THE BAR CONSTRUCTION AND AFFINE STACKS

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ABSTRACT. We discuss the equivalence of two constructions of a unipotent group scheme attached to a differential graded algebra over a $\mathbb{Q}$-algebra. The first construction is using the bar resolution and the second is using Toen's theory of affine stacks. We use this to establish the equivalence of two approaches to the comparison theorem in p-adic Hodge theory for the unipotent fundamental group of varieties defined over p-adic fields.

1. INTRODUCTION

The purpose of this note is to clarify a technical point which arose in our work on nonabelian p-adic Hodge theory in [Ol]. While the main content of this paper is purely algebraic and not concerned with p-adic Hodge theory, let us first recall the motivation.

1.1. Let V be a complete discrete valuation ring with perfect residue field of characteristic $p > 0$ and fraction field K of characteristic 0. Fix an algebraic closure $K \subset \overline{K}$. Let X/V be a smooth proper scheme, and let $D \subset X$ be a divisor with normal crossings over V. Assume that $X_{\overline{K}}$ is connected. Let X° denote $X - D$ and assume given a point $x \in X^\circ(V)$. We can then consider the pro-unipotent fundamental group of (X°, x) in the various motivic realizations (Betti, étale, de Rham, and crystalline). The étale realization $\pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x)$ is the pro-unipotent group scheme over $\mathbb{Q}_p$ which is Tannaka dual to the category of unipotent $\mathbb{Q}_p$-local systems on the geometric generic fiber $X_{\overline{K}}^\circ$. There is a natural action of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ on $\pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x)$, and it is shown in [Ol] that the coordinate ring $\mathscr{O}_{\pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x)}$ is an ind-crystalline representation (that is, a direct limit of crystalline representations).

Let us review what this means (see [Fo] for more details). Let W denote the ring of Witt vectors of the residue field of V, and let $K_0 \subset K$ denote the field of fractions of W. Let $\sigma : W \to W$ be the canonical lifting of Frobenius, and write also σ for the induced automorphism of K_0. Let $\mathrm{B}_{\mathrm{cris}}(V)$ denote Fontaine's K_0-algebra defined for example in [Fo]. Recall that $\mathrm{B}_{\mathrm{cris}}(V)$ comes equipped with an action of G_K, and a semi-linear automorphism φ. The ring $\mathrm{B}_{\mathrm{cris}}(V)$ is contained in a K-algebra $\mathrm{B}_{\mathrm{dR}}(V)$ which is a discrete valuation field. In particular, the valuation on $\mathrm{B}_{\mathrm{dR}}(V)$ defines a filtration on $\mathrm{B}_{\mathrm{dR}}(V)$. For a continuous finite dimensional G_K-representation M over $\mathbb{Q}_p$, define

$$\mathbf{D}(M) := (M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V))^{G_K},$$

where the action of G_K on $M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V)$ is the diagonal action. Then $\mathbf{D}(M)$ is a finite dimensional K_0-vector space with a semilinear automorphism induced by the automorphism of $\mathrm{B}_{\mathrm{cris}}(V)$. There is a natural map

$$\alpha_M : \mathbf{D}(M) \otimes_{K_0} \mathrm{B}_{\mathrm{cris}}(V) \to M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}(V).$$

The representation M is called *crystalline* if α_M is an isomorphism. In this case we obtain an isomorphism

$$\mathbf{D}(M) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}(V) \simeq M \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(V),$$

which induces a filtration on $\mathbf{D}(M) \otimes_{K_0} K$.

1.2. It is shown in [Ol] that there is a canonical isomorphism of Hopf-algebras over K_0

$$\mathbf{D}(\mathscr{O}_{\pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x)}) \simeq \mathscr{O}_{\pi_1^{\mathrm{crys}}(X^\circ, x)},$$

where $\pi_1^{\mathrm{crys}}(X^\circ, x)$ denotes the Tannaka dual of the category of unipotent log isocrystals on $(X, D)/W$. This isomorphism is compatible with the Frobenius automorphisms, where the Frobenius automorphism on $\pi_1^{\mathrm{crys}}(X^\circ, x)$ is induced by Frobenius pullback on the category of unipotent log isocrystals.

We also have an isomorphism

$$\pi_1^{\mathrm{crys}}(X^\circ, x) \otimes_{K_0} K \simeq \pi_1^{\mathrm{dR}}(X_K^\circ, x),$$

where $\pi_1^{\mathrm{dR}}(X_K^\circ, x)$ is the Tannaka dual of the category of unipotent modules with integrable connection on X_K°. In particular, the filtration on $\mathbf{D}(\mathscr{O}_{\pi_1^{\mathrm{et}}(X_{\overline{K}}^\circ, x)}) \otimes_{K_0} K$, obtained as above, induces a filtration on $\mathscr{O}_{\pi_1^{\mathrm{dR}}(X_K^\circ, x)}$.

1.3. On the other hand, Wojtkowiak has defined in [Wo] another filtration on $\mathscr{O}_{\pi_1^{\mathrm{dR}}(X_K^\circ, x)}$ using the bar construction. The main aim of this note is to show that these two filtrations agree. We also treat the case of spaces of unipotent paths between two different points of X°.

The motivation for this verification comes from the work of Kim [Ki1, Ki2, Ki3]. In this work, one needs the p-adic Hodge theory of the fundamental group developed in [Ol], but Wojtkowiak's definition of the Hodge filtration is the one used. To complete some of the proofs in Kim's papers, it therefore becomes necessary to know the equality of the two filtrations.

1.4. The verification that the two filtrations agree reduces to an abstract statement about differential graded algebras. Let k be a $\mathbb{Q}$-algebra, and let dga_k denote the category of commutative differential graded algebras over k. Given an augmented object $\epsilon : A \to k$ of dga_k there are at least two ways of producing a pro-unipotent group scheme from (A, ϵ).

The first is to use the bar construction as explained for example in [Wo]. This construction produces a simplicial dga $T(A, \epsilon)$. One can then take the total dga of this simplicial dga and then

$$B(A, \epsilon) := H^0(\mathrm{tot}(T(A)))$$

has a natural Hopf algebra structure whose associated group scheme we denote by $\pi_1^{\mathrm{bar}}(A, \epsilon)$.

The second approach is using Toen's theory of affine stacks [To]. Here one first converts (A, ϵ) into an augmented cosimplicial k-algebra $\epsilon' : A' \to k$, and then applies a suitable derived version of the functor $\mathrm{Spec}(-)$ to get a pointed simplicial presheaf $\mathbb{R}\mathrm{Spec}(A)$ on the category of k-algebras. One can then take the sheaf associated to the presheaf

$$R \mapsto \pi_1(|\mathbb{R}\mathrm{Spec}(A)(R)|, *)$$

which by Toen's theory is representable by a pro-unipotent group scheme $\pi_1^{\mathrm{aff}}(\mathbb{R}\mathrm{Spec}(A), *)$. Our main purpose in this paper is to establish a natural isomorphism

$$\pi_1^{\mathrm{bar}}(A, \epsilon) \simeq \pi_1^{\mathrm{aff}}(\mathbb{R}\mathrm{Spec}(A), *),$$

as well as a generalization to spaces of paths, and then use this isomorphism to establish the equality of the two filtrations described above.

Notation 1.6. For a category $\mathscr{C}$, we write $\mathscr{C}^o$ for the opposite category. If $\mathscr{C}$ and $\mathscr{D}$ are categories, we write $\mathscr{C}^{\mathscr{D}}$ for the category of functors

$$\mathscr{D} \to \mathscr{C}.$$

We denote the category of simplicial sets by $sSet$. For a $\mathbb{Q}$-algebra k we write $\mathrm{SPr}(k)$ for the category of simplicial presheaves on the category Aff_k of affine k-schemes. So, the category $\mathrm{SPr}(k)$ is the category of functors

$$\mathrm{Aff}_k^o \to sSet.$$

We write $C(k)$ for the category of complexes of k-modules (possibly unbounded in both directions). By [Hi, 2.2.1] (applied to the identity functor $C(k) \to C(k)$) there is a model category structure on $C(k)$ in which equivalences are quasi-isomorphisms, and fibrations are level-wise surjections.

The standard simplicial category is denoted Δ. The objects of Δ are finite nonempty ordered sets, and morphisms are order preserving maps.

2. THE MODEL CATEGORY STRUCTURE ON dga_k

The main reference for this section is the paper [Hi] (see also [Be, §1]).

Fix a $\mathbb{Q}$-algebra k.

2.1. A *differential graded k-algebra* is a complex $A^\cdot \in C(k)$ together with a structure of a graded k-algebra on $\oplus_{i \in \mathbb{Z}} A^i$ such that for every $a \in A^n$ and $b \in A^m$ we have

$$d(a \cdot b) = d(a)b + (-1)^n a d(b).$$

The algebra $A^\cdot$ is called *commutative* if for $a \in A^n$ and $b \in A^m$ we have

$$a \cdot b = (-1)^{nm} b \cdot a.$$

We denote by dga_k the category of commutative, $\mathbb{Z}$-graded, differential graded algebras, and by $\mathrm{dga}_k^{\geq 0}$ the full subcategory of differential graded algebras A with $A^i = 0$ for $i < 0$.

2.2. For a diagram

$$\begin{array}{ccc} & C & \\ & \scriptstyle a \uparrow & \\ A & \xrightarrow{\ b\ } & B \end{array}$$

in dga_k we can form the pushout, usually referred to as the tensor product, $C \otimes_A B$ in dga_k (see for example [1, Example 2 in Chapter 3 (c)]).

As a graded k-module, $C \otimes_A B$ is the quotient of the tensor product $C \otimes_{A^0} B$ be the k-submodule generated by elements

$$ca \otimes b = c \otimes ab,$$

for homogeneous elements $c \in C$, $a \in A$, and $b \in B$. The differential is determined by the formula

$$d(c \otimes b) = d(c) \otimes b + (-1)^p c \otimes d(b),$$

for $c \in C^p$, and the multiplication by

$$(c \otimes b) \cdot (c' \otimes b') = (-1)^{qp'} (cc' \otimes bb'),$$

where $c \in C^p$, $c' \in C^{p'}$, $b \in B^q$, and $b' \in B^{q'}$.

2.3. If $A \in \mathrm{dga}_k$ is a differential graded k-algebra, we write $A^\sharp$ for the underlying complex of k-modules. The functor

$$\sharp : \mathrm{dga}_k \to C(k)$$

has a left adjoint

$$F : C(k) \to \mathrm{dga}_k,$$

sending a complex M to the free commutative differential graded algebra generated by M (see for example [Be, §1.2]). If M is a complex with zero differentials, we write also

$$\mathrm{Sym}^{\cdot} M$$

for $F(M)$. By the construction in loc. cit., there is a natural direct sum decomposition

$$\mathrm{Sym}^{\cdot} M = \oplus_{k \in \mathbb{Z}} \mathrm{Sym}^k M.$$

If $M = N[-i]$ for some k-module M, then $\mathrm{Sym}^{\cdot} M$ is the symmetric algebra on N if i is even, and the exterior algebra on N if i is odd (and in both cases N is placed in degree i).

2.4. Define a morphism $f : A \to B$ in dga_k to be an *equivalence* (resp. *fibration*) if the underlying morphism of complexes $A^\sharp \to B^\sharp$ is a quasi-isomorphism (resp. if $A^i \to B^i$ is surjective for every i).

Proposition 2.5. *There is a model category structure on dga_k with the above defined equivalences and fibrations, and with cofibrations defined by the left lifting property with respect to trivial fibrations.*

Proof. We apply [Hi, 2.2.1], so we have to verify the conditions (H0) and (H1) in loc. cit:

(H0) dga_k admits finite limits and arbitrary colimits; the functor $\sharp$ commutes with filtered colimits.

(H1) Let I_d denote the complex which in degree d and $d+1$ is k with differential $k \to k$ the identity, and 0 in degrees not equal to d or $d+1$. Then for any $A \in \mathrm{dga}_k$ and integer d the map

$$A^\sharp \to (A \otimes_k F(I_d))^\sharp$$

is an equivalence.

For the statement about finite limits, note that dga_k has a final object (the zero differential graded algebra), finite products, and equalizers. As for colimits, note that dga_k has an initial object (the ring k viewed as a differential graded k-algebra), and clearly has all filtered colimits and that the functor $\sharp$ commutes with filtered colimits. Since dga_k has pushouts and therefore also coequalizers it follows that dga_k has arbitrary colimits. This implies (H0).

To verify condition (H1) note that we have $(A \otimes_k F(I_d))^\sharp \simeq A^\sharp \otimes_k F(I_d)^\sharp$, so it suffices to verify that $F(I_d)^\sharp$ is a filtering union of perfect subcomplexes $F(I_d)^\sharp_{\leq n}$ containing k such that for every n the map $k \to F(I_d)^\sharp_{\leq n}$ is a quasi-isomorphism. If d is even then $F(I_d)$ is the graded k-algebra generated by two variables x and y with $y^2 = 0$ and $xy = yx$, where x has degree d, y has degree $d+1$, and the differential is determined by $dx = y$. If d is odd then $F(I_d)$ is the graded k-algebra generated by two variables x and y with $x^2 = 0$ and $xy = yx$, where x has degree d, y has degree $d+1$, and the differential is determined by $dx = y$. In both cases, an elementary verification shows the desired properties (using that k is a $\mathbb{Q}$-algebra). $\square$

2.6. In fact in [Hi, 2.2.3] there is also a description of cofibrations in dga_k. Let $M \in C(k)$ be a complex of free k-modules with zero differentials, and let $\alpha : M \to A^\sharp$ be a morphism for some $A \in \mathrm{dga}_k$. We can then form a morphism

$$A \to A\langle M, \alpha \rangle$$

as follows. The underlying algebra of $A\langle M, \alpha \rangle$ is the tensor product

$$A \otimes_k \mathrm{Sym}^\cdot(M[1]),$$

with the differential given by the differential on A, and such that

$$d|_M = \alpha|_M.$$

Then $A \to A\langle M, \alpha \rangle$ is a cofibration. We call a morphism $A \to A\langle M, \alpha \rangle$ obtained in this way an *elementary cofibration*. We say that a map $A \to B$ is a *standard cofibration* if it is a direct limit of a sequence

$$A = A_1 \to A_2 \to A_3 \to \cdots,$$

where each $A_i \to A_{i+1}$ is an elementary cofibration. By [Hi, 2.2.5] every cofibration in dga_k is a retract of a standard cofibration.

2.7. For $A \in \mathrm{dga}_k$, let Mod_A denote the category of differential graded A-modules. As in [Hi, 3.1], the category Mod_A has a model category structure in which a morphism $M \to N$ in Mod_A is an equivalence (resp. fibration) if the underlying morphism in $C(k)$ is an equivalence (resp. fibration). For $M \in \mathrm{Mod}_A$ we write $M^\sharp$ for the underlying object of $C(k)$.

2.8. Several constructions from standard homological algebra can be carried through in the category Mod_A (see for example [SP, Tag 09JD]). A *homotopy* between two maps $f, g : M \to N$ in Mod_A is a map of A-modules $h : M \to N$ sending M^n to N^{n-1} for all n such that for every n and $x \in M^n$ we have

$$f(x) - g(x) = d_N(h(x)) + h(d_M(x))$$

in N^n.

If $f : N \to M$ is a morphism in Mod_A then the *cone of f*, denoted $C(f)$, is the object of Mod_A given by the usual cone in the category $C(k)$ and with $a \in A^n$ multiplying an element

$(x_m, y_{m+1}) \in C(f)^n = M^m \oplus N^{m+1}$ to give $(ax_m, ay_{m+1}) \in C(f)^{n+m} = M^{n+m} \oplus N^{n+m+1}$. The cone $C(f)$ represents the functor

$$H : \mathrm{Mod}_A \to \mathrm{Set}$$

sending $S \in \mathrm{Mod}_A$ to the set of pairs (g, h), where $g : M \to S$ is a morphism in Mod_A and h is a homotopy between gf and 0.

A variant of this is the following. For an object $M \in \mathrm{Mod}_A$ let M_I denote $C(-\mathrm{id}_M)[-1]$ so we have a morphism $\pi_M : M_I \to M$. Then M_I represents the functor

$$H' : \mathrm{Mod}_A^{\mathrm{op}} \to \mathrm{Set}$$

sending an object $N \in \mathrm{Mod}_A^{\mathrm{op}}$ to the set of pairs (g, h), where $g : N \to M$ is a morphism in Mod_A and h is a homotopy between g and 0.

Lemma 2.9. *Let $f : N \to M$ be a morphism in Mod_A with N cofibrant. Then the map $M \to C(f)$ is a cofibration.*

Proof. Let $p : S' \to S$ be a trivial fibration and consider a commutative diagram of solid arrows

$$
\begin{array}{ccc}
M & \xrightarrow{\;a\;} & S' \\
\downarrow & \nearrow{\scriptstyle e} & \downarrow{\scriptstyle p} \\
C(f) & \xrightarrow{\;b\;} & S.
\end{array}
$$

We have to show that there exists a morphism e filling in the diagram. By the universal property of $C(f)$ the morphism b is defined by a homotopy h between paf and 0 and finding the dotted arrow e is equivalent to lifting h to a homotopy h' between af and 0. Equivalently we need to find a dotted arrow filling in the following diagram:

$$
\begin{array}{ccc}
 & & S_I' \\
 & \nearrow & \downarrow \\
N & \xrightarrow{\;\tilde{b}\;} & S_I,
\end{array}
$$

where $\tilde{b}$ is the map corresponding to (paf, h). Since N is cofibrant and $S_I' \to S_I$ is a trivial fibration (indeed it is clearly termwise surjective and both complexes are acyclic) such a lifting of $\tilde{b}$ exists. $\qquad\square$

2.10. Cofibrations in Mod_A can also be described using the general recipe in [Hi, 2.2.4]. Let $M \in \mathrm{Mod}_A$ be an A-module, $N \in C(k)$ a complex of free k-modules with zero differentials, and $\beta : N \to M^\sharp$ a morphism in $C(k)$. We define

$$M\{N, \beta\}$$

to be the cone of the morphism $A \otimes_k N \to M$ induced by β. So $M\{N, \beta\}$ is the object of Mod_A with underlying A-module

$$M \oplus (A \otimes_k N[1])$$

and differential given by the differential on M and the map β on N. We say that a morphism $M \to M'$ in Mod_A is an *elementary cofibration* if $M' \simeq M\{N, \beta\}$ for some pair (N, β), where N is a complex of free modules with zero differentials, and $\beta : N \to M^\sharp$ is a morphism of

complexes. A morphism $M \to M'$ is called a *standard cofibration* if it is the direct limit of a sequence

$$M = M_0 \to M_1 \to M_2 \to \cdots ,$$

where each $M_i \to M_{i+1}$ is an elementary cofibration. By [Hi, 2.2.5] every cofibration in Mod_A is a retract of an elementary cofibration.

Lemma 2.11. *Let $M \in \mathrm{Mod}_A$ be a cofibrant object. Then the functor*

$$M \otimes_A (-) : \mathrm{Mod}_A \to C(k)$$

takes equivalences to equivalences.

Proof. Since any cofibration is a retract of a standard cofibration, it suffices to consider the case when M is the limit of a sequence

$$0 = M_0 \to M_1 \to M_2 \to \cdots ,$$

where each $M_i \to M_{i+1}$ is an elementary cofibration. Since a filtering direct limit of equivalences is an equivalence, it therefore suffices by induction to prove that if $M \in \mathrm{Mod}_A$ is an object such that $M \otimes_A (-)$ takes equivalences to equivalences, then for an elementary cofibration

$$M \to M\{N, \beta\}$$

the functor $M\{N, \beta\} \otimes_A (-)$ takes equivalences to equivalences. For this note that if $S \in \mathrm{Mod}_A$, then

$$M\{N, \beta\} \otimes_A S \simeq (M \otimes_A S) \oplus (N[1] \otimes_k S)$$

with differential induced by the map β. That is, we have an isomorphism in $C(k)$

$$M\{N, \beta\} \otimes_A S \simeq \mathrm{Cone}(\beta : N \otimes_k S \to M \otimes_A S).$$

It follows that it suffices to show that the functor $(-) \otimes_k N$ takes equivalences to equivalences, which is immediate as each N^i is a free k-module. $\square$

2.12. Let $\mathrm{dga}_{k, A/}$ denote the category of morphisms $A \to B$ in dga_k. There is a natural forgetful functor

$$\mathrm{dga}_{k, A/} \to \mathrm{Mod}_A, \quad (A \to B) \mapsto B^\dagger,$$

where $B^\dagger$ denotes the underlying differential graded A-module of B.

Proposition 2.13. *For any cofibration $A \to B$ in dga_k, the module $B^\dagger \in \mathrm{Mod}_A$ is cofibrant.*

Proof. By [Hi, 2.2.5], the map $A \to B$ is a retract of a standard cofibration, so it suffices to consider the case when $A \to B$ is a standard cofibration. By induction it therefore suffices to show that if $A' \in \mathrm{dga}_{k, A/}$, $M \in C(k)$ is a complex of free k-modules with zero differential, and $\alpha : M \to (A')^\sharp$ is a morphism in $C(k)$, then the induced map

$$(A')^\dagger \to (A'\langle M, \alpha \rangle)^\dagger$$

is a cofibration in Mod_A.

Note that $A'\langle M, \alpha \rangle$ is an inductive limit of a sequence

$$A' \to A'_1 \to A'_2 \to \cdots ,$$

where $A'_i \to A'_{i+1}$ is of the form $A'_i \to A'_i \langle M_i, \alpha_i \rangle$ for some $M_i = N[-i]$ with N a free k-module. Therefore we may further assume that $M = N[-i]$ for some free k-module N.

For $n \geq 0$ define an object of Mod_A

$$\Sigma^{(n)} := \oplus_{k=0}^{n} A'^{,\dagger} \otimes_k \mathrm{Sym}^k(N[-i+1]),$$

with differential given by the differential on $A'^{,\dagger}$ and the map

$$A'^{,\dagger} \otimes_k \mathrm{Sym}^k(N[-i+1]) \to A'^{,\dagger} \otimes_k \mathrm{Sym}^{k-1}(N[-i+1])$$

induced by the map α. We then have a sequence in Mod_A with limit $A'^{,\dagger} \to (A'\langle M, \alpha \rangle)^\dagger$

$$A'^{,\dagger} = \Sigma^{(0)} \to \Sigma^{(1)} \to \Sigma^{(2)} \to \cdots,$$

where each $\Sigma^{(i)} \to \Sigma^{(i+1)}$ is obtained as the cone of the morphism

$$A'^{,\dagger} \otimes_k \mathrm{Sym}^{i+1}(N[-i+1])[-1] \to A'^{,\dagger} \otimes_k \mathrm{Sym}^i(N[-i+1]).$$

By 2.9 each $\Sigma^{(i)} \to \Sigma^{(i+1)}$ is a cofibration, whence $A'^{,\dagger} \to (A'\langle M, \alpha \rangle)^\dagger$ is also a cofibration. $\qquad \square$

Corollary 2.14. *Consider a diagram in* dga_k

(2.14.1)
$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & C \\
\downarrow{\scriptstyle b} & & \\
B, & &
\end{array}
$$

where a is a cofibration. Then $B \otimes_A C$ represents the homotopy colimit in dga_k *of the diagram 2.14.1.*

Proof. By 2.13, the object $C^\dagger \in \mathrm{Mod}_A$ is cofibrant, and therefore the functor

$$(-) \otimes_A C^\dagger : \mathrm{Mod}_A \to C(k)$$

takes equivalences to equivalences (using 2.11). In particular, given a factorization of b

$$A \xrightarrow{\ c\ } E \xrightarrow{\ d\ } B,$$

where c is a cofibration and d is an equivalence, the induced map

$$E \otimes_A C \to B \otimes_A C$$

is an equivalence. $\qquad \square$

2.15. For $M \in \mathrm{Mod}_A$, we can also compute the derived functor

$$M \otimes_A^{\mathbb{L}} (-) : \mathrm{Ho}(\mathrm{Mod}_A) \to D(k)$$

as follows (the following was also remarked in [Hi, §3.4]).

For a simplicial object $P_{\cdot} \in \mathrm{Mod}_A^{\Delta^o}$, let $\mathrm{tot}(P_{\cdot})$ denote the (sum) total complex defined by

$$(\mathrm{tot}(P_{\cdot}))^n := \oplus_{i+j=n} P_{-i}^j$$

and the standard differential induced by the simplicial structure (see for example [Il, I.1.2.1]). The complex $\mathrm{tot}(P_{\cdot})$ is naturally an object of Mod_A with $a \in A^k$ acting by the given multiplication maps

$$a \cdot (-) : P_{-i}^j \to P_{-i}^{j+k}.$$

Proposition 2.16. *Assume $P_{\cdot} \in \mathrm{Mod}_A^{\Delta^o}$ is a simplicial A-module, with each P_i cofibrant. Then the functor*

$$\mathrm{tot}(P_{\cdot}) \otimes_A (-) : \mathrm{Mod}_A \to C(k)$$

preserves arbitrary equivalences. In particular, for any $M \in \mathrm{Mod}_A$ the complex $\mathrm{tot}(P_{\cdot}) \otimes_A M \in C(k)$ represents $\mathrm{tot}(P_{\cdot}) \otimes_A^{\mathbb{L}} M \in D(k)$.

Proof. Note first that if $P_{\cdot} \to P'_{\cdot}$ is a morphism in $\mathrm{Mod}_A^{\Delta^o}$ such that the map $P_i \to P'_i$ is an equivalence for all i, then the induced map

$$\mathrm{tot}(P_{\cdot}) \to \mathrm{tot}(P'_{\cdot})$$

is also an equivalence. Furthermore, by construction we have

$$\mathrm{tot}(P_{\cdot}) \otimes_A M \simeq \mathrm{tot}(P_{\cdot} \otimes_A M).$$

If $M \to M'$ is an equivalence, we therefore get (again using 2.11) that the map

$$\mathrm{tot}(P_{\cdot}) \otimes_A M \simeq \mathrm{tot}(P_{\cdot} \otimes_A M) \to \mathrm{tot}(P_{\cdot} \otimes_A M') \simeq \mathrm{tot}(P_{\cdot}) \otimes_A M'$$

is an equivalence. $\square$

2.17. If $P_{\cdot} \in \mathrm{dga}_k^{\Delta^o}$ is a simplicial object in dga_k, then the total complex $\mathrm{tot}(P^{\dagger})$ has a natural structure of an object of dga_k, which we will denote by $\mathrm{tot}(P_{\cdot})$. The multiplication map is given by the shuffle product (see for example [Wo, §2]).

Corollary 2.18. *Suppose $P_{\cdot} \in \mathrm{dga}_{k,A/}^{\Delta^o}$ for some $A \in \mathrm{dga}_k$, and that for every i the underlying object $P_i^{\dagger} \in \mathrm{Mod}_A$ is cofibrant. Then for any morphism $A \to B$ the homotopy colimit in dga_k of the diagram*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \\
\mathrm{tot}(P_{\cdot}) & &
\end{array}
$$

is represented by $\mathrm{tot}(P_{\cdot}) \otimes_A B$.

Proof. Indeed it suffices to verify that $(\mathrm{tot}(P_{\cdot}))^{\dagger} \otimes_A B^{\dagger} \in C(k)$ represents $(\mathrm{tot}(P_{\cdot}))^{\dagger} \otimes_A^{\mathbb{L}} B^{\dagger}$ which follows from 2.16. $\square$

3. Filtered Thom-Sullivan cochains

The content of this section is essentially contained in [N-A].

3.1. Let k be a $\mathbb{Q}$-algebra.

For an object $A \in \mathrm{dga}_k$, a *filtration* on A is a decreasing filtration $F^{\cdot}$ on the underlying complex of A such that for every $i, j \in \mathbb{Z}$ we have

$$A^i \cdot F^j \subset F^{i+j}.$$

A morphism $(A, F) \to (A', F')$ of filtered differential graded algebras is a morphism $f : A \to A'$ in dga_k such that $f(F^i) \subset (F')^i$ for all $i \in \mathbb{Z}$. Let fdga_k denote the category of

filtered differential graded k-algebras. A morphism $f : (A, F) \to (A', F')$ is a *filtered quasi-isomorphism* if $A \to A'$ is an equivalence and if for every $i \in \mathbb{Z}$ the map $F^i \to F'^i$ is a quasi-isomorphism.

Let $\mathrm{Ho}(\mathrm{fdga}_k)$ be the category obtained by inverting filtered quasi-isomorphisms.

The functor of Thom-Sullivan cochains, discussed in [Ol, §2], can be extended to the filtered context as follows.

3.2. Let us begin by recalling the definition of the functor of Thom-Sullivan cochains. For $p \geq 0$, let $\nabla(p, \bullet) \in \mathrm{dga}_k$ denote the de Rham complex of

$$\mathrm{Spec}(k[t_0, \ldots, t_p]/(\sum_{i=0}^{p} t_i = 1))$$

over k. As explained in [Ol, 2.13], as p varies we obtain a simplicial differential graded algebra

$$\Omega : \Delta^o \to \mathrm{dga}_k, \quad [p] \mapsto \nabla(p, \bullet).$$

Let $\mathcal{M}_\Delta$ be the category whose objects are morphisms $f : [i] \to [j]$ in Δ, and for which a morphism

$$(f : [i] \to [j]) \to (g : [i'] \to [j'])$$

is a commutative diagram in Δ

$$\begin{array}{ccc} [i] & \xrightarrow{f} & [j] \\ \uparrow & & \downarrow \\ [i'] & \xrightarrow{g} & [j']. \end{array}$$

If $A. \in \mathrm{dga}_k^\Delta$ is a cosimplicial differential graded algebra, we define

$$A. \otimes \Omega : \mathcal{M}_\Delta \to \mathrm{dga}_k, \quad (f : [i] \to [j]) \mapsto A_i \otimes \Omega_j.$$

The functor of Thom-Sullivan cochains

$$T : \mathrm{dga}_k^\Delta \to \mathrm{dga}_k$$

is defined by

$$T(A) := \varprojlim_{\mathcal{M}_\Delta} A. \otimes \Omega.$$

3.3. This construction also extends to the filtered context. View $\nabla(p, \bullet)$ as an object of fdga_k with filtration given by

$$F^i \nabla(p, \bullet) := \begin{cases} \nabla(p, \bullet) & \text{if } i \leq 0 \\ 0 & \text{if } i > 0. \end{cases}$$

With this definition the transition maps for Ω are compatible with the filtrations, so we obtain a functor

$$\Omega^f : \Delta^o \to \mathrm{fdga}_k.$$

Furthermore, for a cosimplicial object $A. \in \mathrm{fdga}_k^\Delta$ we obtain a functor

$$A. \otimes \Omega^f : \mathcal{M}_\Delta \to \mathrm{fdga}_k$$

and we define
$$T^f(A.) := \varprojlim_{\mathcal{M}_\Delta} A. \otimes \Omega^f \in \mathrm{fdga}_k.$$

As explained in [N-A, 6.3], integration defines a natural filtered quasi-isomorphism of filtered complexes
$$\int : T^f(A.) \to \mathrm{Tot}(A.).$$
In particular, T^f passes to a functor
$$T^f : \mathrm{Ho}(\mathrm{fdga}_k^\Delta) \to \mathrm{Ho}(\mathrm{fdga}_k).$$

4. THE BAR CONSTRUCTION AND DERIVED TENSOR PRODUCT

4.1. Let k be a $\mathbb{Q}$-algebra and let $A \in \mathrm{dga}_k$ be a cofibrant object with two augmentations $x, y : A \to k$. The bar construction provides a representative for the homotopy pushout of the diagram (which will be of special importance for spaces of paths)

(4.1.1)
$$\begin{array}{ccc} A \otimes_k A & \xrightarrow{x \otimes y} & k \\ \downarrow{\scriptstyle m} & & \\ A, & & \end{array}$$

where m denotes the multiplication map.

4.2. Let $\mathscr{C}$ and $\mathscr{D}$ be categories, and let
$$U : \mathscr{C} \to \mathscr{D}, \quad T : \mathscr{D} \to \mathscr{C},$$
be functors with T left adjoint to U. Write
$$a : \mathrm{id}_\mathscr{D} \to UT, \quad b : TU \to \mathrm{id}_\mathscr{C}$$
for the adjunction morphisms. Given an object $X \in \mathscr{C}$, we can form its *canonical simplicial resolution* $(T, U).(X) \in \mathscr{C}^{\Delta^o}$ as follows (for more details see [Il, I, §1.5]).

Define
$$(T, U)_n(X) := (TU)^{[n]}(X),$$
where we write $(TU)^{[n]}$ for the $n + 1$-fold composite
$$(TU) \circ (TU) \circ \cdots \circ (TU).$$
For $i = 0, \dots, n + 1$, let
$$d_i : (T, U)_{n+1}(X) \to (T, U)_n(X)$$
be the map induced by the morphism of functors
$$\mathrm{id}_{(TU)^{[0,i-1]}} \circ b \circ \mathrm{id}_{(TU)^{[i+1,n]}},$$
and for $i = 0, \dots, n$, let
$$s_i : (T, U)_n(X) \to (T, U)_{n+1}(X)$$
be the map induced by the morphism of functors
$$\mathrm{id}_{(TU)^{[0,i-1]}} \circ \mathrm{id}_T \circ a \circ \mathrm{id}_U \circ \mathrm{id}_{(TU)^{[i+1,n]}}.$$
We then obtain a simplicial object in $\mathscr{C}$ denoted by $(T, U).(X)$.

The map b induces a natural augmentation

$$\pi : (T, U).(X) \to X.$$

4.3. We will apply this construction as follows. Let $A \to A'$ be a morphism in dga_k. We can then consider the restriction functor

$$U : \mathrm{dga}_{k, A'/} \to \mathrm{dga}_{k, A/},$$

which has a left adjoint

$$A' \otimes_A (-) : \mathrm{dga}_{k, A/} \to \mathrm{dga}_{k, A'/}.$$

Lemma 4.4. *For any object $B \in \mathrm{dga}_{k, A'/}$ the map in $\mathrm{dga}_{k, A'/}$*

$$\mathrm{tot}((T, U).(B)) \to B,$$

induced by the augmentation $(T, U).(B) \to B$, is an equivalence.

Proof. It suffices to verify that the map in $\mathrm{dga}_{k, A/}$ obtained by applying the functor U is an equivalence. This follows from [Il, I, 1.5.3 (i)]. $\square$

Lemma 4.5. *Let $B \in \mathrm{dga}_{k, A'/}$ be an object, and suppose $A \to B$ and $A \to A'$ are cofibrations. Then for every n the morphism $A' \to (T, U)_n(B)$ is a cofibration in dga_k.*

Proof. We proceed by induction on n.

For the case $n = 0$, note that $(T, U)_0(B) = A' \otimes_A B$, with A'-module structure given by the morphism

$$A' \to A' \otimes_A B, \quad x \mapsto x \otimes 1.$$

Since a pushout of a cofibration is again a cofibration, the assumption that $A \to B$ is a cofibration implies that $A' \to (T, U)_0(B)$ is also a cofibration.

For the inductive step, assume that $A' \to (T, U)_{n-1}(B)$ is a cofibration. Then the composite

$$A \to A' \to (T, U)_{n-1}(B)$$

is also a cofibration, and $(T, U)_n(B)$ is the pushout of the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \\
(T, U)_{n-1}(B). & &
\end{array}
$$

$\square$

Corollary 4.6. *Consider a diagram in dga_k*

(4.6.1)
$$
\begin{array}{ccc}
A' & \longrightarrow & C \\
\downarrow & & \\
B, & &
\end{array}
$$

and let $A \to A'$ be a cofibration such that the composite $A \to A' \to B$ is also a cofibration. Then the homotopy colimit of the diagram 4.6.1 in dga_k is represented by

$$\mathrm{tot}((T, U).(B)) \otimes_{A'} C \simeq \mathrm{tot}((T, U).(B) \otimes_{A'} C).$$

Proof. Combine 2.13, 2.18, and 4.5. □

4.7. Let A be a cofibrant object in dga_k with two augmentations $x, y : A \to k$. Then we can apply the previous corollary to the map

$$A \to A \otimes_k A, \quad x \mapsto x \otimes 1,$$

and the diagram

(4.7.1)
$$
\begin{array}{ccc}
A \otimes_k A & \xrightarrow{x \otimes y} & k \\
\Big\downarrow{\scriptstyle m} & & \\
A. & &
\end{array}
$$

The simplicial differential graded algebra

$$\mathrm{tot}((T, U).(A) \otimes_{A \otimes A} k)$$

is called *the bar complex* of (A, x, y), and will be denoted by $B(A, x, y)$ (see for example [Wo, §3]). It follows from the above discussion that $B(A, x, y)$ represents the homotopy colimit in the category dga_k of the diagram 4.7.1.

The simplicial differential graded algebra

$$(T, U).(A) \otimes_{A \otimes A} k$$

can be described more explicitly as follows (compare with [Wo, §3]). We have

$$(T, U)_n(A) \otimes_{A \otimes A} k = \underbrace{A \otimes_k A \otimes \cdots \otimes A}_{n},$$

for $n \geq 1$, $(T, U)_0(A) \otimes_{A \otimes A} k = k$, and the simplicial structure is given by the maps

$$d_n(w_1 \otimes \cdots \otimes w_n) = x(w_1)w_2 \otimes \cdots \otimes w_n,$$

$$d_{n-i}(w_1 \otimes \cdots \otimes w_n) = w_1 \otimes \cdots \otimes w_i \cdot w_{i+1} \otimes \cdots \otimes w_n, \quad 0 < i < n,$$

$$d_0(w_1 \otimes \cdots \otimes w_n) = y(w_n)w_1 \otimes \cdots \otimes w_{n-1},$$

and

$$s_i(w_1 \otimes \cdots \otimes w_n) = w_1 \otimes \cdots \otimes w_{n-i} \otimes 1 \otimes w_{n-i+1} \otimes \cdots \otimes w_n, \quad 0 \leq i \leq n.$$

Remark 4.8. In the case when k is a field, the assumption that A is cofibrant in 4.7 is unnecessary. Indeed in this case any complex $M \in C(k)$ is K-flat in the sense of [Sp, 5.1]. In particular, if $A' \to A$ is an equivalence in dga_k, then the induced maps

$$\underbrace{A' \otimes_k A' \otimes \cdots \otimes A'}_{n} \to \underbrace{A \otimes_k A \otimes \cdots \otimes A}_{n}$$

are also equivalences, and so the map on bar complexes

$$B(A', x, y) \to B(A, x, y)$$

is also an equivalence.

5. Comparison of dga_k and $\mathrm{dga}_k^{\geq 0}$

5.1. Let k be a $\mathbb{Q}$-algebra, and let $A \in \mathrm{dga}_k^{\geq 0}$ be an object with two augmentations $x, y : A \to k$. Assume that A is cofibrant when viewed as an object of dga_k. We can then consider the diagram

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{x \otimes y} & k \\
\downarrow{\scriptstyle m} & & \\
A, & &
\end{array}
$$

which can be viewed either as a diagram in $\mathrm{dga}_k^{\geq 0}$ or in dga_k. We already know by 4.7 that the homotopy colimit in dga_k is represented by the bar complex $B(A, x, y)$.

5.2. Let

$$
J : \mathrm{dga}_k^{\geq 0} \to \mathrm{dga}_k
$$

be the inclusion functor. This functor has a left adjoint

$$
\tau_{\geq 0} : \mathrm{dga}_k \to \mathrm{dga}_k^{\geq 0},
$$

which sends $A \in \mathrm{dga}_k$ to the quotient of A by the differential graded ideal generated by elements $x \in A^i$, for $i < 0$.

In particular, the forgetful functor

$$
\sharp : \mathrm{dga}_k^{\geq 0} \to C(k)
$$

has a left adjoint given by the composite

$$
C(k) \xrightarrow{F} \mathrm{dga}_k \xrightarrow{\tau_{\geq 0}} \mathrm{dga}_k^{\geq 0}.
$$

Proposition 5.3. *There is a model category structure on $\mathrm{dga}_k^{\geq 0}$ in which a morphism $A \to B$ is an equivalence (resp. fibration) if and only if the induced map $A^\sharp \to B^\sharp$ in $C(k)$ is an equivalence (resp. fibration).*

Proof. We again apply [Hi, 2.2.1]. Condition (H0) is immediate. For condition (H1) we calculate $\tau_{\geq 0}\mathrm{Sym}^{\cdot}(I_d)$ where I_d is as in the proof of 2.5. If $d \geq 0$ then $\tau_{\geq 0}\mathrm{Sym}^{\cdot}(I_d) = \mathrm{Sym}^{\cdot}(I_d)$ and the statement that for $A \in \mathrm{dga}_k^{\geq 0}$ the map

$$
A^\sharp \to (A \otimes_k \tau_{\geq 0}\mathrm{Sym}^{\cdot}(I_d))^\sharp
$$

is an equivalence was already shown in 2.5. For $d < 0$ we have $\tau_{\geq 0}\mathrm{Sym}^{\cdot}(I_d) = k$ so the result also holds in this case. $\square$

5.4. Since the functor J clearly preserves equivalences and fibrations, the functor $\tau_{\geq 0}$ takes cofibrations to cofibrations, and we have derived functors (see for example [Ho, 1.3.4])

$$
\mathbb{L}\tau_{\geq 0} : \mathrm{Ho}(\mathrm{dga}_k) \to \mathrm{Ho}(\mathrm{dga}_k^{\geq 0}), \quad J : \mathrm{Ho}(\mathrm{dga}_k^{\geq 0}) \to \mathrm{Ho}(\mathrm{dga}_k).
$$

Given a diagram in $\mathrm{dga}_k^{\geq 0}$

(5.4.1)
$$
\mathscr{D} : \quad
\begin{array}{ccc}
A & \to & C \\
\downarrow & & \\
B, & &
\end{array}
$$

let hocolim($\mathscr{D}$) (resp. hocolim$^{\geq 0}(\mathscr{D})$) be the homotopy colimit of $\mathscr{D}$ in dga$_k$ (resp. dga$_k^{\geq 0}$). There is then a natural map

(5.4.2) $$\mathrm{hocolim}(\mathscr{D}) \to J(\mathrm{hocolim}^{\geq 0}(\mathscr{D})).$$

Lemma 5.5. *Suppose there exists factorizations in* dga$_k^{\geq 0}$

$$A \xrightarrow{\ a\ } C' \xrightarrow{\ c\ } C$$

and

$$A \xrightarrow{\ b\ } B' \xrightarrow{\ d\ } B$$

of $A \to C$ and $A \to B$ respectively, such that c and d are equivalences, and a and b define cofibrations in dga$_k$. *Then the map 5.4.2 is an isomorphism.*

Proof. We may without loss of generality assume that $C = C'$ and $B = B'$. Then observe that the maps a and b are also cofibrations in dga$_k^{\geq 0}$ as the functor J takes trivial fibrations to trivial fibrations. Therefore both sides of 5.4.2 are represented by $B \otimes_A C$. $\square$

6. SPACES OF PATHS

6.1. Let X be a simplicial set, and let $x, y : * \to X$ be two points. The space of paths between x and y can then be constructed as follows.

First we define a cosimplicial object X^I in $sSet$. Namely let X^I be the functor

$$\Delta \to sSet$$

sending

$$[n] \mapsto \prod_{\mathrm{Hom}_\Delta([n],[1])} X.$$

Note that

$$X^I([0]) = X \times X,$$

so we obtain a diagram in $(sSet)^\Delta$

$$\begin{array}{c} X^I \\ \downarrow \\ * \xrightarrow{\ x \times y\ } X \times X. \end{array}$$

Let $\widetilde{P}_{x,y} \in (sSet)^\Delta$ be the fiber product of this diagram, and define $P_{x,y} \in sSet$ to be the homotopy limit over Δ of $\widetilde{P}_{x,y}$. Then as explained in [Bo, X, 3.3 (i)] the simplicial set $P_{x,y}$ has geometric realization the space of paths between the two points x and y of $|X|$.

6.2. Either projection

$$X^I \longrightarrow X \times X \xrightarrow{\ \mathrm{pr}_i\ } X$$

is a homotopy equivalence. Indeed it suffices to show this after applying the geometric realization functor, in which case it follows from the fact that if T is a topological space then

the space of maps $[0,1] \to T$ is homotopy equivalent to T. It follows that the space of paths between x and y is equivalent to the homotopy fiber product of the diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow{\scriptstyle \Delta} \\
* & \xrightarrow{x \times y} & X \times X.
\end{array}
$$

6.3. We can perform a similar construction in $\mathrm{SPr}(k)$, where k is a $\mathbb{Q}$-algebra. Namely, let $\mathrm{SPr}(k)^\Delta$ denote the category of cosimplicial objects in $\mathrm{SPr}(k)$. This category has a simplicial model category structure in which cofibrations and equivalences are defined level-wise (see for example [To, §1.2]). If $X \in \mathrm{SPr}(k)$ is a fibrant object and $x, y : * \to X$ are two points, then as above we can form X^I, and the fiber product $\widetilde{P}_{x,y} \in \mathrm{SPr}(k)^\Delta$. We can then form $P_{x,y} := \mathrm{holim}\, \widetilde{P}_{x,y} \in \mathrm{Ho}(\mathrm{SPr}(k))$. By [To, 1.2.1], for every k-algebra R, the image of $P_{x,y}$ under the evaluation map

$$
\mathrm{Ho}(\mathrm{SPr}(k)) \to \mathrm{Ho}(sSet), \quad F \mapsto F(R)
$$

is isomorphic to the space of paths between x and y in $X(R)$.

As above we can also compute the space of paths between x and y as the homotopy fiber product of the diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow{\scriptstyle \Delta} \\
* & \xrightarrow{x \times y} & X \times X.
\end{array}
$$

6.4. In the case when $x, y : * \to X$ is obtained from a cofibrant object $A \in \mathrm{Alg}_k^\Delta$ with two augmentations $\epsilon_x, \epsilon_y : A \to k$, the path space $P_{x,y}$ admits the following algebraic description.

Recall from [To, 2.2.2] that the functor

$$
\mathrm{Spec}(-) : \mathrm{Alg}_k^{\Delta,o} \to \mathrm{SPr}(k)
$$

sending $A. \in \mathrm{Alg}_k^\Delta$ to the simplicial presheaf

$$
B \mapsto ([n] \mapsto \mathrm{Hom}_k(A_n, B))
$$

is a right Quillen functor and has a derived functor

$$
\mathbb{R}\mathrm{Spec} : \mathrm{Ho}(\mathrm{Alg}_k^\Delta)^o \to \mathrm{Ho}(\mathrm{SPr}(k)).
$$

The functor $\mathbb{R}\mathrm{Spec}(-)$ preserves homotopy limits (being a right Quillen functor), and therefore it follows that the path space is given by applying $\mathbb{R}\mathrm{Spec}(-)$ to the homotopy pushout of the diagram in Alg_k^Δ

$$
\begin{array}{ccc}
A \otimes_k A & \xrightarrow{m} & A \\
\downarrow{\scriptstyle x \otimes y} & & \\
k. & &
\end{array}
$$

6.5. If D denotes the differential graded algebra obtained from A using the functor of Thom-Sullivan cochains, and if we still write $x, y : D \to k$ for the augmentations, then it follows

that the differential graded algebra corresponding to $P_{x,y}$ is given by the homotopy pushout, in the category $\mathrm{dga}_k^{\geq 0}$, of the diagram

$$\begin{array}{ccc} D \otimes_k D & \xrightarrow{\ m\ } & D \\ \downarrow{\scriptstyle x \otimes y} & & \\ k. & & \end{array}$$

6.6. For $F \in \mathrm{SPr}(k)$, let $\pi_0^{\mathrm{sh}}(F)$ denote the sheaf associated to the presheaf

$$R \mapsto \pi_0(X(R)).$$

Let $\mathrm{Sh}(k)$ denote the category of sheaves on Aff_k, and let

$$j : \mathrm{Sh}(k) \hookrightarrow \mathrm{SPr}(k)$$

be the inclusion, sending a sheaf $\mathcal{F}$ to the constant simplicial sheaf

$$[n] \mapsto \mathcal{F}.$$

The functor j is right adjoint to the functor $\pi_0^{\mathrm{sh}}(-)$.

Lemma 6.7. *If $A \in \mathrm{Alg}_k^\Delta$ is an object such that the sheaf $\pi_0^{\mathrm{sh}}(\mathbb{R}\mathrm{Spec}(A))$ is representable, then $\pi_0^{\mathrm{sh}}(\mathbb{R}\mathrm{Spec}(A))$ is represented by $\mathrm{Spec}(H^0(A))$.*

Proof. If we view $\mathrm{Sh}(k)$ as a model category in which cofibrations are inclusions, fibrations are epimorphisms, and equivalences are isomorphisms, then the projection $\mathrm{Sh}(k) \to \mathrm{Ho}(\mathrm{Sh}(k))$ is an equivalence of categories, and the functor j is a right Quillen functor. Note also that $\pi_0^{\mathrm{sh}}(-)$ preserves arbitrary equivalences. We therefore obtain that for any $\mathcal{F} \in \mathrm{Sh}(k)$ and $\mathcal{G}. \in \mathrm{Ho}(\mathrm{SPr}(k))$ there is a natural isomorphism

(6.7.1) $$\mathrm{Hom}_{\mathrm{Sh}(k)}(\pi_0^{\mathrm{sh}}(G.), \mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{Ho}(\mathrm{SPr}(k))}(G., j(\mathcal{F})).$$

Similarly we have an inclusion

$$i : \mathrm{Alg}_k \hookrightarrow \mathrm{Alg}_k^\Delta$$

sending an algebra B to the constant cosimplicial algebra

$$[n] \mapsto B.$$

The functor i has a right adjoint sending A to $H^0(A)$. If we give Alg_k a model category structure in which fibrations are epimorphisms, equivalences are isomorphisms, and cofibrations are inclusions, then i is a right Quillen functor. Note also that

$$A \mapsto H^0(A)$$

preserves arbitrary equivalences. From this we obtain a natural isomorphism

(6.7.2) $$[B, A]_{\mathrm{Ho}(\mathrm{Alg}_k^\Delta)} \simeq \mathrm{Hom}_k(B, H^0(A)),$$

for any $B \in \mathrm{Alg}_k$ and $A \in \mathrm{Alg}_k^\Delta$.

Suppose S is a k-algebra such that $\mathrm{Spec}(S)$ represents the sheaf $\pi_0^{\mathrm{sh}}(\mathbb{R}\mathrm{Spec}(A))$. Then for any k-algebra T we have canonical isomorphisms

$$
\begin{aligned}
\mathrm{Hom}_k(T, S) \ &\simeq \ \mathrm{Hom}_{\mathrm{Aff}_k}(\mathrm{Spec}(S), \mathrm{Spec}(T)) \\
&\simeq \ \mathrm{Hom}_{\mathrm{Sh}(k)}(\mathrm{Spec}(S), \mathrm{Spec}(T)) \\
&\simeq \ [\mathbb{R}\mathrm{Spec}(A), \mathrm{Spec}(T)]_{\mathrm{Ho}(\mathrm{SPr}(k))} \quad \text{by (6.7.1)} \\
&\simeq \ [T, A]_{\mathrm{Ho}(\mathrm{Alg}_k^{\Delta})} \quad \text{by [To, 2.2.3]} \\
&\simeq \ \mathrm{Hom}_k(T, H^0(A)) \quad \text{by (6.7.2).}
\end{aligned}
$$

$\square$

6.8. Let $x, y : A \to k$ be an augmented object of Alg_k^{Δ} with associated augmented differential graded algebra D, and assume that the map $k \to H^0(D)$ is an isomorphism. Suppose that there exists a field k_0 and a pointed object $x_0, y_0 : A_0 \to k_0$ of Alg_k^{Δ} such that (A, x, y) is obtained by base change along a morphism $k_0 \to k$ from (A_0, x_0, y_0). Then the sheaf $\pi_0^{\mathrm{sh}}(P_{x,y})$ associated to the presheaf

$$R \mapsto \pi_0(P_{x,y}(R))$$

is representable, as this sheaf is a torsor under the sheaf associated to the presheaf

$$R \mapsto \pi_1(\mathbb{R}\mathrm{Spec}(A)(R), x),$$

which is representable by [To, 2.4.5].

We conclude that the sheaf $\pi_0^{\mathrm{sh}}(P_{x,y})$ is represented by the spectrum of the ring

$$H^0(E),$$

where E is the homotopy colimit is taken in the category $\mathrm{dga}_k^{\geq 0}$ of the diagram

$$
\begin{array}{ccc}
D \otimes_k D & \xrightarrow{\ m\ } & D \\
{\scriptstyle x \otimes y}\big\downarrow & & \\
k. & &
\end{array}
$$

7. Constructing cofibrant representatives when k is a field.

7.1. In general, cofibrations in $\mathrm{dga}_k^{\geq 0}$ are mysterious. However, in the case when k is a field of characteristic 0, which we assume for the remainder of this section, one can construction cofibrant representatives using the theory of minimal models (see for example [BG]). For the convenience of the reader we recall the argument. The main result of this section is the following proposition.

Proposition 7.2. *If k is a field and $a : A \to B$ is a morphism in $\mathrm{dga}_k^{\geq 0}$ with $H^0(A) = H^0(B) = k$, then there exists a factorization of a in $\mathrm{dga}_k^{\geq 0}$*

$$A \xrightarrow{\ a'\ } A' \xrightarrow{\ b\ } B,$$

where a' is a cofibration in dga_k and b is a trivial fibration.

The proof is in steps (7.3)–(7.5).

Lemma 7.3. *Let $A \in \mathrm{dga}_k^{\geq 0}$ be an object, let V be a k-vector space, and let $\alpha : V \to Z_A^q$ be a map to the closed elements Z_A^q in A^q. If the induced map*

$$V \to H^q(A)$$

is injective, then the natural map

$$A \to A\langle V[-q], \alpha \rangle$$

induces an isomorphism

$$H^i(A) \to H^i(A\langle V[-q], \alpha \rangle)$$

for $i < q$, where $A\langle V[-q], \alpha \rangle$ is defined as in 2.6.

Proof. Let us first consider the case $q = 1$. In this case

$$(A\langle V[-q], \alpha \rangle)^0 = A^0 \otimes_k \mathrm{Sym}^{\cdot} V.$$

For $s \geq 0$, let $F^s \subset \mathrm{Sym}^{\cdot} V$ be the k-subspace generated by monomials of degree $\leq s$. Then $F^{s-1} \subset F^s$, and the differential on $A^0 \langle V[-q], \alpha \rangle$ sends $A^0 \otimes F^s$ to $A^1 \otimes F^s$. In particular, if we write gr_F^s for F^s/F^{s-1}, then the differential on $A\langle V[-q], \alpha \rangle$ induces a map

$$A^0 \otimes \mathrm{gr}_F^s \to A^1 \otimes \mathrm{gr}^s F.$$

By construction this map is equal to $d_A \otimes 1$.

Let $z \in A^0 \otimes \mathrm{Sym}^{\cdot} V$ be a closed element, and write

$$z = z_0 + z_1 + \cdots + z_n,$$

where $z_i \in A^0 \otimes \mathrm{Sym}^i V$. Let $[z_n] \in A^0 \otimes \mathrm{gr}_F^n$ be the image of z_n. Then $(d_A \otimes 1)[z_n] = 0$ since z is closed. The assumption that $H^0(A) = k$ then implies that $z_n \in k \otimes_k \mathrm{Sym}^n V \subset A^0 \otimes \mathrm{Sym}^n V$. Let

$$\partial : \mathrm{Sym}^n V \to V \otimes \mathrm{Sym}^{n-1}(V)$$

be the map sending

$$v_1 \otimes \cdots \otimes v_n \mapsto \sum_{i=1}^n v_i \otimes (v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n).$$

Then it follows that the image under the differential of z_n in $A^1 \otimes \mathrm{Sym}^{\cdot} V$ is equal to the image of $\partial(z_n)$ under the inclusion

$$V \otimes \mathrm{Sym}^{n-1} V \hookrightarrow A^1 \otimes \mathrm{Sym}^{\cdot} V.$$

In particular, $\partial(z_n) \in A^1 \otimes \mathrm{Sym}^{n-1} V$. Furthermore, since z is closed we must have $[\partial(z_n)] = [-(d_a \otimes 1)[z_{n-1}]]$ in

$$A^1 \otimes \mathrm{gr}_F^{n-1}.$$

View $A^{\cdot} \otimes \mathrm{gr}_F^{n-1}$ as a complex with differential $d_A \otimes 1$. Then

$$H^i(A^{\cdot} \otimes \mathrm{gr}_F^{n-1}) \simeq H^i(A^{\cdot}) \otimes \mathrm{gr}_F^{n-1}.$$

In particular, since $V \to H^1(A)$ is injective, we must have $\partial(z_n) = 0$. This is impossible if $n > 0$. We conclude that $z \in A^0$, and therefore is in k.

Next consider the case $q > 1$. Note first of all that the map

$$A^i \to (A\langle V[-q], \alpha \rangle)^i$$

is an isomorphism for $i < q - 1$ and injective for all i. It follows that the map

$$H^i(A) \to H^i(A\langle V[-q], \alpha\rangle)$$

is an isomorphism for $i < q-1$, and injective for $i = q-1$. It remains to verify the surjectivity on H^{q-1}. For this observe that

$$(A\langle V[-q], \alpha\rangle)^{q-1} = A^{q-1} \oplus A^0 \otimes_k V,$$

and

$$(A\langle V[-q], \alpha\rangle)^q = \begin{cases} A^q \oplus A^1 \otimes V & \text{if } q > 2 \\ A^q \oplus A^1 \otimes V \oplus A^0 \otimes \bigwedge^2 V & \text{if } q = 2. \end{cases}$$

Let $(z, w) \in A^{q-1} \oplus A^0 \otimes V$ be a closed element. The image under the differential (the first component in the caes $q = 2$) is then equal to

$$(d_A z + \alpha(w), (d_A \otimes 1)(w)) \in A^q \oplus A^1 \otimes V.$$

Here we write $\alpha(w)$ (somewhat abusively) for the image of w under the composite

$$A^0 \otimes V \xrightarrow{1 \otimes \alpha} A^0 \otimes A^q \xrightarrow{\quad\cdot\quad} A^q,$$

where the second map is multiplication. We conclude that w is in the kernel of the map

$$(d_A \otimes 1) : A^0 \otimes V \to A^1 \otimes V,$$

and therefore since $H^0(A) = k$ we conclude that $w \in V = k \otimes V \subset A^0 \otimes V$. Therefore the image of w under

$$\alpha : V \to A^1$$

is a boundary. Since $V \to H^1(A)$ is injective this implies that $w = 0$. Therefore the map

$$H^{q-1}(A) \to H^{q-1}(A\langle V[-q], \alpha\rangle)$$

is surjective. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.4. We now construct inductively a sequence of morphisms in $\mathrm{dga}_k^{\geq 0}$

$$A = A'_{-1} \to A'_0 \to A'_1 \to \cdots,$$

with compatible morphisms $\rho_q : A'_q \to B$ such that the composite

$$A \longrightarrow A'_q \xrightarrow{\rho_q} B$$

is equal to a, and such that the following hold:

(i) For all q the map

$$A'_q \to A'_{q+1}$$

is a cofibration in dga_k.

(ii) The map

$$H^i(A'_q) \to H^i(B),$$

induced by ρ_q, is an isomorphism for $i \leq q$ and an injection for $i = q + 1$.

For $q = -1$ we take $A'_{-1} = A$.

So assume

$$A \longrightarrow A'_q \xrightarrow{\rho_q} B$$

has been constructed.

First choose a k-subspace $j : W \subset Z_B^{q+1}$ such that the induced map

$$H^{q+1}(A'_q) \oplus W \to H^{q+1}(B)$$

is an isomorphism. Let $\alpha : W \to A^{q+2}$ be the zero map, and set

$$C_{q+1} := A'_q \langle W[-q-2], \alpha \rangle.$$

We then have a factorization of ρ_q

$$A'_q \longrightarrow C_{q+1} \longrightarrow B,$$

where the second map is induced by j. Furthermore the induced map

$$H^i(C_{q+1}) \to H^i(B)$$

is an isomorphism for $i \le q + 1$. It then remains to arrange for the map on H^{q+2} to be injective.

For this let $V \subset H^{q+2}(C_{q+1})$ be the kernel of the map to $H^{q+2}(B)$, and choose a lifting $\alpha : V \to Z_{C_{q+1}}^{q+2}$. We can then find a factorization of $C_{q+1} \to B$ as

$$C_{q+1} \to C_{q+1}\langle V[-q-2], \alpha \rangle \to B.$$

By 7.3, the map $H^i(C_{q+1}) \to H^i(C_{q+1}\langle V[-q-2], \alpha \rangle)$ is an isomorphism for $i \le q+1$, and for $i = q+2$ the map sends V to zero. Repeating this construction inductively we get a sequence of cofibrations in dga_k

$$C_{q+1} = D_0 \to D_1 \to D_2 \to \cdots,$$

together with compatible maps $\sigma_t : D_t \to B$ such that the following hold:

(1) For every t, the map $H^i(D_t) \to H^i(D_{t+1})$ is an isomorphism for $i \le q + 1$.
(2) The map $H^{q+2}(D_t) \to H^{q+2}(D_{t+1})$ sends the kernel of $H^{q+2}(D_t) \to H^{q+2}(B)$ to zero.

We now define A'_{q+1} to be $\varinjlim_t D_t$ with the map $\rho_{q+1} : A'_{q+1} \to B$ the map defined by the σ_t.

7.5. Finally to get the factorization in 7.2 define $A' := \varinjlim A'_q$. $\qquad \square$

The main application for us of 7.2 is the following:

Corollary 7.6. *Let k be a field of characteristic 0, and let R be a k-algebra. Let $A \in \mathrm{dga}_R^{\ge 0}$ be an object with two augmentations $x, y : A \to R$, and consider the following diagram, which we shall denote $\mathscr{D}$,*

$$
\begin{array}{ccc}
A \otimes_R A & \xrightarrow{x \otimes y} & R \\
\downarrow{\scriptstyle m} & & \\
A. & &
\end{array}
$$

Suppose further that (A, x, y) is obtained by base change from an object $A_0 \in \mathrm{dga}_k^{\geq 0}$ with two augmentations $x_0, y_0 : A_0 \to k$, and such that $H^0(A_0) = k$. Then the natural map defined in 5.4.2

$$\mathrm{hocolim}(\mathscr{D}) \to J(\mathrm{hocolim}^{\geq 0}(\mathscr{D}))$$

is an isomorphism. In particular, if $\pi_0(P_{x,y})$ denotes the functor

$$\mathrm{Alg}_R \to \mathrm{Set}, \quad S \mapsto \pi_0(P_{x,y}(S)),$$

where $P_{x,y}$ is the path space of the augmented cosimplicial algebra associated to (A, x, y), then there is a canonical isomorphism

$$\pi_0(P_{x,y}) \simeq \mathrm{Spec}(H^0(B(A, x, y))).$$

Proof. Note that we also have $H^0(A_0 \otimes_k A_0) = k$. It then follows from 7.2 that the assumptions of 5.5 are satisfied for the diagram $\mathscr{D}$. $\qquad\square$

8. Application to p-adic Hodge theory

Let $(X, D)/V$ be as in 1.1, and let $x, y \in X^\circ(V)$ be two points. Assume further that X is geometrically connected. We can then consider various differential graded algebras associated to X°.

8.1. *De Rham realization.* Let $U. \to X_K$ be an étale hypercover with each U_n affine. We then obtain a cosimplicial differential graded algebra $DR(U.)$ by

$$[n] \mapsto \Gamma(U_n, \Omega^\cdot_{U_n/K}(\log)),$$

where $\Omega^\cdot_{U_n/K}(\log)$ denotes the de Rham complex with log poles along the preimage of D. We let A_{DR} denote the differential graded K-algebra obtained by applying the functor of Thom-Sullivan cochains to this cosimplicial differential graded algebra. The cohomology of A_{DR} is the de Rham cohomology of X_K^o/K.

Write also $x, y \in X(K)^o$ for the two K-valued points induced by the given V-valued points. For suitable choice of the hypercover U, we can lift these points to morphisms

$$x, y : \mathrm{Spec}(K) \to U..$$

For example we can take the hypercovering associated to a Zariski covering of X_K. These two points then define augmentations

$$x, y : A_{\mathrm{DR}} \to K.$$

Let $P_{x,y}^{\mathrm{DR}}$ be the space of paths obtained as in section 6 from (A_{DR}, x, y) by first converting A_{DR} into a cosimplicial algebra with augmentations, then applying the functor $\mathbb{R}\mathrm{Spec}$, and then taking π_0 of the space of paths between x and y. It follows from [Ol, 8.30] that $P_{x,y}^{\mathrm{DR}}$ has the following interpretation.

Let $\mathscr{C}_{\mathrm{DR}}$ denote the category of unipotent modules with integrable connection on X_K^o. It is a Tannakian category. The two points $x, y \in X^o(K)$ defines fiber functors

$$\omega_x, \omega_y : \mathscr{C}_{\mathrm{DR}} \to \mathrm{Vec}_K,$$

and the scheme $P_{x,y}^{\mathrm{DR}}$ represents the functor

$(K\text{-algebras}) \to \mathrm{Set}, \quad R \mapsto \{\text{set of isomorphisms of fiber functors } \omega_x \otimes R \to \omega_y \otimes R\}.$

8.2. The bar complex $B(A_{\mathrm{DR}}, x, y)$ can be described as follows. By 4.8, it is given by the total complex associated to the simplicial differential graded algebra

$$T(A_{\mathrm{DR}}, x, y) : [n] \mapsto \underbrace{A_{\mathrm{DR}} \otimes \cdots \otimes A_{\mathrm{DR}}}_{n}$$

described in 4.7. This simplicial differential graded algebra can also be described as follows. Let $(X, x, y)^{\cdot}$ be the cosimplicial scheme

$$[n] \mapsto X^n,$$

with transition maps given by

(8.2.1) $\qquad d^0(x_1, \ldots, x_n) = (x_1, \ldots, x_n, x), \quad d^{n+1}(x_1, \ldots, x_n) = (y, x_1, \ldots, x_n),$

$$d^i(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-i+1}, x_{n-i+1}, \ldots, x_n), \quad 0 < i < n+1,$$

and the codegeneracy operators defined by the projections.

We can then consider the de Rham complex of $(X, x, y)^{\cdot}_K$. This is again computed using a hypercovering $U_{\cdot} \to X$ as above. Namely, we can also consider the cosimplicial simplicial scheme

$$[n] \mapsto U^n_{\cdot}.$$

Taking the de Rham complex with log poles of each U^n_m defines a functor

$$\Delta^o \times \Delta \mapsto \mathrm{dga}_K, \quad ([n], [m]) \mapsto \Gamma(U^n_{m,K}, \Omega^{\cdot}_{U^n_{m,K}/K}).$$

Applying the functor of Thom-Sullivan cochains in the cosimplicial direction we obtain a simplicial differential graded algebra $T(U_{\cdot}, x, y)$. Note that there is a natural morphism

(8.2.2) $\qquad\qquad\qquad T(A_{\mathrm{DR}}, x, y) \to T(U_{\cdot}, x, y)$

which is a level-wise equivalence. We therefore obtain a canonical isomorphism

$$P^{\mathrm{DR}}_{x,y} \simeq \mathrm{Spec}(H^0(\mathrm{tot}(T(U_{\cdot}, x, y)))).$$

8.3. The advantage of considering the simplicial scheme $U_{\cdot}$ is that each differential graded algebra

$$\Gamma(U^n_{m,K}, \Omega^{\cdot}_{U^n_{m,K}/K})$$

is naturally an object of fdga_K, with filtration given by the Hodge filtration. Applying the functor of filtered Thom-Sullivan cochains, the simplicial differential graded algebra $T(U_{\cdot}, x, y)$ is naturally a simplicial object in fdga_K. We conclude that

$$H^0(\mathrm{tot}(T(U_{\cdot}, x, y))) \simeq \mathscr{O}_{P^{\mathrm{DR}}_{x,y}}$$

has a natural filtration, which we call the *Hodge filtration*.

Remark 8.4. A straightforward verification, which we leave to the reader, shows that the Hodge filtration on $\mathscr{O}_{P^{\mathrm{DR}}_{x,y}}$ is independent of the choices in the above construction.

8.5. *Étale realization.* Here we again start with an étale hypercover $U_{\cdot} \to X_K$ such that each $U^o_{\overline{K}}$ is a $K(\pi, 1)$ (see [Ol2, §5.1] for what this means). Given a collection E of geometric generic points of $X_{\overline{K}}$, the construction in [Ol, §5] defines a cosimplicial differential graded algebra

$$[n] \mapsto GC(U^o_{n,\overline{K}}, E).$$

Applying the functor of Thom-Sullivan cochains we obtain a differential graded algebra $A_{\mathrm{et}} \in \mathrm{dga}_{\mathbb{Q}_p}$. Moreover, there is a natural action of $\mathrm{Gal}_{\overline{K}/K}$ on the image of this algebra in $\mathrm{Ho}(\mathrm{dga}_{\mathbb{Q}_p})$

Two points $x, y : \mathrm{Spec}(K) \to X^o$ define augmentations $A_{\mathrm{et}} \to \mathbb{Q}_p$. Let $P^{\mathrm{et}}_{x,y}$ be the affine scheme associated to (A_{et}, x, y) as in section 4.

The Tannakian interpretation of $P^{\mathrm{et}}_{x,y}$ is the following. Let $\mathscr{C}_{\mathrm{et}}$ denote the category of unipotent sheaves of $\mathbb{Q}_p$-spaces on $X^o_{\overline{K}}$. This is a Tannakian category and the two points x and y define fiber functors

$$\omega_x, \omega_y : \mathscr{C}_{\mathrm{et}} \to \mathrm{Vec}_{\mathbb{Q}_p}.$$

By [Ol, 8.30], the scheme $P^{\mathrm{et}}_{x,y}$ represents the functor

$$(\mathbb{Q}_p\text{-algebras}) \to \mathrm{Set}, \quad R \mapsto \{\text{set of isomorphisms of fiber functors } \omega_x \otimes R \to \omega_y \otimes R\}.$$

On the other hand, by our earlier discussion there is a canonical isomorphism

$$P^{\mathrm{et}}_{x,y} \simeq \mathrm{Spec}(H^0(B(A_{\mathrm{et}}, x, y))).$$

8.6. We would like to again relate this to the cosimplicial scheme $(X^o_{\overline{K}}, x, y)^{\cdot}$, and then consider the p-adic comparison isomorphism for the cosimplicial scheme $(X, x, y)^{\cdot}$. This, however, requires some care with base points so let us begin by making some general observations.

Let

$$f : Y := \mathrm{Spec}(B) \to X := \mathrm{Spec}(A)$$

be a morphism of geometrically connected smooth V-schemes, and assume given divisors $D_Y \subset Y$ and $D_X \subset X$ with normal crossings relative to V such that $f^{-1}(D_X) \subset D_Y$. As usual we set

$$X^o := X - D_X, \quad Y^o := Y - D_Y.$$

Fix also geometric generic points

$$\eta_X : \mathrm{Spec}(k(\eta_X)) \to X_{\overline{K}}, \quad \eta_Y : \mathrm{Spec}(k(\eta_Y)) \to Y_{\overline{K}}.$$

Let $A \subset \overline{A}$ (resp. $B \subset \overline{B}$) be the integral closure of A (resp. B) in the maximal field extension $k(X) \subset L \subset k(\eta_X)$ (resp. $k(Y) \subset L \subset k(\eta_Y)$) such that the normalization of $X^o_{\overline{K}}$ (resp. $Y^o_{\overline{K}}$) in L is étale over $X^o_{\overline{K}}$ (resp. $Y^o_{\overline{K}}$).

Consider the fiber functors

$$\omega_X : \mathrm{Fet}(X^o_{\overline{K}}) \to \mathrm{Set}, \quad \omega_Y : \mathrm{Fet}(Y^o_{\overline{K}}) \to \mathrm{Set}$$

defined by the geometric points η_X and η_Y respectively.

Lemma 8.7. *There is a canonical bijection between morphisms $\bar{f} : \mathrm{Spec}(\overline{B}) \to \mathrm{Spec}(\overline{A})$ such that the diagram*

(8.7.1)
$$\begin{array}{ccc} \mathrm{Spec}(\overline{B}) & \xrightarrow{\bar{f}} & \mathrm{Spec}(\overline{A}) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

commutes, and isomorphisms σ between the two fiber functors

(8.7.2)
$$\omega_X, \omega_Y \circ f^* : \mathrm{Fet}(X^o_{\overline{K}}) \to \mathrm{Set}.$$

Proof. Let $A' \subset k(\eta_X)$ (resp. $B' \subset k(\eta_Y)$) be the coordinate ring of the integral closure of $X_{\overline{K}}^o$ (resp. $Y_{\overline{K}}^o$) in the maximal extension $L \subset k(\eta_X)$ (resp. $L \subset k(\eta_Y)$) such that the normalization is étale over $X_{\overline{K}}^o$ (resp. $Y_{\overline{K}}^o$). We have $\overline{A} \subset A'$ and $\overline{B} \subset B'$. Then $\mathrm{Spec}(A')$ (resp. $\mathrm{Spec}(B')$) is the universal covering of $X_{\overline{K}}^o$ (resp. $Y_{\overline{K}}^o$). We therefore have a canonical bijection between the set of isomorphisms of fiber functors σ between the two functors in 8.7.2, and dotted arrows filling in the following diagram

$$
\begin{array}{ccc}
\mathrm{Spec}(B') & \overset{f'}{\dashrightarrow} & \mathrm{Spec}(A') \\
\downarrow & & \downarrow \\
Y_{\overline{K}}^o & \longrightarrow & X_{\overline{K}}^o.
\end{array}
$$

Thus it remains to observe that there is a canonical bijection between the set of dotted arrows f' and the set of arrows $\bar{f}$ filling in the diagram 8.7.1. $\qquad\square$

Remark 8.8. Note that the bijection in 8.7 is compatible with the natural actions of the fundamental groups. Namely, by construction there is a natural action of $\pi_1(X_{\overline{K}}^o, \omega_X)$ (resp. $\pi_1(Y_{\overline{K}}^o, \omega_Y)$) on $\mathrm{Spec}(\overline{A})$ over X (resp. $\mathrm{Spec}(\overline{B})$ over Y). Given an isomorphism σ between the fiber functors in 8.7.2 we obtain a morphism of fundamental groups

$$\sigma_* : \pi_1(Y_{\overline{K}}^o, \omega_Y) \to \pi_1(X_{\overline{K}}^o, \omega_X)$$

and by the construction the morphism $\bar{f}$ corresponding to σ is compatible with the group actions.

8.9. Consider now a functor

$$U_\cdot^{(\cdot)} : \Delta \times \Delta^o \to (V\text{-schemes}), \quad ([n], [m]) \mapsto U_m^{(n)},$$

such that each $U_m^{(n)}$ is a finite disjoint union of affine smooth V-schemes with connected geometric generic fiber.

Consider also a second functor

$$E_\cdot^{(\cdot)} : \Delta \times \Delta^o \to (\overline{K}\text{-schemes})$$

such that for every n and m the scheme $E_n^{(m)}$ is a finite disjoint union of separably closed fields. Suppose further given a morphism of simplicial cosimplicial schemes

$$E_\cdot^{(\cdot)} \to U_{\cdot, \overline{K}}^{(\cdot)}$$

such that for every n and m the map

$$E_n^{(m)} \to U_{n, \overline{K}}^{(m)}$$

is a family of (not necessarily generic) geometric points of $U_{n, \overline{K}}^{(m)}$, in the sense of [Ol, 5.13].

Choose for every n and m, and for each connected component Γ of $U_{n, \overline{K}}^{(m)}$ a geometric generic point

$$\eta_{n, m, \Gamma} : \mathrm{Spec}(\Omega_{n, m, \Gamma}) \to \Gamma_{\overline{K}} \subset U_{n, \overline{K}}^{(m)}.$$

Choose furthermore for every $e \in E_n^{(m)}$ mapping to Γ_e an isomorphism

$$\iota_e : \omega_e \to \omega_{\eta_{n, m, \Gamma_e}}$$

between the two fiber functors

$$\mathrm{FEt}(\Gamma_{e,\overline{K}}) \to \mathrm{Set}.$$

Let π_{n,m,Γ_e} denote the fundamental group of $\Gamma_{e,\overline{K}}$ with respect to the base point η_{n,m,Γ_e}. Define

$$GC(U_{n,\overline{K}}^{(m)}, \mathbb{Q}_p) := \oplus_{e \in E_n^{(m)}} GC(U_{n,\overline{K}}^{(m)}, \eta_{n,m,\Gamma_e}, \mathbb{Q}_p).$$

This is a differential graded $\mathbb{Q}_p$-algebra. As n and m vary these algebras form a simplicial cosimplicial algebra as follows.

Consider morphisms in Δ

$$\delta_1 : [n'] \to [n], \quad \delta_2 : [m] \to [m']$$

defining a morphism

$$\delta : U_n^{(m)} \to U_{n'}^{(m')}.$$

For any $e \in E_n^{(m)}$ with image $\delta(e) \in E_{n'}^{(m')}$ we then have an isomorphism of fiber functors

$$\sigma_e : \omega_{\delta(e)} \to \omega_e \circ \delta^*.$$

Using the isomorphisms ι_e and $\iota_{\delta(e)}$ this defines an isomorphism of fiber functors

$$\sigma_e : \omega_{n',m',\Gamma_{\delta(e)}} \to \omega_{n,m,\Gamma_e} \circ \delta^*.$$

This in turn induces a morphism of differential graded algebras

$$GC(U_{n',\overline{K}}^{(m')}, \eta_{n',m',\Gamma_{\delta(e)}}, \mathbb{Q}_p) \to GC(U_{n,\overline{K}}^{(m)}, \eta_{n,m,\Gamma_e}, \mathbb{Q}_p).$$

Putting all these together we obtain a morphism of differential graded algebras

$$GC(U_{n',\overline{K}}^{(m')}, \mathbb{Q}_p) \to GC(U_{n,\overline{K}}^{(m)}, \mathbb{Q}_p).$$

It follows from the construction that these maps are compatible with composition so we obtain a cosimplicial simplicial differential graded algebra. Applying the functor of Thom-Sullivan cochains we obtain a simplicial differential graded algebra, which we shall denote by

$$T(U_{\cdot}^{(\cdot)}, \mathbb{Q}_p).$$

8.10. The advantage of this construction is that it works well also with other coefficient rings, such as Fontaine's ring A_{cris}.

Let $(X, D)/V$ be as in 1.1, and let $x, y : \mathrm{Spec}(V) \to X^o$ be two points. Choose an étale hypercovering $U_{\cdot} \to X$ with each U_n very small, in the sense of [Ol, 6.1]. Choose furthermore liftings

$$\mathrm{Spec}(V) \to U_{\cdot}$$

of x and y (this is possible as we can take U_0 as obtained from a Zariski covering of X). We then obtain a functor

$$(U_{\cdot}, x, y)^{\cdot} : \Delta \to \text{(simplicial log schemes)}$$

by

$$[n] \mapsto U_{\cdot}^n,$$

with transition maps defined as in 8.2.1.

We can also apply this construction with $\mathrm{Spec}(V) \coprod \mathrm{Spec}(V)$ with the two inclusions of $\mathrm{Spec}(V)$. Let

$$\Sigma^{\cdot} : \Delta \to \text{(schemes)}$$

be the resulting functor, so that

$$\Sigma^n = \coprod_{i=0}^{n} \mathrm{Spec}(V).$$

There is a natural map

$$\Sigma^{\cdot} \to (X, x, y)^{\cdot}$$

induced by x and y. For $[n], [m] \in \Delta$ define E_n^m to be

$$\coprod_{\mathrm{Hom}_{(X,x,y)^m}(\Sigma^m,(U_n,x,y)^m)} \Sigma^m.$$

The E_n^m define a simplicial cosimplicial scheme, there is a natural morphism

(8.10.1) $$E_{\cdot}^{\cdot} \to (U_{\cdot}, x, y)^{\cdot},$$

and each E_n^m is isomorphic to a finite disjoint union of copies of $\mathrm{Spec}(V)$. Furthermore, the induced map

$$E_{\cdot,\overline{K}}^{\cdot} \to (U_{\cdot,\overline{K}}^o, x, y)^{\cdot}$$

is a family of geometric points. Choosing generic base points

$$\eta_{n,m,\Gamma} : \mathrm{Spec}(\Omega_{n,m,\Gamma}) \to (U_{n,\overline{K}}^o, x, y)^m$$

and isomorphisms ι_e as in 8.9, we therefore obtain a simplicial differential graded algebra

$$T((U_{\cdot,\overline{K}}^o, x, y)^{\cdot}, \mathbb{Q}_p),$$

and a natural equivalence

(8.10.2) $$B(A_{\mathrm{et}}, x, y) \to \mathrm{tot}(T((U_{\cdot,\overline{K}}^o, x, y)^{\cdot}, \mathbb{Q}_p)).$$

8.11. Let $(U_{\cdot}^{\wedge}, x, y)^{\cdot}$ be the simplicial cosimplicial space sending $([n], [m]) \in \Delta \times \Delta^o$ to the p-adic completion of $(U_n, x, y)^m$. Note that because V is p-adically complete the morphism 8.10.1 lifts uniquely to a morphism

$$E_{\cdot}^{\cdot} \to (U_{\cdot}^{\wedge}, x, y)^{\cdot}.$$

We can furthermore choose geometric generic points

$$\hat{\eta}_{n,m,\Gamma} : \mathrm{Spec}(\widehat{\Omega}_{n,m,\Gamma}) \to (U_{n,\overline{K}}^{\wedge o}, x, y)^m$$

extending the $\eta_{n,m,\Gamma}$. We then obtain a simplicial differential graded algebra

$$T((U_{\cdot,\overline{K}}^{\wedge o}, x, y)^{\cdot}, \mathbb{Q}_p)$$

and there is a natural map

$$T((U_{\cdot,\overline{K}}^o, x, y)^{\cdot}, \mathbb{Q}_p) \to T((U_{\cdot,\overline{K}}^{\wedge o}, x, y)^{\cdot}, \mathbb{Q}_p).$$

For every n and m, we also have the Galois module

$$\mathrm{B}_{\mathrm{cris}}((U_n^{\wedge}, x, y)^m).$$

By 8.7, the Galois cohomology groups of these modules are functorial and we obtain another simplicial differential graded algebra

$$T((U_{\cdot}^{\wedge}, x, y)^{\cdot}, \mathrm{B}_{\mathrm{cris}}((U_{\cdot}^{\wedge}, x, y)^{\cdot})).$$

There are natural maps

$$T((U^o_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p) \to T((U^{\wedge o}_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p) \to T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, \mathrm{B}_{\mathrm{cris}}((U^{\wedge}_{\cdot}, x, y)^{\cdot})).$$

8.12. We can also consider the standard resolution $\mathscr{K} \to \mathbb{R}^{\bullet}$ of the structure sheaf in the convergent topos of $(X, M_X)/K$. By the same reasoning as in [Ol, 6.17] (and see also this reference for the notation), we then get a diagram of simplicial filtered differential graded algebras

$$T((U^o_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p) \otimes \mathrm{B}_{\mathrm{dR}}(V)$$

$$\downarrow$$

$$T((U^{\wedge o}_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p) \otimes \mathrm{B}_{\mathrm{dR}}(V)$$

$$\downarrow$$

$$T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, \mathrm{B}_{\mathrm{cris}}((U^{\wedge}_{\cdot}, x, y)^{\cdot})) \otimes_{\mathrm{B}_{\mathrm{cris}}(V)} \mathrm{B}_{\mathrm{dR}}(V)$$

$$\downarrow$$

$$T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, \mathbb{R}^{\bullet}(\mathrm{B}_{\mathrm{cris}}((U^{\wedge}_{\cdot}, x, y)^{\cdot}))) \otimes_{\mathrm{B}_{\mathrm{cris}}(V)} \mathrm{B}_{\mathrm{dR}}(V)$$

$$\uparrow$$

$$T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, \mathbb{R}^{\bullet}((U^{\wedge}_{\cdot}, x, y)^{\cdot}/K_0)_{\mathrm{cris}}) \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}(V)$$

$$\downarrow$$

$$T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, DR(\mathbb{R}^{\bullet})(U^{\wedge}_{\cdot}, x, y)^{\cdot}) \otimes_K \mathrm{B}_{\mathrm{dR}}(V)$$

$$\uparrow$$

$$T((U^{\wedge}_{\cdot}, x, y)^{\cdot}, DR(\mathscr{K})(U^{\wedge}_{\cdot}, x, y)^{\cdot}) \otimes_K \mathrm{B}_{\mathrm{dR}}(V)$$

$$\uparrow$$

$$T^{\mathrm{dR}}((U_{\cdot}, x, y)^{\cdot}) \otimes_K \mathrm{B}_{\mathrm{dR}}(V).$$

In degree n, this sequence of morphisms is the sequence defining the p-adic comparison isomorphism for X^n. In particular, all the maps in the above diagram are filtered quasi-isomorphisms, compatible with the action of Galois in the homotopy category. We therefore obtain an isomorphism

(8.12.1) $$H^0(T^{\mathrm{dR}}(U_{\cdot}, x, y)^{\cdot}) \otimes_K \mathrm{B}_{\mathrm{dR}}(V) \simeq H^0(T((U^o_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p)) \otimes \mathrm{B}_{\mathrm{dR}}(V)$$

compatible with the the Galois action and filtrations. On the other hand, we have a diagram

(8.12.2) $$H^0(T^{\mathrm{dR}}(U_{\cdot}, x, y)^{\cdot}) \otimes_K \mathrm{B}_{\mathrm{dR}}(V) \xrightarrow{(8.12.1)} H^0(T((U^o_{\cdot,\overline{K}}, x, y)^{\cdot}, \mathbb{Q}_p)) \otimes \mathrm{B}_{\mathrm{dR}}(V)$$

$$\downarrow{\scriptstyle(8.2.2)} \qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle(8.10.2)}$$

$$H^0(B(A_{\mathrm{dR}}, x, y)) \otimes_K \mathrm{B}_{\mathrm{dR}} \longrightarrow H^0(B(A_{\mathrm{et}}, x, y)) \otimes \mathrm{B}_{\mathrm{dR}}(V),$$

where the bottom horizontal row is the map obtained from the isomorphism in the homotopy category constructed in [Ol]

$$A_{\mathrm{dR}} \otimes_K \mathrm{B}_{\mathrm{dR}}(V) \simeq A_{\mathrm{et}} \otimes \mathrm{B}_{\mathrm{dR}}(V).$$

It follows from the construction that 8.12.2 commutes. In particular, we obtain the following:

Corollary 8.13. *The filtration on $\mathscr{O}_{P_{x,y}^{\mathrm{dR}}}$ induced by the isomorphism*

$$\mathscr{O}_{P_{x,y}^{\mathrm{dR}}} \otimes_K \mathrm{B}_{\mathrm{dR}}(V) \simeq \mathscr{O}_{P_{x,y}^{\mathrm{et}}} \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}(V)$$

constructed in [Ol], *agrees with the filtration defined using the bar construction in 8.3.*

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