^j(P \otimes_A^{\mathbb{L}} C) = 0,$$

and the natural map

$$\mathscr{H}^0(P) \otimes_E \mathscr{H}^0(C) \to \mathscr{H}^0(P \otimes_A^{\mathbb{L}} C)$$

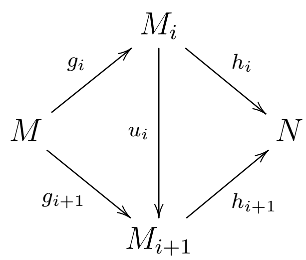
is injective.

Proposition 5.14. *Let $f : M \to N$ be a morphism in $\mathcal{D}^{\geq 0}$ which is an isomorphism on $\mathscr{H}^0$ and surjective on $\mathscr{H}^i$ for $i > 0$. If M is $\mathcal{D}^{\geq 0}$-acyclic then N is also $\mathcal{D}^{\geq 0}$-acylic.*

Proof. We inductively construct a factorization of f in $D(X,A)$ for $i \geq 0$

$$M \xrightarrow{g_i} M_i \xrightarrow{h_i} N \qquad (5.14.1)$$

such that M_i is $\mathcal{D}^{\geq 0}$-acyclic and the map $M_i \to N$ induces an isomorphism $\mathscr{H}^s(M_i) \to \mathscr{H}^s(N)$ for $s \leq i$. Furthermore, we will construct maps $u_i : M_i \to M_{i+1}$ such that the diagrams



commute. We then get an isomorphism $N \simeq \text{hocolim } M_i$, which implies the result as the homotopy colimit of a sequence of $\mathcal{D}^{\geq 0}$-acyclic objects is again $\mathcal{D}^{\geq 0}$-acyclic .

To construct the M_i we proceed by induction on i. For the base case $i = 0$ we take $M_0 = M$. So we assume the factorization 5.14.1 has been construction for an integer i and construct the corresponding factorization for $i + 1$.

For a complex $T \in D(X, A)$ let $T^\sharp \in D(X, E)$ denote the underlying complex. Let W_0 denote the cone of the morphism $h_i : M_i \to N$. We then have $\mathscr{H}^s(W_0) = 0$ for $s < i$ and a short exact sequence

$$0 \to \mathscr{H}^i(W_0) \to \mathscr{H}^{i+1}(M_i) \to \mathscr{H}^{i+1}(N).$$

Let L_0 denote $\mathscr{H}^i(W_0)$. Since $W_0 \in D^{\geq i}(X, E)$ we get a morphism $L_0[-i] \to W_0^\sharp$ and therefore also a morphism

$$\alpha_0 : L_0[-i-1] \to M_i^\sharp.$$

Let F_{i0} denote the cone of the induced morphism

$$A \otimes_E L_0[-i-1] \to M_i.$$

We then have a factorization

$$M_i \to F_{i0} \to N$$

of h_i such that L_0 maps to 0 under the map $\mathscr{H}^{i+1}(M_i) \to \mathscr{H}^{i+1}(F_{i0})$.

Lemma 5.15. *The object F_{i0} is $\mathcal{D}^{\geq 0}$-acyclic and the map $M_i \to F_{i0}$ induces an isomorphism on cohomology sheaves $\mathscr{H}^s$ for $s \leq i$.*

Proof. We have a distinguished triangle

$$A \otimes_E L_0[-i-1] \to M_i \to F_{i0} \to A \otimes_E L_0[-i],$$

and therefore for every $C \in \mathcal{D}^{\geq 0}$ we get by tensoring this triangle with C and taking cohomology sheaves a long exact sequence

$$\cdots \to \mathscr{H}^{s-i-1}(C) \otimes_E L_0 \to \mathscr{H}^s(C \otimes^{\mathbb{L}}_A M_i) \to \mathscr{H}^s(C \otimes^{\mathbb{L}}_A F_{i0}) \to \mathscr{H}^{s-i}(C) \otimes_E L_0 \to \cdots .$$

Since i is at least 0 it follows that the map

$$\mathscr{H}^s(C \otimes^{\mathbb{L}}_A M_i) \to \mathscr{H}^s(C \otimes^{\mathbb{L}}_A F_{i0})$$

is injective (resp. an isomorphism) for $s = 0$ (resp. $s < 0$).

Taking $C = A$ we also get that the map $\mathscr{H}^s(M_i) \to \mathscr{H}^s(F_{i0})$ is an isomorphism for $s \leq i$ (note that the map $L_0 \to \mathscr{H}^{i+1}(M_i)$ is injective). From this the lemma follows. □

Repeating the construction inductively we then obtain a sequence of morphisms

$$M_i \to F_{i0} \to F_{i1} \to \cdots \to N,$$

such that for all j the map $F_{ij} \to F_{i(j+1)}$ induces an isomorphism on $\mathscr{H}^s$ for $s \leq i$ and such that the map $\mathscr{H}^{i+1}(F_{ij}) \to \mathscr{H}^{i+1}(F_{i(j+1)}$ sends the kernel of $\mathscr{H}^{i+1}(F_{ij}) \to \mathscr{H}^{i+1}(N)$ to 0, and such that F_{ij} is $\mathcal{D}^{\geq 0}$-acyclic. Now define M_{i+1} to be $\text{hocolim }_j F_{ij}$. This completes the proof of 5.14. □

5.16. Let X be a scheme, let ℓ be a prime invertible on X, and let E denote either a finite extension of $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. Let dga denote the category of sheaves of differential graded E-algebras in the pro-étale topos of X, and define $\text{dga}^{\geq 0,*}$ as in 3.53. Fix a diagram in $\text{dga}^{\geq 0,*}$

$$\begin{array}{ccc} A & \xrightarrow{b} & B \\ \downarrow{\scriptstyle c} & & \\ C, & & \end{array}$$

with the maps $\mathscr{H}^i(A) \to \mathscr{H}^i(B)$ surjective for all i.

Proposition 5.17. *Assume that X is quasi-compact and separated, quasi-excellent, and topologically noetherian. Assume further that the underlying complexes $A^a, B^a, C^a \in D(X, E)$ are in $D_c(X, E)$. Then $\mathscr{H}^i(C \otimes^{\mathbb{L}}_A B) = 0$ for $i < 0$.*

Proof. Since X is topologically noetherian there exists a finite stratification of X such that the restrictions of the cohomology sheaves of A, B, and C are lisse on each stratum. Restricting to such a stratum we may further assume that the cohomology sheaves are lisse. Using 3.45, the result now follows from 5.14 applied to the map of A-modules $A \to B$ and noting that A is trivially $\mathcal{D}^{\geq 0}$-acyclic. □

6. Sheaves with unipotent local monodromy

Throughout this section we work over a field k and ℓ denotes a prime invertible in k. Let E denote either a finite extension of $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$.

6.1. Consider a correspondence over k

$$c = (c_1, c_2) : C \to X \times X,$$

with C and X of finite type over k. Let $D \subset X$ be a closed subscheme such that $c_1^{-1}(D) = c_2^{-1}(D)$ (set-theoretically), and let $E \subset C$ denote this common preimage. Let $j : X^\circ \hookrightarrow X$ (resp. $\tilde{j} : C^\circ \hookrightarrow C$) be the complement of D (resp. E). Let A_X (resp. A_C) denote the differential graded on X (resp. C) given by $Rj_*E_{X^\circ}$ (resp. $R\tilde{j}_*E_{C^\circ}$). We then have a correspondence of ℓ-derived schemes

$$c : (C, A_C) \to (X, A_X) \times (X, A_X).$$

Lemma 6.2. *(i) Let X be a finite type k-scheme and let $K \in D_c(X, E)$ be a constructible complex. Suppose that for every geometric point $\bar{x} \to X$ mapping to a closed point of X the pullback of K to the pro-étale site of $\text{Spec}(\mathscr{O}_{X,\bar{x}})$ is acyclic. Then K is acyclic.*

(ii) Let $f : L \to M$ be a morphism in $D_c(X, E)$ such that for any geometric point $\bar{x} \to X$ lying over a closed point the pullback of f to the pro-étale site of $\text{Spec}(\mathscr{O}_{X,\bar{x}})$ is an isomorphism. Then f is an isomorphism.

Proof. For (i), note that by definition (see [3, 6.8.8]) there exists a finite stratification of X such that the restriction of K to each stratum has lisse cohomology sheaves. Restricting to each stratum it therefore suffices to consider the case when K has lisse cohomology sheaves, when the result is immediate.

Statement (ii) follows from (i) applied to the cone of f. □

Definition 6.3. A lisse sheaf V of E-modules on X° has *unipotent local monodromy* if for every geometric point $\bar{x} \to X$ lying over a closed point the pullback of V to the pro-étale site of

$$\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})^\circ := X^\circ \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$$

is unipotent (that is, there exists a finite filtration on the pullback of V whose successive quotients are trivial).

Theorem 6.4. *Let V be a lisse sheaf of E-modules on X° with unipotent local monodromy and let $M \in D(X, A_X)$ denote Rj_*V.*

(i) M is c-quasi-perfect.

(ii) Any morphism $u^\circ : c_1^*V \to c_2^*V$ over C° extends uniquely to a morphism $u : Lc_1^*M \to Lc_2^*M$ in $D_c(C, A_C)$.

The proof occupies the remainder of this section. In the following we work with the notation and assumptions of the theorem.

Lemma 6.5. *Let c_i^*V $(i = 1, 2)$ denote the lisse sheaf on C° obtained from V by pullback along c_i. Then the natural map in $D(C, A_C)$*

$$Lc_i^*M \to R\tilde{j}_*c_i^*V$$

is an isomorphism.

Proof. By 6.2 (ii) it suffices to show that for any geometric point $\bar{y} \to C$ lying over a closed point of C the morphism pulls back to an isomorphism over $\mathrm{Spec}(\mathscr{O}_{C,\bar{y}})$. Let $\bar{x} \to X$ denote the geometric point of X obtained from $\bar{y}$ by composing with c_i. We then have a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathscr{O}_{C,\bar{y}}) & \xrightarrow{\tilde{c}_i} & \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}) \\ \downarrow & & \downarrow \\ C & \xrightarrow{c_i} & X. \end{array}$$

It therefore suffices to prove the analogous statement for the morphism

$$\mathrm{Spec}(\mathscr{O}_{C,\bar{y}}) \to \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$$

and the restriction of M to $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$. In this case the result is immediate as this pullback admits a finite filtration whose successive quotients are trivial by assumption. □

From this lemma and adjunction we get 6.4 (ii).

We use a similar reduction to the unipotent case to prove 6.4 (i).

6.6. By adjunction the natural map

$$M^\vee = R\mathcal{H}om_{A_X}(M, A_X) \to Rj_*V^\vee$$

is an isomorphism, where $V^\vee$ denotes the dual of V, and by 6.5 applied to $V^\vee$ we have

$$Lc_1^*M^\vee \simeq R\tilde{j}_*c_1^*V^\vee.$$

Also by adjunction and using the isomorphism $Lc_2^*M \simeq R\tilde{j}_*V$ we have

$$R\mathcal{H}om_{A_C}(Lc_1^*M, Lc_2^*M) \simeq R\tilde{j}_*(c_1^*V^\vee \otimes_E c_2^*V).$$

Thus the c-quasi-perfection of M is equivalent to the following statement:

Lemma 6.7. *The natural map*

$$(R\tilde{j}_*c_1^*V^\vee)\otimes^{\mathbb{L}}_{A_C}(R\tilde{j}_*c_2^*V)\to R\tilde{j}_*(c_1^*V^\vee\otimes_E c_2^*V)$$

is an isomorphism.

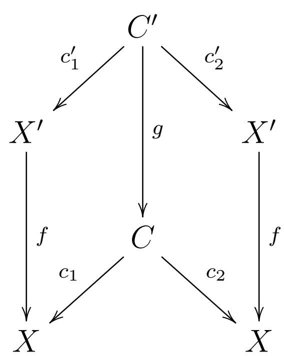
Proof. By 6.2 (ii) it suffices to verify that the map is an isomorphism after pulling back to the pro-étale site of $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ for every geometric closed point $\bar{x}\to X$. The pullbacks of V and $V^\vee$ to such $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ are unipotent by assumption, and in this case the result follows from filtering V or $V^\vee$ reducing to the trivial case. □

This completes the proof of 6.4. □

7. Properties of naive local terms

7.1. **Functoriality.**

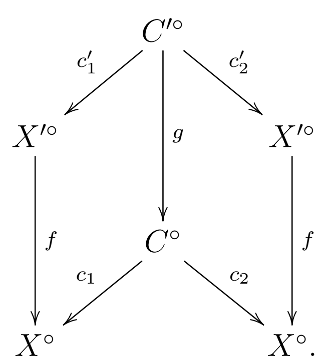
7.2. We continue with the notation of 6.1. Let



be a commutative diagram with

$$c':C'\to X'\times X'$$

a second correspondence with C' and X' quasi-excellent. Let $D'\subset X'$ be a closed subscheme with $c_1'^{-1}(D')=c_2'^{-1}(D')$ (set-theoretically) and let $E'\subset C'$ denote this common preimage. Let $j':X'^\circ\hookrightarrow X'$ (resp. $\tilde{j}':C'^\circ\hookrightarrow C'$) denote the complement of D' (resp. E') and assume that $f^{-1}(D)\subset D'$ so we have a commutative diagram



Let $A_{X'}$ (resp. $A_{C'}$) denote the sheaf of differential graded algebras $Rj'_*E_{X'^\circ}$ (resp. $R\tilde{j}'_*E_{C'^\circ}$) so we have a commutative diagram of ℓ-derived schemes

$$\begin{array}{ccccc} & & (C', A_{C'}) & & \\ & \swarrow^{c'_1} & & \searrow^{c'_2} & \\ (X', A_{X'}) & & \downarrow g & & (X', A_{X'}) \\ \downarrow f & & (C, A_C) & & \downarrow f \\ & \swarrow^{c_1} & & \searrow^{c_2} & \\ (X, A_X) & & & & (X, A_X). \end{array}$$

7.3. Let $M \in D_c(X, A_X)$ be a c-quasi-perfect object and let $u : Lc_1^*M \to Lc_2^*M$ be a naive action so we have a naive local term (see remark 5.5)

$$t_u \in H^0(F, A_F),$$

where F denotes $C \times_{c, X\times X, \Delta_X} X$.

The pullback $M' := f^*M \in D_c(X', A_{X'})$ may not be c'-quasi-perfect, but still the action u pulls back to a section

$$u' \in H^0(C', Lc'^*(M'^\vee \boxtimes M')),$$

and therefore, as noted in 5.6, we can also define a naive local term

$$t_{u'} \in H^0(F', A_{F'}),$$

where $F' := C' \times_{c', X'\times X', \Delta_{X'}} X'$.

Proposition 7.4. *The image of t_u under the natural map*

$$H^0(F, A_F) \to H^0(F', A_{F'})$$

is equal to $t_{u'}$.

Proof. This is immediate from the construction. □

7.5. **Calculations using universal covers.**

7.6. In this subsection we work with the notation and assumptions of 6.1. Assume in addition that X and C are normal.

Let V be a lisse sheaf of E-modules on X° with unipotent local monodromy and let $M \in D_c(X, A_X)$ denote Rj_*V. By [3, 6.8.13] we can write $V = V_{\mathcal{O}} \otimes_{\mathcal{O}_E} E$ for a lisse sheaf of $\mathcal{O}_E$-modules $V_{\mathcal{O}}$. Let $Y_n^\circ \to X^\circ$ denote the finite étale cover defined by the quotient $V_{\mathcal{O}}/\ell^n V_{\mathcal{O}}$, and let $Y_n \to X$ be the normalization of X in Y_n°. Let $Y^\circ \to X^\circ$ (resp. $Y \to X$) denote $\varprojlim Y_n^\circ$ (resp. $\varprojlim Y_n$), the limit taken with respect to the natural transition maps. Then $Y^\circ \to X^\circ$ is a pro-étale covering.

If $\bar{x} \to X^\circ$ is a geometric point and $I \subset GL(V_{\bar{x}})$ is the image of $\pi_1(X^\circ, \bar{x})$, then Y° is a torsor under the sheaf of groups on $X_{\text{pro-et}}$ defined by the pro-finite group I as in [3, 7.4.3].

The action of I on Y° extends to an action on Y which is transitive in fibers.

7.7. Let T denote the fiber product of the diagram

$$\begin{array}{ccc} & & C \\ & & \downarrow \\ Y \times Y & \longrightarrow & X \times X, \end{array}$$

so the restriction $T^\circ \to C^\circ$ of T is a torsor under $I \times I$ and this action extends to an action of $I \times I$ on T over C which is transitive on fibers.

Note that $\tilde{j}' : T^\circ \hookrightarrow T$ is a dense open immersion, since the closure is $I \times I$-invariant and contains a point in every fiber.

Let $j' : Y^\circ \hookrightarrow Y$ be the inclusion and let A_Y (resp. A_T) denote $Rj'_* E_{Y^\circ}$ (resp. $R\tilde{j}'_* E_{T^\circ}$). We then have a commutative diagram of ℓ-derived schemes

$$\begin{array}{ccccc} & & (T, A_T) & & \\ & \swarrow^{c'_1} & & \searrow^{c'_2} & \\ (Y, A_Y) & & \downarrow g & & (Y, A_Y) \\ \downarrow f & & (C, A_C) & & \downarrow f \\ & \swarrow_{c_1} & & \searrow_{c_2} & \\ (X, A_X) & & & & (X, A_X). \end{array}$$

7.8. Let W denote the E-vector space $H^0(Y^\circ, f^*V)$. By construction the pullback f^*V on Y° is trivial, so we have

$$Rj'_*(f^*V) \simeq W \otimes_E A_Y.$$

In particular, there is an induced map in $D(Y, A_Y)$

$$Lf^*M \to W \otimes_E A_Y. \tag{7.8.1}$$

Lemma 7.9. *The map 7.8.1 is an isomorphism.*

Proof. By a similar argument to the one proving 6.2 it suffices to show that the map becomes an isomorphism after base change $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ with $\bar{x} \to X$ a geometric closed point. Since the pullback of V to such $\mathrm{Spec}(\mathscr{O}_{X,\bar{x}})^\circ$ is unipotent this gives the result. □

7.10. Let

$$v : Lc_1'^*(W \otimes_E A_Y) \to Lc_2'^*(W \otimes_E A_Y)$$

denote the pullback of u. This map v is determined by its restriction to T°. This restriction is simply a map of constant sheaves

$$W \to W$$

on the pro-étale site of T°, and such a map is given by a continuous map

$$\rho : \pi_0(T^\circ) \to GL(W).$$

Since T° is a torsor over C° the action of $I \times I$ on T° induces a transitive action of $I \times I$ on $\pi_0(T^\circ)$. Let $\epsilon : I \to GL(W)$ be the representation corresponding to V. Then ρ is compatible with these actions in the sense that for $p \in \pi_0(T^\circ)$ and $(g_1, g_2) \in I \times I$ we have

$$\rho((g_1, g_2) * p) = \epsilon(g_2)^{-1}\rho(p)\epsilon(g_1)^{-1}.$$

Note also that since T is integral the map

$$\pi_0(T^\circ) \to \pi_0(T)$$

is an isomorphism, so we can also view ρ as a map $\pi_0(T) \to GL(W)$.

7.11. Let F' denote $T \times_{Y\times Y, \Delta_Y} Y$. By construction the map v is given by the map of constant sheaves $W \to W$ defined by ρ tensored with the identity map on A_T. It follows that the naive local term $t_v \in H^0(F', A_{F'})$ is equal to the image of the composite map

$$\pi_0(F') \longrightarrow \pi_0(T) \xrightarrow{\rho} GL(W) \xrightarrow{\text{trace}} E,$$

viewed as a section $\mathrm{tr}(\rho) \in H^0(F', E_{F'})$, under the natural map

$$H^0(F', E_{F'}) \to H^0(F', A_{F'}). \tag{7.11.1}$$

Remark 7.12. Note that the map 7.11.1 is injective by 3.54.

7.13. Let $\bar{z} \to F := C \times_{c, X\times X, \Delta_X} X$ be a geometric point, and let $\bar{x} \to X$ be its projection to X. Let $\bar{z}' \to T$ be a lifting of $\bar{z}$. Then $c_1'(\bar{z}')$ may not equal $c_2'(\bar{z}')$ but since the action of I is transitive on fibers of $Y \to X$ there exists an element $\gamma \in I$ so that replacing $\bar{z}'$ by $(1, \gamma) * \bar{z}'$ we obtain a lifting $\bar{z}'$ with $c_1'(\bar{z}') = c_2'(\bar{z}')$. In other words, we can lift $\bar{z}$ to a point $\bar{z}' \to F'$.

Let $[\bar{z}'] \in \pi_0(F')$ denote the component containing $\bar{z}'$, and let $t_{\bar{z}} \in A_{F,\bar{z}}$ be the image of the naive local term t_u.

Lemma 7.14. *The image of $t_{\bar{z}}$ in $A_{F',\bar{z}'}$ is equal to the image of $\mathrm{tr}(\rho)([\bar{z}'])$ under the map 7.11.1.*

Proof. This follows from 7.4. □

Proposition 7.15. *Assume that the local term $t_u \in H^0(F, A_F)$ lies in $H^0(F, E_F)$. Let $\bar{z}_i \to F$ ($i = 1, 2$) be two fixed points, and let $\bar{z}_i' \to F'$ ($i = 1, 2$) be liftings. Let $\gamma : W \to W$ denote the image under ρ of the image of $[\bar{z}_1] \in \pi_0(F')$ in $\pi_0(T)$, and let $(g_1, g_2) \in I \times I$ be an element such that the image of $[\bar{z}_2] \in \pi_0(F')$ in $\pi_0(T)$ is equal to (g_1, g_2) applied to the image of $[\bar{z}_1]$. Then $t_{u,\bar{z}_1} = \mathrm{tr}(\gamma|W)$ and $t_{u,\bar{z}_2} = \mathrm{tr}(g_2^{-1}\gamma g_1|W)$.*

Proof. This follows from 7.14 and the discussion in 7.10. □

8. Comparison with true local terms

For a discussion of true local terms and their basic properties we refer to [22, §1.2].

8.1. Continuing with the notation and assumptions of 6.1, assume in addition that X° is smooth of dimension d and that C is normal of the same dimension d. Let F denote $C \times_{c, X\times X, \Delta_X} X$ so we have a cartesian diagram

$$\begin{array}{ccc} F & \xrightarrow{\delta} & C \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ X & \xrightarrow{\Delta_X} & X \times X. \end{array}$$

Let $h : F \to X \times X$ denote $\Delta_X \circ c' = c \circ \delta$. Fix also a factorization

$$\Delta^{-1}(A_{X\times X}) \xrightarrow{a} \widetilde{A}_X \xrightarrow{b} A_X$$

of the natural map

$$\Delta^{-1}(A_{X\times X}) \to A_X,$$

with a a cofibration and b a trivial fibration. Let A_F denote $c'^{-1}\widetilde{A}_X \otimes_{h^{-1}A_{X\times X}} \delta^{-1}A_C$ so we have a homotopy cartesian diagram of ℓ-derived schemes

$$\begin{array}{ccc} (F, A_F) & \xrightarrow{\delta} & (C, A_C) \\ \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c} \\ (X, A_X) & \xrightarrow{\Delta_X} & (X \times X, A_{X\times X}). \end{array}$$

8.2. There is a canonical morphism in $D(C, A_C)$

$$v : Lc_1^* A_X = A_C \to Rc_2^! A_X.$$

Indeed by the base change formula the natural map

$$Rc_2^! A_X \to R\tilde{j}_* Rc_2^! E_{X^\circ} \simeq R\tilde{j}_* \Omega_{C^\circ}(-d)[-2d]$$

is an isomorphism, and v is the map corresponding to the fundamental class of C (see [16, 2.7]). As in [22, 1.2.2] we can view v as a global section in

$$H^0(C, Rc^!(\Omega_X^{A_X} \boxtimes A_X)).$$

Observe that $\Omega_X^{A_X}$ is isomorphic to $j_! E_{X^\circ}(d)[2d]$, and therefore there is a unique map

$$L\Delta_X^*(\Omega_X^{A_X} \boxtimes A_X) \to \Omega_X^{A_X} \tag{8.2.1}$$

in $D_c(X, A_X)$ restricted to the canonical map over X°. The composition of this map with the map $\Omega_X^{A_X} \to \Omega_X$ in $D(X, E)$ is the evaluation map

$$D(A_X) \otimes_\Lambda A_X \to \Omega_X.$$

8.3. The adjoint of 8.2.1 is a morphism in $D_c(X \times X, A_{X\times X})$

$$\Omega_X^{A_X} \boxtimes A_X \to R\Delta_{X*}\Omega_X^{A_X}.$$

Applying $c^!$ to this map and using base change we get a morphism in $D_c(C, A_C)$

$$Rc^!(\Omega_X^{A_X} \boxtimes A_X) \to R\delta_* c'^! \Omega_X^{A_X} \simeq R\delta_* \Omega_F^{A_F}.$$

Applying the global section functor and evaluating this map on v we therefore get an element

$$\sigma \in H^0(F, \Omega_F^{A_F}).$$

By construction the image of σ under the natural map

$$H^0(F, \Omega_F^{A_F}) \to H^0(F, \Omega_F)$$

is the class denoted $\mathrm{Tr}(v)$ in [22, 1.2.2], giving rise to the so-called true local terms of v.

8.4. Let V be a lisse sheaf of E-modules on X° with unipotent local monodromy and let $M \in D_c(X, A_X)$ denote Rj_*V. Let $u^\circ : c_1^*V \to c_2^*V$ be a morphism over C° and let $u : Lc_1^*M \to Lc_2^*M$ be the induced morphism (see 6.4 (ii)).

Lemma 8.5. *The natural map in $D_c(C, A_C)$*

$$Lc_2^*M \otimes_{A_C}^{\mathbb{L}} Rc_2^! A_X \to Rc_2^! M$$

is an isomorphism.

Proof. By base change we have $Rc_2^! A_X \simeq R\tilde{j}_* E_{C^\circ} \simeq A_C$ and $Rc_2^! M \simeq R\tilde{j}_* c_2^* V$. The result then follows from 6.5 which shows that $Lc_2^* M \simeq R\tilde{j}_* c_2^* V$ as well. □

8.6. We therefore get a morphism

$$Lc_1^*M \to Rc_2^! M \tag{8.6.1}$$

from the tensor product $u \otimes v$.

Let $M^a \in D_c(X, E)$ denote the underlying complex of E-modules of M. There is a natural map

$$c_1^*(M^a) \to (Lc_1^*M)^a$$

and therefore 8.6.1 induces a morphism

$$w : c_1^*(M^a) \to (Rc_2^! M)^a = c_2^!(M^a),$$

which is an action of c on M^a in the usual sense. As in [22, 1.2.2] we therefore get a class

$$\mathrm{Tr}(w) \in H^0(F, \Omega_F).$$

8.7. The evaluation map in $D_c(F, E_F)$

$$A_F \otimes_E \Omega_F^{A_F} \to \Omega_F$$

induces a pairing

$$\langle \cdot, \cdot \rangle : H^0(F, A_F) \times H^0(F, \Omega_F^{A_F}) \to H^0(F, \Omega_F).$$

Theorem 8.8. *Let $t_u \in H^0(F, A_F)$ be the naive local term of u. Then*

$$\mathrm{Tr}(w) = \langle t_u, \sigma \rangle.$$

Proof. We unwind the definition of $\mathrm{Tr}(w)$. Notice that $D(M^a) \simeq j_! V^\vee$ and $\Omega_X^{A_X} \simeq j_! E_{X^\circ}(d)[2d]$ and therefore we have isomorphisms

$$D(M^a) \simeq (\Omega_X^{A_X} \otimes_{A_X}^{\mathbb{L}} M^\vee)^a,$$

$$D(M^a) \otimes_E^{\mathbb{L}} M^a \simeq (\Omega_X^{A_X} \otimes_{A_X}^{\mathbb{L}} M^\vee \otimes_{A_X}^{\mathbb{L}} M)^a,$$

and

$$D(M^a) \boxtimes M^a \simeq ((\Omega_X^{A_X} \boxtimes A_X) \otimes_{A_{X \times X}}^{\mathbb{L}} (M^\vee \boxtimes M))^a.$$

Furthermore, the diagram

$$\begin{array}{ccccc}
D(M^a) \boxtimes M^a & \longrightarrow & \Delta_{X*}(D(M^a) \otimes^{\mathbb{L}} M^a) & \xrightarrow{\text{ev}} & \Delta_{X*}\Omega_X \\
\simeq \downarrow & & \simeq \downarrow & & \uparrow \\
((\Omega_X^{A_X} \boxtimes A_X) \otimes^{\mathbb{L}}_{A_{X\times X}} (M^\vee \boxtimes M))^a & \longrightarrow & \Delta_{X*}(\Omega_X^{A_X} \otimes^{\mathbb{L}}_{A_X} M^\vee \otimes^{\mathbb{L}}_{A_X} M)^a & \xrightarrow{\text{id}\otimes\text{ev}} & \Delta_{X*}\Omega_X^{A_X}
\end{array}$$

commutes, where "ev" indicates the evaluation maps. Note also that we have a commutative diagram

$$\begin{array}{ccccc}
Rc^!((\Omega_X^{A_X} \boxtimes A_X) \otimes^{\mathbb{L}}_{A_{X\times X}} (M^\vee \boxtimes M)) & \longrightarrow & Rc^!\Delta_{X*}(\Omega_X^{A_X} \otimes^{\mathbb{L}}_{A_X} M^\vee \otimes^{\mathbb{L}}_{A_X} M) & \xrightarrow{\text{id}\otimes\text{ev}} & Rc^!\Delta_{X*}\Omega_X^{A_X} \\
\uparrow & & \uparrow & & \downarrow \text{bc} \\
 & & & & \delta_*\Omega_F^{A_F} \\
 & & & & \uparrow \text{mult} \\
(Rc^!(\Omega_X^{A_X} \boxtimes A_X) \otimes^{\mathbb{L}}_{A_C} Lc^*(M^\vee \boxtimes M) & \longrightarrow & (Rc^!\Delta_{X*}\Omega_X^{A_X}) \otimes^{\mathbb{L}}_{A_C} Lc^*\Delta_{X*}(M^\vee \otimes_{A_X} M) & \xrightarrow{\text{bc}\otimes\text{ev}} & \delta_*\Omega_F^{A_F} \otimes^{\mathbb{L}}_{A_C} A_F,
\end{array}$$

where the notation "bc" indicates the base change isomorphisms and "mult" indicates the map induced by the A_F-module structure.

It follows that $\text{Tr}(w)$ is equal to the image in $H^0(F, \Omega_F)$ of the section

$$\sigma \otimes u \in H^0(C, Rc^!(\Omega_X^{A_X} \boxtimes A_X) \otimes^{\mathbb{L}}_{A_C} Lc^*(M^\vee \boxtimes M))$$

under the map on sections induced by the composite

$$\begin{array}{c}
(Rc^!(\Omega_X^{A_X} \boxtimes A_X) \otimes^{\mathbb{L}}_{A_C} Lc^*(M^\vee \boxtimes M) \\
\downarrow \\
(Rc^!\Delta_{X*}\Omega_X^{A_X}) \otimes^{\mathbb{L}}_{A_C} Lc^*\Delta_{X*}(M^\vee \otimes_{A_X} M) \\
\downarrow \text{bc}\otimes\text{ev} \\
\delta_*\Omega_F^{A_F} \otimes^{\mathbb{L}}_{A_C} A_F, \\
\downarrow \text{mult} \\
\delta_*\Omega_F^{A_F} \\
\downarrow \text{ev} \\
\delta_*\Omega_F.
\end{array}$$

This is exactly the definition of the class $\langle t_u, \sigma\rangle$. □

Corollary 8.9. *If* $t_u \in H^0(F, E) \subset H^0(F, A_F)$ *then* $\text{Tr}(w)$ *is equal to the class* $\text{Tr}(v)$ *multiplied by the scalar* t_u.

Proof. This is immediate from 8.8. □

9. Filtered actions

9.1. Continuing with the notation and assumptions of 6.1, let V be a lisse sheaf of E-modules on X°, and let $u^\circ : c_1^*V \to c_2^*V$ be a morphism over C° defining a morphism $u : \mathrm{L}c_1^*M \to \mathrm{L}c_2^*M$, where $M := Rj_*V \in D(X, A_X)$.

For any fixed point $y \in C$ mapping $x \in X$, we then get a diagram

$$\begin{array}{ccc} & C_{(\bar{y})} & \\ {\scriptstyle c_{1,\bar{y}}} \swarrow & & \searrow {\scriptstyle c_{2,\bar{y}}} \\ X_{(\bar{x})} & & X_{(\bar{x})}, \end{array}$$

where $X_{(\bar{x})}$ (resp. $C_{(\bar{y})}$) denotes the spectrum of the strict henselization $\mathscr{O}_{X,\bar{x}}$ (resp. $\mathscr{O}_{Y,\bar{y}}$), and a map

$$u^\circ_{(\bar{y})} : c_{1,\bar{y}}^* V_{(\bar{x})} \to c_{2,\bar{y}}^* V_{(\bar{x})}$$

over $C^\circ_{(\bar{y})}$.

Definition 9.2. The action u of C on V is *filtered* if for every fixed point $y \in C$ mapping to $x \in X$, there exists a finite filtration $\mathrm{Fil}^{\cdot}$ on $V_{(\bar{x})}$ with each successive quotient $\mathrm{Fil}^i/\mathrm{Fil}^{i+1}$ isomorphic to a constant sheaf E^{r_i} for some integer r_i and such that the map $u^\circ_{(\bar{y})}$ respects the filtration.

Let F denote $C \times_{c, X\times X, \Delta_X} X$.

Assumption 9.3. *Assume that* $\mathscr{H}^i(A_F) = 0$ *for* $i < 0$.

Remark 9.4. By 5.17 this holds for example if C and X are normal and $X^\circ \subset X$ and $C^\circ \subset C$ are dense.

9.5. Assuming that u is filtered and that 9.3 holds we can calculate the element $t_u \in H^0(F, A_F)$ as follows. Since

$$H^0(F, A_F) \simeq H^0(F, \mathscr{H}^0(A_F))$$

it suffices to calculate the stalk $t_{u,\bar{y}}$ for any fixed geometric point $\bar{y} \to F$. Now since u is filtered, this is equal to the sum

$$\sum_i \mathrm{tr}(\mathrm{gr}_i(u) : \Gamma(C^\circ_{(\bar{y})}, \mathrm{Fil}^i/\mathrm{Fil}^{i+1}) \to \Gamma(C^\circ_{(\bar{y})}, \mathrm{Fil}^i/\mathrm{Fil}^{i+1})). \tag{9.5.1}$$

Definition 9.6. The number (9.5.1) is called the *naive trace at* $\bar{y}$.

Corollary 9.7. *For any connected component* $\Gamma \subset F$*, the restriction of* t_u *to* Γ *is in* E *and equal to the naive trace at any fixed point of* Γ.

9.8. Filtered actions arise naturally as we now discuss.

First let us recall some results from [5]. Let $X_0/\mathbb{F}_q$ be a smooth geometrically connected scheme, and let $D_0 \subset X_0$ be a divisor with normal crossings. Fix a separable closure $\mathbb{F}_q \hookrightarrow k$ and let $x \in X_0(\mathbb{F}_q)$ be a point with resulting geometric point $\bar{x} : \mathrm{Spec}(k) \to X$. Let $X_{0,(x)}$ denote the henselization of X_0 at x and let $X_{(\bar{x})}$ be the strict henselization.

Fix also a geometric generic point $\bar{\eta} \to X_{0,(x)}$, and assume further that the pullback $D_{0,(x)}$ of D_0 to $X_{0,(x)}$ is a divisor with simple normal crossings with components I. Fix for each

$i \in I$ a local generator $t_i \in \Gamma(X_{0,(x)}, \mathscr{O}_{X_{0,(x)}})$ for the component D_i corresponding to i. As discussed in [5, 1.7.11] we then have an extension

$$0 \to \mathbb{Z}'(1)^I \to \pi_1(X^\circ_{0,(x)}, \bar{\eta})' \to \mathrm{Gal}(k/\mathbb{F}_q) \to 0,$$

where $\mathbb{Z}'(1) := \varprojlim_{(n,p)=1} \mu_n$ and $\pi_1(X^\circ_{0,(x)}, \bar{\eta})'$ is the tame fundamental group of $X^\circ_{0,(x)} := X_{0,(x)} - D_{0,(x)}$. Let $W(X_{0,(x)} - D_{0,(x)}, \bar{\eta})'$ denote the *Weil group* obtained by pulling back this sequence along the morphism $\mathbb{Z} \to \mathrm{Gal}(k/\mathbb{F}_q)$ induced by the q-power Frobenius morphism on k, so we have an exact sequence

$$0 \to \mathbb{Z}'(1)^I \to W(X^\circ_{0,(x)}, \bar{\eta})' \to \mathbb{Z} \to 0.$$

Let R be an ℓ-adic representation of $W(X^\circ_{0,(x)}, \bar{\eta})'$. We then have the following facts:

(1) ([5, 1.7.12.1]) The restriction of R to $\mathbb{Z}'(1)^I$ is quasi-unipotent.
(2) ([5, 1.7.12.2]) If F' and F'' are two lifts to $W(X_{0,(x)} - D_{0,(x)}, \bar{\eta})'$ of $1 \in \mathbb{Z}$ then the eigenvalues of F' and F'' agree, up to multiplication by a root of unity. In particular, if $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ is an isomorphism then we can define the ι-weights of R.
(3) ([5, 1.7.12.3]) If the ι-weights of R are integers then there exists a unique filtration Fil_w on R, called the *weight filtration*, whose successive quotients are ι-pure. If R is pure then the action of $\mathbb{Z}'(1)^I$ factors through a finite quotient.

Remark 9.9. If V is a mixed $\overline{\mathbb{Q}}_\ell$-sheaf on X°_0 with pointwise integral ι-weights and tame local monodromy along D_0, and $R(V)$ is the resulting representation of $W(X^\circ_{0,(x)}, \bar{\eta})'$ then the ι-weights of $R(V)$ are also integral (this can be seen by reducing to the case of curves and using [5, 1.8.5]).

Remark 9.10. Let $\mathscr{O}$ be the ring of integers of a finite extension of $\overline{\mathbb{Q}}_\ell$, and suppose given a lisse Weil sheaf $V_{\mathscr{O}}$ of $\mathscr{O}$-modules on X°. Let V denote the sheaf $V_{\mathscr{O}} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_\ell$. Assume that the sheaf $V_{\mathscr{O}}/\ell^2 V_{\mathscr{O}}$ is trivial. In this case the sheaf V is tamely ramified along D (since the monodromy group of V is a pro-ℓ-group). Furthermore, it follows from the proof of [20, Proposition on p. 515] that if $R(V)$ denotes the resulting representation of $W(X^\circ_{0,(x)}, \bar{\eta})'$ then the restriction of $R(V)$ to $\mathbb{Z}'(1)^I$ is unipotent. In particular, if Fil_w denotes the weight filtration on $R(V)$ then $\mathbb{Z}'(1)^I$ acts trivially on $\mathrm{gr}^i_{\mathrm{Fil}_w}(V)$ for all i.

9.11. Let $f : C_0 \to X_0$ be a morphism of smooth $\mathbb{F}_q$-schemes, and assume that the preimage $E_0 := f^{-1}(D_0)$ endowed with the reduced structure is a divisor with simple normal crossings on C_0. Let $y \in C_0(\mathbb{F}_q)$ be a point mapping to $x \in X_0(\mathbb{F}_q)$. We then get an induced commutative diagram (after choosing compatible specialization maps)

$$\begin{array}{ccc} W(C^\circ_{0,(y)}, \bar{\eta})' & & \\ \downarrow{\scriptstyle f_*} & \searrow & \\ & & \mathbb{Z} \\ & \nearrow & \\ W(X^\circ_{0,(x)}, \bar{\eta})' & & \end{array}$$

In particular, a lifting of Frobenius to $W(C^\circ_{0,(y)}, \bar{\eta})'$ maps to a lifting of Frobenius in $W(X^\circ_{0,(x)}, \bar{\eta})'$. It follows that if R is a representation of $W(X^\circ_{0,(x)}, \bar{\eta})'$ with integral ι-weights and f^*R is the induced representation of $W(C^\circ_{0,(y)}, \bar{\eta})'$, then the weight filtration on f^*R is equal to the restriction of the weight filtration on R (i.e. the weight filtration defined over $X^\circ_{0,(x)}$ pulls back to the weight filtration defined over $C^\circ_{0,(y)}$).

9.12. One way to ensure that the action u is filtered is the following. Suppose that k is the algebraic closure of a finite field $\mathbb{F}_q$, and that c is obtained by base change from a morphism

$$C_0 \to X_0 \times X_0$$

over $\mathbb{F}_q$. Assume further that X is smooth and that D (resp. $E := c_2^{-1}(D)_{\text{red}}$) is a divisor with simple normal crossings on X (resp. C).

Proposition 9.13. *Let $\mathscr{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_\ell$ and let $V_\mathscr{O}$ be a lisse Weil sheaf (see* [5, 1.1.10]*) of $\mathscr{O}$-modules on X° with induced $\overline{\mathbb{Q}}_\ell$-sheaf V. Assume that V is pointwise pure and that the sheaf $V_\mathscr{O}/\ell^2 V_\mathscr{O}$ is trivial. Let $u^\circ : c_1^*V \to c_2^*V$ be an action over C° commuting with Frobenius in the sense that the diagram*

$$\begin{array}{ccc}
F^*_{C^\circ} c_1^* V & \xrightarrow{F_C^* u^\circ} & F^*_{C^\circ} c_2^* V \\
\simeq \downarrow & & \downarrow \simeq \\
c_1^* F^*_{X^\circ} V & & c_2^* F^*_{X^\circ} V \\
\downarrow \varphi & & \varphi \downarrow \\
c_1^* V & \xrightarrow{u^\circ} & c_2^* V
\end{array}$$

commutes, where F_{X° and F_{C° denote the relative Frobenius morphisms and φ is the Weil sheaf structure. Then u° is filtered.

Proof. Let $y \in \text{Fix}(c)$ be a point mapping to $x \in X$. After replacing $\mathbb{F}_q$ by a field extension we may assume that these points are defined over $\mathbb{F}_q$. Let $\text{Fil}^{\cdot}_w$ be the weight filtration on $V_{(x)}$. Then using remark 9.10 the successive quotients $\text{Fil}^i_w/\text{Fil}^{i+1}_w$ are trivial. Furthermore, since the weight filtration is functorial and u° commutes with Frobenius, which implies that the corresponding map of representations is a morphism of $W(C^\circ_{0,(y)}, \bar{y})$-representations, the map $u^\circ_{(y)} : c^*_{1,y} V_{(x)} \to c^*_{2,y} V_{(x)}$ respects this filtration. □

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