A BOUNDEDNESS THEOREM FOR HOM-STACKS

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1. Statement of Theorem

The purpose of this note is to prove the following "boundedness" stated in [Ol]. Let $\mathcal{X}$ and $\mathcal{Y}$ be separated Deligne–Mumford stacks of finite presentation over an algebraic space S and define $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ as in [Ol, 1.1]. Assume that $\mathcal{X}$ is flat and proper over S, and that locally in the fppf topology on S, there exists a finite flat surjection $Z \to \mathcal{X}$ from an algebraic space Z. Let $\mathcal{Y} \to W$ be a quasi-finite proper surjection over S to a separated algebraic space W over S of finite presentation. By [Ol, 1.1] we then have Deligne–Mumford stacks $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ and $\underline{\mathrm{Hom}}_S(\mathcal{X}, W)$.

Theorem 1.1. *The natural map*

$$\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}_S(\mathcal{X}, W) \tag{1.1.1}$$

is of finite type.

Remark 1.2. For a simple example to illustrate this theorem, consider the case when $\mathcal{X} = X$ is a smooth proper scheme over S, $\mathcal{Y} = BG$ for some finite group G, and $W = S$. Then the stack $\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y})$ classifies G–torsors on $\mathcal{X}$, and Theorem 1.1 essentially amounts to [SGA4, XVI.2.2] (this special case is in fact used in the proof; see section 3).

Note that in the case when S is the spectrum of a field, then $\mathcal{X}$ has a coarse moduli space $\pi : \mathcal{X} \to X$ and the formation of this moduli space commutes with arbitrary base change (see [Ol, 2.11] for a discussion of this). In this case the right side of 1.1.1 is canonically isomorphic to $\underline{\mathrm{Hom}}_S(X, W)$ by the universal property of coarse moduli spaces.

Our interest in this theorem comes from the theory of moduli spaces for twisted stable maps. As explained in [Ol2] the above theorem combined with the existence of a universal twisted curve (constructed in loc. cit.) yields a very quick proof of boundedness for the Abramovich-Vistoli moduli space of twisted stable maps [A-V]. Considering the very general nature of 1.1 we also hope that it will have interesting applications elsewhere in proving boundedness for moduli spaces (already some other applications have been found in recent work of Lieblich and Kovács [LK]).

The proof of 1.1 is a rather complicated devissage to the result [SGA4, XVI.2.2]. In section 2 we explain a construction (well-known to experts) called "rigidification" which enables one to"kill off" generic stabilizers of stacks. In section 3 we study a key special case of 1.1, from which the general case will be deduced. In section 4 we collect together various rather general results which will be used for the devissage. In section 5 we then start the proof of 1.1 with some preliminary reductions, and then in section 6 we explain the devissage to the special case of section 3.

Remark 1.3. In subsequent work [A-O-V2, Appendix B] we will generalize 1.1 to a certain class of Artin stacks with finite diagonal called *tame Artin stacks* [A-O-V] (note that the result [Ol, 1.1] holds not just for Deligne-Mumford stacks but also for Artin stacks with finite diagonal).

Acknowledgements: I am grateful to D. Abramovich for numerous helpful discussions which led to the results of this paper. I also want to thank M. Lieblich, A. Vistoli and J. Starr for several helpful conversations. In addition, Brian Conrad provided a large number of very helpful comments on earlier drafts of this paper. The referee also gave several helpful comments (in particular helped reorganize the paper to improve exposition).

The author was partially supported by NSF grant DMS-0714086 and an Alfred P. Sloan fellowship.

1.4. *Notation:* For an algebraic stack $\mathcal{X}$ over a scheme S we denote by $I(\mathcal{X})$ the *inertia stack* of $\mathcal{X}$. By definition $I(\mathcal{X})$ is the fiber product of the diagram

$$\begin{array}{ccc} & & \mathcal{X} \\ & & \downarrow{\scriptstyle \Delta} \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times_S \mathcal{X}. \end{array}$$

The stack $I(\mathcal{X})$ associates to any S-scheme T the groupoid of pairs (t, α), where $t \in \mathcal{X}(T)$ and α is an automorphism of t in $\mathcal{X}(T)$. In particular $I(\mathcal{X})$ is a relative group space over $\mathcal{X}$.

Let $f : \mathcal{G} \to \mathcal{X}$ be a morphism between Artin stacks of finite presentation and with finite diagonals over an algebraic space S , and assume that $\mathcal{X}$ is flat and proper over S and that locally in the fppf topology on S there exists a finite flat surjection $Z \to \mathcal{X}$ from an algebraic space Z. We denote by $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ the stack which to any S-scheme T associates the groupoid of sections $\mathcal{X}_T \to \mathcal{G}_T$ of the base change $\mathcal{G}_T \to \mathcal{X}_T$ of f to T. The stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is an Artin stack locally of finite presentation over S with quasi-compact and separated diagonal, as it is equal to the fiber product of the diagram

(1.4.1)
$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle \mathrm{Id}_{\mathcal{X}}} \\ \underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{G}) & \xrightarrow{g \mapsto f \circ g} & \underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{X}). \end{array}$$

When $\mathcal{G}$ and $\mathcal{X}$ are Deligne-Mumford stacks then $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is even a Deligne-Mumford stack, again by [Ol, 1.1].

2. Rigidification

We begin by recording the following result which is well-known to experts:

Proposition 2.1. *Let $\mathcal{X}$ be a normal Deligne–Mumford stack separated and of finite presentation over a locally noetherian base scheme S, and let X denote the (separated) coarse moduli space of $\mathcal{X}$. Then there exists a canonical factorization*

(2.1.1)
$$\mathcal{X} \xrightarrow{\alpha} \overline{\mathcal{X}} \xrightarrow{\beta} X$$

of the projection $\mathcal{X} \to X$, where $\overline{\mathcal{X}}$ is a separated Deligne–Mumford stack of finite presentation over S with coarse moduli space $\beta : \overline{\mathcal{X}} \to X$, α makes $\mathcal{X}$ an étale gerbe over $\overline{\mathcal{X}}$, and β is an isomorphism over some dense open subspace of X.

Proof. Consider first the special case when $\mathcal{X} = [V/G]$ where V is a normal connected scheme and G is a finite group acting on V. Let $H \subset G$ be the subgroup of elements that acts trivially on V. Since V is integral, H is equal to the stabilizer of the generic point of V. Also H is normal in G. Let $G \to \overline{G}$ be the quotient of G by H so the action of G on V factors through an action of $\overline{G}$ on V. In this case set $\overline{\mathcal{X}} = [V/\overline{G}]$. The natural projection

$$\alpha : [V/G] \to [V/\overline{G}]$$

realizes $\mathcal{X}$ as a gerbe over $\overline{\mathcal{X}}$, and in particular the map α induces an isomorphism on coarse moduli spaces. We therefore obtain a factorization 2.1.1 of the map $\mathcal{X} \to X$ from $\mathcal{X}$ to its coarse moduli space. Note also that by construction the generic stabilizer of the action of $\overline{G}$ on V is trivial, so the map $\overline{\mathcal{X}} \to X$ is an isomorphism over a dense open subset of X. This gives the proposition in this special case.

The $\underline{\text{Isom}}$-spaces for $\overline{\mathcal{X}}$ admit the following description which we will use in the proof of the general case below. The stack $\mathcal{X}$ can be viewed as the stack which to any scheme T associates the groupoid of pairs (P, s), where P is an fppf G–torsor over T and $s : P \to V$ is a G–equivariant map. Note that since H acts trivially on V this is equivalent to the groupoid of pairs $(P, \bar{s})$, where P is a G–torsor on T and $\bar{s} : P_H \to V$ is a $\overline{G}$–equivariant map. Here P_H is the $\overline{G}$–torsor obtained by taking the quotient of P by the H–action.

Consider two objects $t_i = (P_i, \bar{s}_i)$ $(i = 1, 2)$ of $\mathcal{X}(T)$, and let $I(t_1, t_2)$ be the functor on T-schemes which to any $g : T' \to T$ associates the set of isomorphisms $\sigma : g^*t_1 \to g^*t_2$ in $\mathcal{X}(T')$. There is a natural action of H on $I(t_1, t_2)$ induced by the action of H on P_1. Let $\overline{I}(t_1, t_2)(T)$ be the quotient of $I(t_1, t_2)(T)$ by this action. The set $\overline{I}(t_1, t_2)(T)$ is the quotient of the set of isomorphisms $\sigma : P_1 \to P_2$ of G–torsors compatible with the maps s_i by the natural action of H. Equivalently, $\overline{I}(t_1, t_2)(T)$ is equal to the set of isomorphisms $\bar{\sigma} : P_{1,H} \to P_{2,H}$ of $\overline{G}$–torsors compatible with the maps $\bar{s}_i$ which lift to an isomorphism of G–torsors $\sigma : P_1 \to P_2$. Since any isomorphism of $\overline{G}$–torsors $\bar{\sigma} : P_{1,H} \to P_{2,H}$ locally lifts to an isomorphism of G–torsors $\sigma : P_1 \to P_2$, this implies that the space $\underline{\text{Isom}}_{\overline{\mathcal{X}}}(t_1, t_2)$ is equal to the sheaf associated to the presheaf sending $(g : T' \to T)$ to $\overline{I}(g^*t_1, g^*t_2)(T')$.

The general case is obtained as follows.

Considering each of the connected components of $\mathcal{X}$ separately, we can without loss of generality assume that $\mathcal{X}$ is connected, and hence irreducible since $\mathcal{X}$ is normal [L-MB, 4.13]. First we claim that there is a dense open subset $U \subset X$ such that $\mathcal{X}_U := \mathcal{X} \times_X U \to U$ is an étale gerbe. For this, let n_0 be the minimum of the orders of the stabilizer groups of geometric points of $\mathcal{X}$. Consider the substack $\mathcal{U} \subset \mathcal{X}$ which to any T associates the sub–groupoid of $\mathcal{X}(T)$ consisting of 1–morphisms $t : T \to \mathcal{X}$ such that the order of the stabilizer group of t at every geometric point of T is n_0.

Lemma 2.2. *The substack $\mathcal{U} \subset \mathcal{X}$ is an open substack.*

Proof. It suffices to show that for any scheme T and morphism $t : T \to \mathcal{X}$ the fiber product $V := \mathcal{U} \times_{\mathcal{X},t} T$ is an open subset of T. Let G denote the group scheme of automorphisms of t. Then G is finite over T, for every point $z \in T$ the fiber G_z has length at least n_0, and $V \subset T$ is the subset of points $z \in T$ where the fiber has length exactly n_0. This is clearly an open condition. □

Since the map $\mathcal{X} \rightarrow X$ is proper, the image of $\mathcal{X} - \mathcal{U}$ in X is a closed subset whose complement U is the coarse moduli space of $\mathcal{U}$ (note that $\mathcal{U} = U \times_X \mathcal{X}$ by [Ol, 2.4 (i)]). We claim that the map $\mathcal{U} \rightarrow U$ is an étale gerbe. This assertion is étale local on U so by [Ol, 2.12] we may assume that $\mathcal{U} = [V/G]$ for some U–scheme V separated over S with action of a group G of order n_0. Since the action of G fixes every point of V and V is reduced (being étale over $\mathcal{U}$), the action of G on V is trivial and so $\mathcal{U}$ is étale locally on U isomorphic to $U \times BG$ for some finite group G.

Let $I(\mathcal{X}) \rightarrow \mathcal{X}$ be the intertia stack of $\mathcal{X}$. Let $\Gamma \subset I(\mathcal{X})$ denote the scheme–theoretic closure of the inclusion $I(\mathcal{U}) \hookrightarrow I(\mathcal{X})$. We claim that Γ is a normal sub–group scheme of $I(\mathcal{X})$ which is finite étale over $\mathcal{X}$. To see this we may work étale locally on X and hence as before we can assume that $\mathcal{X} = [V/G]$ for some finite group G acting on a connected normal X–scheme V separated over S. In this case, let $H \subset G$ be the normal subgroup of elements stabilizing every point of V as above, and let $W \subset V$ be the subset of points at which the stabilizer is equal to H. The subset W is invariant under the action of G. Furthermore, $I(\mathcal{X}) \times_{\mathcal{X}} V$ is isomorphic to the fiber product of the diagram

$$\begin{array}{ccc} & & V \\ & & \downarrow \Delta \\ V \times G & \longrightarrow & V \times V. \end{array}$$

In particular, $I(\mathcal{X}) \times_{\mathcal{X}} V$ is closed in $V \times G$ and so $\Gamma \times_{\mathcal{X}} V$ is isomorphic to the closure of $W \times H \subset V \times G$ which is just $V \times H$ mapping to V by the projection.

We now construct $\overline{\mathcal{X}}$ by "killing off" the group scheme Γ as in [A-C-V, 5.1.5]. For any morphism $T \rightarrow \mathcal{X} \times \mathcal{X}$ corresponding to objects $t_1, t_2 \in \mathcal{X}(T)$, the finite étale group scheme $t_1^*\Gamma$ acts on the T–scheme $\mathrm{Isom}_{\mathcal{X}}(t_1, t_2)$. An element $h \in t_1^*\Gamma(T)$ sends an isomorphism $\iota : t_1 \rightarrow t_2$ to $\iota \circ h$. This action is faithful, and hence we can form the quotient $\overline{I}(t_1, t_2)$ of $\mathrm{Isom}_{\mathcal{X}}(t_1, t_2)$ by this action. This quotient $\overline{I}(t_1, t_2)$ is an algebraic space separated over T since $t_1^*\Gamma$ is finite over T. Furthermore, since the map $\overline{I}(t_1, t_2) \rightarrow T$ is quasi–finite the algebraic space $\overline{I}(t_1, t_2)$ is in fact a scheme [Kn, II.6.16]. As explained in the proof of [A-C-V, 5.1.5], for three objects $t_1, t_2, t_3 \in \mathcal{X}(T)$ there is a unique composition law

$$\overline{I}(t_1, t_2) \times_T \overline{I}(t_2, t_3) \longrightarrow \overline{I}(t_1, t_3)$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Isom}(t_1, t_2) \times_T \mathrm{Isom}(t_2, t_3) & \longrightarrow & \mathrm{Isom}(t_1, t_3) \\ \downarrow & & \downarrow \\ \overline{I}(t_1, t_2) \times_T \overline{I}(t_2, t_3) & \longrightarrow & \overline{I}(t_1, t_3) \end{array}$$

commutes. We define $\overline{\mathcal{X}}^{\mathrm{ps}}$ to be the prestack which to any T associates the groupoid whose objects are the objects of $\mathcal{X}(T)$ but whose morphisms $t_1 \rightarrow t_2$ are the elements of $\overline{I}(t_1, t_2)(T)$, and then define $\overline{\mathcal{X}}$ to be the stack associated to $\overline{\mathcal{X}}^{\mathrm{ps}}$.

To prove the remaining statements in the proposition, we may by the construction of $\overline{\mathcal{X}}$ work étale locally on the coarse moduli space X. It therefore suffices to consider the case when $\mathcal{X} = [V/G]$, where V is a finite X–scheme and G is a finite group. The remaining statements therefore follow from the special case considered at the beginning of the proof. □

Remark 2.3. The factorization 2.1.1 has the following universal property. Let $\mathcal{Y}$ be an algebraic stack and $\rho : \mathcal{X} \to \mathcal{Y}$ a morphism of algebraic stacks such that the induced map on inertia stacks $I(\mathcal{X}) \to I(\mathcal{Y})$ sends $\Gamma \subset I(\mathcal{X})$ to the identity in $I(\mathcal{Y})$ (where Γ is defined as in the proof of 2.1). Then ρ factors uniquely through a morphism $\overline{\mathcal{X}} \to \mathcal{Y}$. This is shown in [A-G-V, C.1.1].

3. A special case

Proposition 3.1. *Let $f : X \to S$ be a smooth proper scheme over a locally noetherian base scheme S, and let $\pi : \mathcal{G} \to X$ be an étale gerbe over X with $\mathcal{G}$ a Deligne–Mumford stack with finite diagonal. Assume that $\mathcal{G} \to X$ admits a section s. Then the Deligne–Mumford stack $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ is of finite type.*

Remark 3.2. By writing $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ as a fiber product as in 1.4.1, one sees that 1.1 implies 3.1. On the other hand, in section 6 we will reduce the proof of 1.1 to 3.1.

Proof. By [L-MB, 3.21], the section s identifies $\mathcal{G}$ with BG, where $G = \underline{\mathrm{Aut}}(s)$ denotes the finite étale (since $\mathcal{G}$ has finite diagonal and is a gerbe) group scheme over X of automorphisms of the section s. In this case, the stack $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ is isomorphic to the stack which to any S–scheme T associates the groupoid of étale p^*G–torsors P over $X_T := X \times_S T$ (where $p : X_T \to X$ is the projection). In particular, the sheaf associated (with respect to the big étale topology) to the presheaf which associates to an S–scheme T the set of isomorphism classes of elements in $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ is isomorphic to the sheaf associated to the presheaf which to such a scheme T associates the set $H^1((X \times_S T)_{\mathrm{et}}, \mathrm{pr}_1^*G)$, and the group of automorphisms of any object $P \to T$ of $\underline{\mathrm{Sec}}(\mathcal{G}/X)(T)$ is canonically isomorphic to $H^0((X \times_S T)_{\mathrm{et}}, P \times^{\mathrm{pr}_1^*G,\mathrm{conj}} \mathrm{pr}_1^*G)$, where $P \times^{\mathrm{pr}_1^*G,\mathrm{conj}} \mathrm{pr}_1^*G$ denote the quotient of $P \times \mathrm{pr}_1^*G$ by the action of pr_1^*G given on scheme-valued points by

$$\gamma \cdot (p, g) = (\gamma \cdot p, \gamma g \gamma^{-1}), \quad \gamma, g \in p_1^*G, \quad p \in P.$$

If $f : X \to S$ denotes the structure morphism, then by [SGA4, XVI.2.2] (note that in the statement of loc. cit. one should further assume that the sheaf F is locally constant on X) the sheaf R^1f_*G is locally constant constructible on S_{et} and its formation commutes with arbitrary base change. Replace S by an étale cover over which R^1f_*G is constant and over which there exist representative G–torsors $P_i \to X$ $(i \in R^1f_*G)$. For each P_i, the sheaf $f_*(P_i \times^{\mathrm{pr}_1^*G,\mathrm{conj}} \mathrm{pr}_1^*G)$ is again by [SGA4, XVI.2.2] locally constant constructible, and its formation commutes with arbitrary base change on S. After replacing S by another étale cover, we may assume that these sheaves are in fact constant. Set $H_i := f_*(P_i \times^{\mathrm{pr}_1^*G,\mathrm{conj}} \mathrm{pr}_1^*G)$.

Each P_i induces a map $BH_i \to \underline{\mathrm{Sec}}(\mathcal{G}/X)$. By the preceding discussion the induced map

$$F : \coprod_i BH_i \longrightarrow \underline{\mathrm{Sec}}(\mathcal{G}/X). \tag{3.2.1}$$

induces a bijection on the sheaves associated to the presheaves of isomorphism classes of objects of the stacks, and in particular any object of $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ is locally in the image of 3.2.1. Furthermore, for any two objects $a, b \in \coprod_i BH_i$ the natural map of sheaves

$$\underline{\mathrm{Isom}}_{\coprod_i BH_i}(a, b) \to \underline{\mathrm{Isom}}_{\underline{\mathrm{Sec}}(\mathcal{G}/X)}(F(a), F(b))$$

is locally surjective and hence an isomorphism since if these sheaves are nonempty then this is a morphism of fppf H_i–torsors if $a \in BH_i$. Thus F is fully faithful and every object of $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ is locally in the essential image. Since both sides of 3.2.1 are stacks it follows that F is an equivalence. □

4. Some results about modifications

In this section we isolate four propositions (two of which are just quoted from [Ol]) that will be used in the proof of 1.1 in the following sections.

4.1. The stack of proper descent data.

4.2. Let S be a noetherian scheme, and let $\mathcal{G} \to \mathcal{X}$ be a morphism of Artin stacks of finite type over S with finite diagonals. Assume that $\mathcal{X}$ is flat and proper over S and that fppf-locally on S the stack $\mathcal{X}$ admits a finite flat surjection $Z \to \mathcal{X}$ with Z a scheme.

Let $f : \mathcal{R} \to \mathcal{X}$ be a proper surjection, with $\mathcal{R}/S$ an algebraic stack flat and of finite type over S and with finite diagonal. Assume that $\mathcal{R}$ also admits a finite flat cover by an algebraic space fppf locally on S (note that this holds automatically if f is representable). For $i \geq 1$, let $\mathcal{R}^{(i)}$ denote the i-fold fiber product of $\mathcal{R}$ with itself over $\mathcal{X}$. Define $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ to be the stack which to any S-scheme T associates the groupoid of pairs (w, ι), where $w : \mathcal{R} \to \mathcal{G}$ is a morphism over $\mathcal{X}$ and $\iota : \mathrm{pr}_1^* w \to \mathrm{pr}_2^* w$ is an isomorphism in $\mathcal{G}(\mathcal{R}^{(2)})$ projecting to the canonical isomorphism $\mathrm{pr}_1 \circ f \simeq \mathrm{pr}_2 \circ f$ in $\mathcal{X}(\mathcal{R}^{(2)})$, which satisfies the usual cocycle condition when pulled back to $\mathcal{G}(\mathcal{R}^{(3)})$.

Proposition 4.3. *(i) The stack* $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ *is an Artin stack locally of finite type over* S *with quasi-compact and separated diagonal.*

(ii) If the stack $\underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{X}} \mathcal{R}/\mathcal{R})$ *is of finite type over* S*, then the stack* $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ *is of finite type over* S*.*

(iii) If $\mathcal{G}$*,* $\mathcal{X}$*, and* $\mathcal{R}$ *are Deligne-Mumford stacks, then* $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ *is a Deligne-Mumford stack.*

Proof. Let $\mathcal{G}_{\mathcal{R}^{(i)}}$ denote the fiber product $\mathcal{R}^{(i)} \times_{\mathcal{X}} \mathcal{G}$.

The stack $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ can be described as follows. As remarked in 1.4, the stacks

$$\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(i)}}/\mathcal{R}^{(i)})$$

are Artin stacks locally of finite type over S with separated and quasi-compact diagonals. Furthermore, with the assumptions of (iii) the stack $\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}}/\mathcal{R})$ is even a Deligne-Mumford stack.

Let $\mathcal{P}$ be the fiber product of the diagram

$$\begin{array}{ccc} & & \underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}}/\mathcal{R}) \\ & & \downarrow{\scriptstyle \mathrm{pr}_1^*\times\mathrm{pr}_2^*} \\ \underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(2)}}/\mathcal{R}^{(2)}) \times \underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(2)}}/\mathcal{R}^{(2)}) & \xleftarrow{\Delta} & \underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(2)}}/\mathcal{R}^{(2)}). \end{array}$$

The morphism Δ is of finite type, and hence $\mathcal{P}$ is of finite type over $\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}}/\mathcal{R})$, whence also locally of finite type over S with separated and quasi-compact diagonal. Furthermore with the assumptions of (ii) (resp. (iii)) the stack $\mathcal{P}$ is of finite type over S (resp. a Deligne-Mumford stack).

For any scheme $T \to S$, the groupoid $\mathcal{P}(T)$ is equivalent to the category of pairs (s, ι), where $s : \mathcal{R}_T \to \mathcal{G}_{\mathcal{R}_T}$ is a section and $\iota : \mathrm{pr}_1^*(s) \simeq \mathrm{pr}_2^*(s)$ is an isomorphism in $\mathcal{G}(\mathcal{R}_T^{(2)})$, projecting to the canonical isomorphism in $\mathcal{X}(\mathcal{R}_T^{(2)})$. It follows that $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ is isomorphic to the fiber product of the diagram

$$\begin{array}{ccc} \mathcal{P} & & \\ \downarrow{\scriptstyle \tau} & & \\ I(\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(3)}}/\mathcal{R}^{(3)})) & \xleftarrow{e} & \underline{\mathrm{Sec}}(\mathcal{Y}_{\mathcal{R}^{(3)}}/\mathcal{R}^{(3)}), \end{array}$$

where $I(\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(3)}}/\mathcal{R}^{(3)}))$ denotes the inertia stack of $\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(3)}}/\mathcal{R}^{(3)})$ over S, τ sends (s, ι) to

$$(\pi_1^*(s), \pi_{12}^*(\iota) \circ \pi_{23}^*(\iota) \circ \pi_{13}^{*-1}(\iota)),$$

where $\pi_i : \mathcal{R}^{(3)} \to \mathcal{R}$ (resp. $\pi_{ij} : \mathcal{R}^{(3)} \to \mathcal{R}^{(2)}$) denotes the projection to the i–th factor (resp. i–th and j–th factor) and e sends t to (t, id). Since the morphism e is a closed immersion (since $\underline{\mathrm{Sec}}(\mathcal{G}_{\mathcal{R}^{(3)}}/\mathcal{R}^{(3)})$ has separated diagonal), it follows that $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ is a closed substack of $\mathcal{P}$. In particular, $\underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$ is locally of finite type over S with separated and quasi-compact diagonal, and with the assumptions of (ii) (resp. (iii)) of finite type (resp. a Deligne-Mumford stack). □

Remark 4.4. Note that by descent theory, if $\mathcal{R} \to \mathcal{X}$ is flat then the pullback functor

$$\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X}) \to \underline{\mathrm{Des}}(\mathcal{R}/\mathcal{X})$$

is an isomorphism of stacks. In particular, in this case it follows from (4.3 (ii)) that if $\underline{\mathrm{Sec}}(\mathcal{G} \times_{\mathcal{X}} \mathcal{R}/\mathcal{R})$ is of finite type over S, then the stack $\underline{\mathrm{Sec}}(\mathcal{G}/\mathcal{X})$ is also of finite type over S.

4.5. **Passage to the maximal reduced subspace.**

4.6. Let S be a noetherian scheme, X/S an algebraic S-space of finite type, and let $\mathcal{G} \to X$ be a quasi-finite and proper morphism from a finite type Deligne-Mumford stack $\mathcal{G}/S$ with finite diagonal. Assume further that X is flat and proper over S.

Proposition 4.7 ([Ol, 5.11])**.** *Let $X_0 \hookrightarrow X$ be a closed immersion defined by a nilpotent ideal $\mathcal{J} \subset \mathcal{O}_X$, and assume X_0 is flat over S. Let $\mathcal{G}_0$ denote the base change $\mathcal{G} \times_X X_0$. Then the natural map*

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to \underline{\mathrm{Sec}}(\mathcal{G}_0/X_0)$$

is of finite type.

4.8. **The case of a finite morphism of spaces.**

4.9. Let S be a noetherian scheme, and X/S a proper flat algebraic space over S. Let $\mathcal{G} \rightarrow X$ be a finite morphism (so $\mathcal{G}$ is also an algebraic space).

Proposition 4.10 ([Ol, 5.10])**.** *The algebraic space* $\underline{\mathrm{Sec}}(\mathcal{G}/X)$ *is of finite type over* S*.*

4.11. **Behavior with respect to proper modifications of X.**

4.12. Let S be a noetherian scheme, X/S an algebraic space of finite type, and let $\mathcal{Y}$ be a Deligne-Mumford stack of finite type over S with finite diagonal. Assume that the following conditions hold:

(i) The formation of the coarse space $\pi_{\mathcal{Y}} : \mathcal{Y} \rightarrow Y$ commutes with arbitrary base change $S' \rightarrow S$.
(ii) X is proper and flat over S.

Proposition 4.13. *Let* $m : X' \rightarrow X$ *be a proper surjection with* X' *a proper and flat algebraic space over* S*. Let* $f : X \rightarrow Y$ *be a morphism and let* $\mathcal{G} \rightarrow X$ *(resp.* $\mathcal{G}' \rightarrow X'$*) denote the pullback along* f *(resp.* $f \circ m$*) of* $\mathcal{Y}$*. Then the pullback map*

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \rightarrow \underline{\mathrm{Sec}}(\mathcal{G}'/X') \tag{4.13.1}$$

is of finite type.

Proof. By noetherian induction it suffices to show that 4.13.1 is of finite type after making a dominant base change $S' \rightarrow S$ of finite type. We may therefore assume that S is integral. Note also that the projections $\mathcal{G}' \rightarrow X'$ and $\mathcal{G} \rightarrow X$ are proper and quasi-finite since the morphism $\mathcal{Y} \rightarrow Y$ is proper and quasi-finite.

Let

$$X' \xrightarrow{\ p\ } Z \xrightarrow{\ q\ } X$$

be the Stein factorization of m so that p is proper with $p_*\mathcal{O}_{X'} = \mathcal{O}_Z$ and q is finite. After shrinking on S we may assume that Z is also flat over S and that the map $\mathcal{O}_Z \rightarrow p_*\mathcal{O}_{X'}$ remains an isomorphism after arbitrary base change on S. If $\mathcal{G}_Z$ denotes $\mathcal{G} \times_X Z$ then we have maps

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \rightarrow \underline{\mathrm{Sec}}(\mathcal{G}_Z/Z) \rightarrow \underline{\mathrm{Sec}}(\mathcal{G}'/X').$$

From this it follows that it suffices to consider the following two special cases.

Case 1: $m_*\mathcal{O}_{X'} = \mathcal{O}_X$ *and the formation of* $m_*\mathcal{O}_{X'}$ *commutes with arbitrary base change on* S*.*

Fix a section $s' : X' \rightarrow \mathcal{G}'$ and define $\mathcal{A}_{s'}$ to be the coherent sheaf of algebras on $\mathcal{G}$ obtained by pushing forward $\mathcal{O}_{X'}$ via the composite $X' \rightarrow \mathcal{G}' \rightarrow \mathcal{G}$. After shrinking on S we may assume that the formation of $\mathcal{A}_{s'}$ commutes with arbitrary base change on S. If s' is obtained from a morphism $s : X \rightarrow \mathcal{G}$, then since $m_*\mathcal{O}_{X'} = \mathcal{O}_X$ we have $\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}_{s'})$ mapping isomorphically to X. The fiber product

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \times_{\underline{\mathrm{Sec}}(\mathcal{G}'/X'),s} S$$

is therefore represented by the condition that the map $\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}_s) \rightarrow X$ is an isomorphism. If Z denotes the proper X–stack $\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}_s)$ and $g : Z \rightarrow X$ the projection, then the condition that g is étale is represented by an open subscheme of S and when g is étale the condition that g is an isomorphism is represented by an open of S.

Case 2: m is finite. By 4.7 it suffices to consider the case when X is reduced. In this case by the same argument proving [SGA6, XII.2.6] there exists a factorization of m

$$X' = X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} X_0 = X$$

such that for every i if $g_i : Z_i \to X_{i-1}$ denotes $X_i \times_{X_{i-1}} X_i$ then the sequence

$$\mathcal{O}_{X_{i-1}} \to p_{i*}\mathcal{O}_{X_i} \rightrightarrows g_{i*}\mathcal{O}_{Z_i} \tag{4.13.2}$$

is exact. After shrinking on S we may assume that all the X_i and Z_i are flat over S. Let $\mathcal{G}_i$ denote $\mathcal{G} \times_X X_i$. Then it suffices to show that each of the morphisms

$$\underline{\mathrm{Sec}}(\mathcal{G}_{i-1}/X_{i-1}) \to \underline{\mathrm{Sec}}(\mathcal{G}_i/X_i)$$

is of finite type.

In addition to assuming m finite, we may therefore also assume that the sequence

$$\mathcal{O}_X \to m_*\mathcal{O}_{X'} \rightrightarrows g_*\mathcal{O}_{X' \times_X X'}$$

is exact, and that the same holds after arbitrary base change on S (after possibly further shrinking on S). Let $X^{(i)}$ denote the $i+1$–fold fiber product of X' over X. After shrinking on S we may assume that $X^{(1)}$ and $X^{(2)}$ are flat over S. Define $\underline{\mathrm{Des}}(X'/X)$ as in 4.2. By 4.3 the stack $\underline{\mathrm{Des}}(X'/X)$ is algebraic over S with separated and quasi–compact diagonal.

The map 4.13.1 then factors as

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \xrightarrow{F} \underline{\mathrm{Des}}(X'/X) \xrightarrow{G} \underline{\mathrm{Sec}}(\mathcal{G}'/X')$$

where F is the functor sending a morphism $s : X \to \mathcal{G}$ to the pullback of s to X' with the tautological descent datum. The morphism G sends (s', σ) to s'. We show that both F and G are of finite type.

That F is of finite type can be seen as follows. Let $(s', \sigma) \in \underline{\mathrm{Des}}(X'/X)(S)$ denote an object and consider the fiber product

$$\mathcal{P} := \underline{\mathrm{Sec}}(\mathcal{G}/X) \times_{\underline{\mathrm{Des}}(X'/X)} S.$$

Define $\mathcal{A}$ to be the equalizer of the two maps of sheaves on $\mathcal{G}$

$$s'_*\mathcal{O}_{X'} \rightrightarrows s'_*\mathrm{pr}_{1*}\mathcal{O}_{X^{(1)}}$$

induced by σ. After shrinking on S we may assume that the formation of $\mathcal{A}$ commutes with arbitrary base change on S. If (s', σ) is induced by a morphism $s : X \to \mathcal{G}$ then by the exactness of 4.13.2 the projection $\mathrm{Spec}_{\mathcal{G}}(\mathcal{A}) \to X$ is an isomorphism. It follows, as in case 1, that $\mathcal{P}$ is represented by an open subscheme of S.

To see that G is of finite type, consider a section $s' : X' \to \mathcal{G}'$ and let I denote the finite $X^{(1)}$–space classifying isomorphisms $\mathrm{pr}_1^*\sigma \to \mathrm{pr}_2^*\sigma$. Then the fiber product

$$\underline{\mathrm{Des}}(X'/X) \times_{\underline{\mathrm{Sec}}(\mathcal{G}'/X')} S \tag{4.13.3}$$

is equal to the subfunctor of $\underline{\mathrm{Sec}}(I/X^{(1)})$ classifying isomorphisms $\sigma : \mathrm{pr}_1^*s' \to \mathrm{pr}_2^*s'$ which satisfy the cocycle condition on $X^{(2)}$. By 4.10 the space $\underline{\mathrm{Sec}}(I/X^{(1)})$ is quasi–compact with separated and quasi–compact diagonal. Let A denote the finite $X^{(2)}$–space classifying automorphisms of pr_1^*s'. Then 4.13.3 is equal to the fiber product of the diagram

$$\begin{array}{ccc} \underline{\mathrm{Sec}}(I/X^{(1)}) & \xrightarrow{w} & \underline{\mathrm{Sec}}(A/X^{(2)}) \\ & & \uparrow \mathrm{id} \\ & & S, \end{array}$$

where w is the map sending σ to $\mathrm{pr}_{13}^*(\sigma)^{-1}\circ \mathrm{pr}_{23}^*(\sigma)\circ \mathrm{pr}_{12}^*(\sigma)$. It follows that 4.13.3 is of finite type over S. □

5. Preliminary reductions

5.1. We now begin the proof of 1.1 by making some preliminary reductions. Using [Ol, 2.3], we can assume that S is a noetherian scheme. By noetherian induction, to prove that the map

$$\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \to \underline{\mathrm{Hom}}_S(\mathcal{X}, W) \tag{5.1.1}$$

is of finite type, it suffices to find a dominant morphism $S' \to S$ of finite type such that the pullback of 5.1.1 to S' is of finite type. In particular we may assume S is integral. By [Ol, 2.11], we may also assume that the formation of the coarse moduli spaces of $\mathcal{X}$ and $\mathcal{Y}$ commute with arbitrary base change on S. In this case the right side of 5.1.1 is canonically isomorphic to $\underline{\mathrm{Hom}}_S(X, W)$, where X is the coarse moduli space for $\mathcal{X}$ (with X proper over S by [Ol, 2.10]). Further shrinking on S lets us assume that X is flat over S.

Let $\pi_{\mathcal{Y}} : \mathcal{Y} \to Y$ denote the coarse moduli space of $\mathcal{Y}$. The universal property of the coarse moduli space [Ol, 2.4 (ii)] gives a map $Y \to W$ under $\mathcal{Y}$, and hence a factorization

$$\underline{\mathrm{Hom}}_S(\mathcal{X}, \mathcal{Y}) \xrightarrow{a} \underline{\mathrm{Hom}}_S(X, Y) \xrightarrow{b} \underline{\mathrm{Hom}}_S(X, W) \tag{5.1.2}$$

of 5.1.1. Since the morphism $\pi_{\mathcal{Y}}$ is proper and surjective [Ol, 2.6 (i)], and the map $\mathcal{Y} \to W$ is proper and quasi-finite, the morphism $Y \to W$ is proper and quasi–finite, whence finite. By 4.10 the map b in 5.1.2 is of finite type since for any morphism $T \to \underline{\mathrm{Hom}}_S(X, W)$ from a noetherian scheme T corresponding to an S-morphism $f : X_T \to W$, the fiber product

$$\underline{\mathrm{Hom}}_S(X, Y) \times_{\underline{\mathrm{Hom}}_S(X,W)} T$$

is isomorphic to $\underline{\mathrm{Sec}}_T(Y \times_{W,f} X_T/X_T)$. To prove that 5.1.1 is of finite type it therefore suffices to prove that the map a is of finite type.

To prove 1.1 we may therefore make the following additional assumptions which will be in effect for the rest of this paper:

(i) The formation of the coarse space $\pi_{\mathcal{Y}} : \mathcal{Y} \to Y$ commutes with arbitrary base change $S' \to S$.
(ii) The space W is equal to Y.
(iii) S is noetherian.

6. Devissage to the case of a gerbe

Let the notation be as in 1.1.

6.1 (Reduction to the case $\mathcal{X} = X$)**.** After replacing S by an fppf cover, there exists by assumption a finite flat cover $Z \to \mathcal{X}$ with Z an algebraic space. In particular, Z is proper over S. By 4.4 and 4.3 (ii), it suffices to prove 1.1 for Z, and so replacing $\mathcal{X}$ by Z we may assume that $\mathcal{X} = X$.

6.2. To prove that 1.1.1 is of finite type, it suffices to show that for any morphism $S' \to S$ of finite type and morphism $S' \to \underline{\mathrm{Hom}}_S(X, Y)$ corresponding to a map $f : X_{S'} \to Y_{S'}$, the fiber product

$$\underline{\mathrm{Sec}}(\mathcal{G}/X_{S'}) = \underline{\mathrm{Hom}}_S(X, \mathcal{Y}) \times_{\underline{\mathrm{Hom}}_S(X,Y)} S' \tag{6.2.1}$$

is of finite type over S', where $\mathcal{G} = \mathcal{Y} \times_{Y,f} X_{S'}$. To prove this we can without loss of generality replace S by S' and hence may assume that $S = S'$. Note that $\mathcal{G} \to X$ is proper and quasi-finite. Note also that without loss of generality, we can assume that S is the spectrum of a noetherian integal domain. Furthermore, by noetherian induction it suffices to exhibit a dominant morphism $S' \to S$ of finite type such that $\underline{\mathrm{Sec}}(\mathcal{G}/X) \times_S S'$ is of finite type over S'.

6.3 (Reduction to the case when X is reduced with geometrically reduced generic fiber)**.** After possibly making a base change along a dominant generically finite morphism $S' \to S$ of finite type, where S' is an integral affine scheme, we may assume that the maximal reduced closed subscheme $X_0 \subset X$ has geometrically reduced generic fiber. After shrinking on S we may further assume that X_0 is flat over S. By 4.7 the map

$$\underline{\mathrm{Sec}}(\mathcal{G}/X) \to \underline{\mathrm{Sec}}(\mathcal{G}_0/X_0)$$

is of finite type (where $\mathcal{G}_0 := \mathcal{G} \times_X X_0$). Replacing X by X_0 we may therefore assume that X is reduced with geometrically reduced generic fiber.

6.4 (Reduction to the case when X/S is smooth and $\mathcal{G}/X$ has a section)**.** Let $\bar{k}$ denote an algebraic closure of $k(S)$. By Chow's lemma [L-MB, 16.6.1] there exists a proper surjection $Z_1 \to \mathcal{G} \times_S \mathrm{Spec}(\bar{k})$ with Z_1 a scheme. Then by [deJ, 4.1] there exists a proper surjective morphism $Z \to Z_1$, where $Z/\bar{k}$ is smooth. Note that $\mathcal{G} \to X$ is a proper surjection, so $Z_1 \to X \times_S \mathrm{Spec}(\bar{k})$ is a proper surjection. By a standard limit argument, it follows that we can find an integral affine scheme S', a dominant generically finite morphism $S' \to S$ of finite type, a proper surjection $Z \to X_{S'}$ such that Z/S' is smooth and proper, and a section $Z \to \mathcal{G}$ over X. Replacing S by S' we may therefore assume that there exists a proper surjection $\pi : Z \to X$ with Z/S smooth and proper, and $\mathcal{G} \times_X Z \to Z$ having a section. By 4.13, it therefore suffices to consider $X = Z$.

6.5. Assume that X/S is smooth and that there exist a section $s : X \to \mathcal{G}$. Shrinking the excellent S if necessary we may assume that S is regular. Hence X is also regular. We can also assume X is connected so irreducible. Then $\mathcal{G}$ is also irreducible. Let $\mathcal{G}^*$ denote the normalization of the reduced and irreducible stack $\mathcal{G}_{\mathrm{red}}$. The stack $\mathcal{G}^*$ is also irreducible. Let $\mathcal{U} \subset \mathcal{G}_{\mathrm{red}}$ be the dense open substack where $\mathcal{G}_{\mathrm{red}}$ is normal. After shrinking S, we may assume that $\mathcal{G}_{\mathrm{red}}$ is flat over S, and hence the image of $\mathcal{U}$ in S is open. Furthermore by a standard reduction to the case of schemes, we can by [EGA, IV.13.1.3] shrink some more on S so that $\mathcal{G}$ and $\mathcal{G}^*$ have equidimensional fibers over S of some pure dimension, say d. We can assume

X is nonempty, so that $\mathcal{G}$ is nonempty, and hence also $\mathcal{U}$ is nonempty. Also by reduction to the case of schemes, one sees that by [EGA, IV.9.6] we can after shrinking some more on S assume that the open immersion $\mathcal{U} \hookrightarrow \mathcal{G}_{\mathrm{red}}$ is fiberwise dense. This reduction implies that for any point $s \in S$ the map $\mathcal{G}^*_s \to \mathcal{G}_{\mathrm{red},s}$ is generically an isomorphism.

Lemma 6.6. (i) *The morphism $\mathcal{G}_{\mathrm{red}} \to X$ is a coarse moduli space for $\mathcal{G}_{\mathrm{red}}$.*
(ii) *The projection $p : \mathcal{G}^* \to X$ is a coarse moduli space for $\mathcal{G}^*$.*

Proof. Since X is reduced, the section $s : X \to \mathcal{G}$ is induced by a unique morphism $s_{\mathrm{red}} : X \to \mathcal{G}_{\mathrm{red}}$ over X. Therefore (i) follows from [Ol, 2.9 (i)]. For (ii), let $\mathcal{Z} \subset \mathcal{G}_{\mathrm{red}}$ be the complement of $\mathcal{U}$ and let $Z \subset X$ be the closed image in X. Then the complement $U \subset X$ of Z is dense and satisfies the assumptions of [Ol, 2.9 (ii)]. □

In fact the section $s : X \to \mathcal{G}$ lifts uniquely to a section $s^* : X \to \mathcal{G}^*$. This follows from the following general result:

Proposition 6.7. *Let $\mathcal{X}$ be an integral Artin stack of finite type over a locally noetherian base scheme S, and let $\pi : \mathcal{X}^* \to \mathcal{X}$ be its normalization. Then for any normal integral scheme Z and dominant morphism $f : Z \to \mathcal{X}$ there exists a unique lifting $f^* : Z \to \mathcal{X}^*$ of f.*

Proof. By smooth descent theory and the uniqueness part of the proposition, it suffices to construct the morphism locally in the smooth topology on $\mathcal{X}$. We may therefore assume that $\mathcal{X}$ is a scheme, in which case the result is standard [EGA, II.6.3.9]. □

Let $\mathcal{G}^* \to \overline{\mathcal{G}}^* \to X$ be the factorization given by 2.1.

Lemma 6.8. *The proper, quasi-finite, and surjective map $\overline{\mathcal{G}}^* \to X$ is an isomorphism.*

Proof. To see that the morphism $\pi' : \overline{\mathcal{G}}^* \to X$ is an isomorphism, we may work étale locally on X and hence by [Ol, 2.12] can write $\overline{\mathcal{G}}^* = [V/G]$ for some finite X–scheme V. Since $\overline{\mathcal{G}}^*$ is normal and V is étale over $\overline{\mathcal{G}}^*$, the scheme V is also normal. Moreover, the section s^* induces a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{id}} & X, \end{array}$$

where P is an étale G–torsor and α is G–equivariant. Since $\overline{\mathcal{G}}^* \to X$ is an isomorphism over a dense open subset of X, the map α is an isomorphism over a dense open subset of X. Since P and V are both finite over X, the morphism α is also finite. Hence α is a finite birational morphism between normal schemes whence an isomorphism. □

6.9. It follows that $\mathcal{G}^*$ is an étale gerbe over X, and in particular is smooth over S. For any point $s \in S$, the equidimensional fiber $\mathcal{G}^*_s$ is normal and the map $\mathcal{G}^*_s \to \mathcal{G}_s$ between pure d–dimensional reduced stacks is finite and even an isomorphism over a dense open substack of the target. The same therefore also holds for the map $\mathcal{G}^*_s \to (\mathcal{G}_s)_{\mathrm{red}}$. By [EGA, III.4.4.9] it follows that $\mathcal{G}^*_s$ is equal to the normalization of $(\mathcal{G}_s)_{\mathrm{red}}$. In particular, any section $X_s \to \mathcal{G}_s$

lifts uniquely to a section $X_s \to \mathcal{G}_s^*$ (since $X_s \to \mathcal{G}_s$ necessarily factors through $(\mathcal{G}_s)_{\text{red}}$). It follows that the map

$$\underline{\text{Sec}}(\mathcal{G}^*/X) \longrightarrow \underline{\text{Sec}}(\mathcal{G}/X)$$

is surjective on field valued points. Hence to verify that $\underline{\text{Sec}}(\mathcal{G}/X)$ is quasi–compact it suffices to show that $\underline{\text{Sec}}(\mathcal{G}^*/X)$ is quasi–compact, which follows from 3.1.

This completes the proof of 1.1. □

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