THE SIX OPERATIONS FOR SHEAVES ON ARTIN STACKS II: ADIC COEFFICIENTS

YVES LASZLO AND MARTIN OLSSON

Abstract. In this paper we develop a theory of Grothendieck's six operations for adic constructible sheaves on Artin stacks continuing the study of the finite coefficients case in [12].

1. Introduction

In this paper we continue the study of Grothendieck's six operations for sheaves on Artin stacks begun in [12]. Our aim in this paper is to extend the theory of finite coefficients of loc. cit. to a theory for adic sheaves. In a subsequent paper [13] we will use this theory to study perverse sheaves on Artin stacks.

Throughout we work over a regular noetherian scheme S of dimension ≤ 1. In what follows, all stacks considered will be algebraic locally of finite type over S.

Let Λ be a complete discrete valuation ring and for every n let Λ_n denote the quotient $\Lambda/\mathfrak{m}^n$ so that $\Lambda = \varprojlim \Lambda_n$. We then define for any stack $\mathcal{X}$ a triangulated category $\mathbf{D}_c(\mathcal{X}, \Lambda)$ which we call the *derived category of constructible* Λ*–modules on* $\mathcal{X}$ (of course as in the classical case this is abusive terminology). The category $\mathbf{D}_c(\mathcal{X}, \Lambda)$ is obtained from the derived category of projective systems $\{\mathrm{F}_n\}$ of Λ_n–modules by localizing along the full subcategory of complexes whose cohomology sheaves are *AR-null* (see 2.1 for the meaning of this). For a morphism $f : \mathcal{X} \to \mathcal{Y}$ of finite type of stacks locally of finite type over S we then define functors

$$\mathrm{R}f_* : \mathbf{D}_c^+(\mathcal{X}, \Lambda) \to \mathbf{D}_c^+(\mathcal{Y}, \Lambda), \quad \mathrm{R}f_! : \mathbf{D}_c^-(\mathcal{X}, \Lambda) \to \mathbf{D}_c^-(\mathcal{Y}, \Lambda),$$

$$\mathrm{L}f^* : \mathbf{D}_c(\mathcal{Y}, \Lambda) \to \mathbf{D}_c(\mathcal{X}, \Lambda), \quad \mathrm{R}f^! : \mathbf{D}_c(\mathcal{Y}, \Lambda) \to \mathbf{D}_c(\mathcal{X}, \Lambda),$$

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}} : \mathbf{D}_c^-(\mathcal{X}, \Lambda)^{\mathrm{op}} \times \mathbf{D}_c^+(\mathcal{X}, \Lambda) \to \mathbf{D}_c^+(\mathcal{X}, \Lambda),$$

and

$$(-)\overset{\mathbf{L}}{\otimes}(-) : \mathbf{D}_c^-(\mathcal{X}, \Lambda) \times \mathbf{D}_c^-(\mathcal{X}, \Lambda) \to \mathbf{D}_c^-(\mathcal{X}, \Lambda)$$

satisfying all the usual adjointness properties that one has in the theory for schemes and the theory for finite coefficients.

In order to develop this theory we must overcome two basic problems. The first one is the necessary consideration of unbounded complexes which was already apparent in the finite coefficients case. The second one is the non-exactness of the projective limit functor. It should be noted that important previous work has been done on the subject, especially in [5] and [9] (see also [18] for the adic problems). In particular the construction of the normalization functor (3.3) used in this paper is due to Ekedahl [9]. None of these works, however, give an entirely satisfactory solution to the problem since for example cohomology with compact support and the duality theory was not constructed.

2. $\mathrm{R}\lim$ FOR UNBOUNDED COMPLEXES

Since we are forced to deal with unbounded complexes (in both directions) when considering the functor $\mathrm{R}f_!$ for Artin stacks, we must first collect some results about the unbounded derived category of projective systems of Λ–modules. The key tool is [12], §2.

2.1. **Projective systems.** Let $(\Lambda, \mathfrak{m})$ be a complete local regular ring and $\Lambda_n = \Lambda/\mathfrak{m}^{n+1}$. We demote by $\Lambda_\bullet$ the pro-ring $(\Lambda_n)_{n\geq 0}$. At this stage, we could have take any projective system of rings and Λ the projective limit. Let $\mathcal{X}/\mathrm{S}$ be a stack (by convention algebraic locally of finite type over S). For any topos $\mathcal{T}$, we will denote by $\mathcal{T}^{\mathbf{N}}$ the topos of projective systems of $\mathcal{T}$. These topos will be ringed by $\Lambda, \Lambda_\bullet$ respectively. We denote by π the morphism of ringed topos $\pi : \mathcal{T}^{\mathbf{N}} \to \mathcal{T}$ defined by $\pi^{-1}(\mathrm{F}) = (\mathrm{F})_n$, the constant projective system. One checks the formula

$$\pi_* = \varprojlim .$$

Recall that for any $\mathrm{F} \in \mathrm{Mod}(\mathcal{T}, \Lambda_\bullet)$, the sheaf $\mathrm{R}^i\pi_*\mathrm{F}$ is the sheaf associated to the presheaf $\mathrm{U} \mapsto \mathrm{H}^i(\pi^*\mathrm{U}, \mathrm{F})$. We'll use several times the fundamental exact sequence [8, 0.4.6]

$$0 \to \varprojlim{}^1 \mathrm{H}^{i-1}(\mathrm{U}, \mathrm{F}_n) \to \mathrm{H}^i(\pi^*\mathrm{U}, \mathrm{F}) \to \varprojlim \mathrm{H}^i(\mathrm{U}, \mathrm{F}_n) \to 0. \tag{2.1.1}$$

If $*$ denotes the punctual topos, then this sequence is obtained from the Leray spectral sequence associated to the composite

$$\mathrm{T}^{\mathbb{N}} \to *^{\mathbb{N}} \to *$$

and the fact that $\mathrm{R}^i \varprojlim$ is the zero functor for $i > 1$.

Recall that lisse-étale topos can be defined using the lisse-étale site Lisse-ét$(\mathcal{X})$ whose objects are smooth morphisms $\mathrm{U} \to \mathcal{X}$ such that U is an algebraic space of finite type over S.

Recall (cf. [3, exp. V]). that a projective system $\mathrm{M}_n, n \geq 0$ in an additive category is *AR-null* if there exists an integer r such that for every n the composite $\mathrm{M}_{n+r} \to \mathrm{M}_n$ is zero.

Definition 2.2. A complex M of $\mathrm{Mod}(\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}, \Lambda_\bullet)$ is

- *AR-null* if all the $\mathcal{H}^i(\mathrm{M})$'s are AR-null.

- *constructible* if all the $\mathcal{H}^i(\mathrm{M}_n)$'s ($i \in \mathbb{Z}, n \in \mathbf{N}$) are constructible.
- *almost zero* if for any $\mathrm{U} \to \mathcal{X}$ in Lisse-ét($\mathcal{X}$), the restriction of $\mathcal{H}^i(\mathrm{M})$ to Étale(U) is AR-null.

Observe that the cohomology sheaves $\mathcal{H}^i(\mathrm{M}_n)$ of a constructible complex are by definition cartesian.

Remark 2.3. A constructible complex M is almost zero if and only if its restriction to some presentation $\mathrm{X} \to \mathcal{X}$ is almost zero, meaning that there exists a covering of X by open subschemes U of finite type over S such that the restriction M_{U} of M to Étale(U) is AR-null.

2.4. **Restriction of** $\mathrm{R}\pi_*$ **to** U**.** Let $\mathrm{U} \to \mathcal{X}$ in Lisse-ét($\mathcal{X}$). The restriction of a complex M of $\mathcal{X}$ to Étale(U) is denoted as usual M_{U}.

Lemma 2.5. *One has* $\mathrm{R}\pi_*(\mathrm{M}_{\mathrm{U}}) = (\mathrm{R}\pi_*\mathrm{M})_{\mathrm{U}}$ *in* $\mathcal{D}(\mathrm{U}_{\text{ét}}, \Lambda)$.

Proof. We view U both as a sheaf on $\mathcal{X}$ or as the constant projective system $\pi^*\mathrm{U}$. With this identification, one has $(\mathcal{X}_{\text{lis-ét}|\mathrm{U}})^{\mathbf{N}} = (\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}})_{|\mathrm{U}}$ which we will denote by $\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}|\mathrm{U}}$. The following diagram commutes

$$\begin{tikzcd}
\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "j"] \arrow[d, "\pi"] & \mathcal{X}^{\mathbf{N}}_{\text{lis-ét}} \arrow[d, "\pi"] \\
\mathcal{X}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "j"] & \mathcal{X}_{\text{lis-ét}}
\end{tikzcd}$$

where j denotes the localization morphisms and π is as above. Because the left adjoint $j_!$ of j^* is exact, j^* preserves K-injectivity. We get therefore

$$\mathrm{R}\pi_* j^* = j^*\mathrm{R}\pi_*. \tag{2.5.1}$$

As before, the morphism of sites ϵ^{-1} : Étale(U) $\hookrightarrow$ Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$ and the corresponding one of total sites induces a commutative diagram

$$\begin{tikzcd}
\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "\epsilon"] \arrow[d, "\pi"] & \mathrm{U}^{\mathbf{N}}_{\text{ét}} \arrow[d, "\pi"] \\
\mathcal{X}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "\epsilon"] & \mathrm{U}_{\text{ét}}
\end{tikzcd}$$

Since ϵ_* is exact with an exact left adjoint ϵ^*, one has

$$\mathrm{R}\pi_*\epsilon_* = \epsilon_*\mathrm{R}\pi_*. \tag{2.5.2}$$

One gets therefore

$$\begin{aligned}(\mathrm{R}\pi_*\mathrm{M})_{\mathrm{U}} &= \epsilon_* j^*\mathrm{R}\pi_*\mathrm{M} \\ &= \epsilon_*\mathrm{R}\pi_* j^*\mathrm{M} \text{ by } (2.5.1) \\ &= \mathrm{R}\pi_*\epsilon_* j^*\mathrm{M} \text{ by } (2.5.2) \\ &= \mathrm{R}\pi_*(\mathrm{M}_{\mathrm{U}})\end{aligned}$$

☐

As before, let $\mathcal{T}$ be a topos and let $\mathcal{A}$ denote the category of $\Lambda_\bullet$–modules in $\mathcal{T}^{\mathbb{N}}$.

Proposition 2.6 (Lemma 1.1 of Ekedahl)**.** *Let* M *be a complex of* $\mathcal{A}$.

(1) *If* M *is AR-null, then* $\mathrm{R}\pi_*\mathrm{M} = 0$.
(2) *If* M *almost zero, then* $\mathrm{R}\pi_*\mathrm{M} = 0$.

Proof. Assume M is AR-null. By [9, lemma 1.1] $\mathrm{R}\pi_*\mathcal{H}^j(\mathrm{M}) = 0$ for all j. By [12, 2.1.10] one gets $\mathrm{R}\pi_*\mathrm{M} = 0$. The second point follows from (1) using 2.5. ☐

Lemma 2.7 (Lemma 1.3 iv) of Ekedahl)**.** *Let* M *be complex in* $\mathcal{T}$ *of* Λ_n*-modules. Then, the adjunction morphism* $\mathrm{M} \to \mathrm{R}\pi_*\pi^*\mathrm{M}$ *is an isomorphism.*

Remark 2.8. Here we view π is a morphism of ringed topos $(\mathcal{T}^{\mathbf{N}}, \Lambda_n) \to (\mathcal{T}, \Lambda_n)$. Then functor π^* sends a Λ_n–module M to the constant projective system M. In particular, π^* is exact (in fact equal to π^{-1}) and hence passes to the derived category.

Proof of 2.7: The sheaf $\mathrm{R}^i\pi_*\mathcal{H}^j(\pi^*\mathrm{M})$ is the sheaf associated to the presheaf sending U to $\mathrm{H}^i(\pi^*\mathrm{U}, \mathcal{H}^j(\pi^*\mathrm{M}))$. It follows from 2.1.1 and the fact that the system $\mathrm{H}^{i-1}(\mathrm{U}, \mathcal{H}^j(\pi^*\mathrm{M})_n)$ satisfies the Mittag-Leffler condition that this presheaf is isomorphic to the sheaf associated to the presheaf $\mathrm{U} \mapsto \varprojlim \mathrm{H}^i(\mathrm{U}, \mathcal{H}^j(\pi^*\mathrm{M})_n) = \mathrm{H}^i(\mathrm{U}, \mathcal{H}^j(\pi^*\mathrm{M})$. It follows that $\mathrm{R}^i\pi_*\mathcal{H}^j(\pi^*\mathrm{M}) = 0$ for all $i > 0$ and

$$(*) \qquad \mathcal{H}^j\mathrm{M} = \mathrm{R}\pi_*\mathcal{H}^j(\pi^*\mathrm{M}).$$

By [12, 2.1.10] one can therefore assume M bounded from below. The lemma follows therefore by induction from (*) and from the distinguished triangles

$$\mathcal{H}^j[-j] \to \tau_{\geq j}\mathrm{M} \to \tau_{\geq j+1}\mathrm{M}.$$

☐

In fact, we have the following stronger result:

Proposition 2.9. *Let* $\mathrm{N} \in \mathcal{D}(\mathcal{T}^{\mathbf{N}}, \Lambda_n)$ *be a complex of projective systems such that for every* m *the map*

$$\mathrm{N}_{m+1} \to \mathrm{N}_m \tag{2.9.1}$$

is a quasi–isomorphism. Then the natural map $\pi^*\mathrm{R}\pi_*\mathrm{N} \to \mathrm{N}$ *is an isomorphism. Consequently, the functors* $(\pi^*, \mathrm{R}\pi_*)$ *induce an equivalence of categories between* $\mathcal{D}(\mathcal{T}, \Lambda_n)$ *and the category of complexes* $\mathrm{N} \in \mathcal{D}(\mathcal{T}^{\mathbf{N}}, \Lambda_n)$ *such that the maps 2.9.1 are all isomorphism.*

Proof. By [12, 2.1.10] it suffices to prove that the map $\pi^*\mathrm{R}\pi_*\mathrm{N} \to \mathrm{N}$ is an isomorphism for N bounded below. By devissage using the distinguished triangles

$$\mathcal{H}^j(\mathrm{N})[j] \to \tau_{\geq j}\mathrm{N} \to \tau_{\geq j+1}\mathrm{N}$$

one further reduces to the case when N is a constant projective system of sheaves where the result is standard (and also follows from 2.7. ☐

3. λ-COMPLEXES

Following Behrend and [3, exp. V, VI], let us start with a definition. Let $\mathcal{X}$ be an algebraic stack locally of finite type over S, and let $\mathcal{A}$ denote the category of $\Lambda_\bullet$–modules in $\mathcal{X}^{\mathbb{N}}_{\text{lis-ét}}$.

Definition 3.1. We say that

- a system $\mathrm{M} = (\mathrm{M}_n)_n$ of $\mathcal{A}$ is *adic* if all the M_n's are constructible and moreover all morphisms
$$\Lambda_n \otimes_{\Lambda_{n+1}} \mathrm{M}_{n+1} \to \mathrm{M}_n$$
are isomorphisms; it is called *almost adic* if all the M_n's are constructible and if for every U in Lisse-ét($\mathcal{X}$) there is a morphism $\mathrm{N}_\mathrm{U} \to \mathrm{M}_\mathrm{U}$ with almost zero kernel and cokernel with N_U adic in $\mathrm{U}_{\text{ét}}$.
- a complex $\mathrm{M} = (\mathrm{M}_n)_n$ of $\mathcal{A}$ is called a λ-*complex* if all the cohomology modules $\mathcal{H}^i(\mathrm{M})$ are almost adic. Let $\mathcal{D}_c(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$ denote the full triangulated subcategory whose objects are λ–complexes. The full subcategory of $\mathcal{D}_c(\mathcal{A})$ of complexes concentrated in degree 0 is called the category of λ-*modules.*
- The category $\mathbf{D}_c(\mathcal{X}, \Lambda)$ (sometimes written just $\mathbf{D}_c(\mathcal{X})$ if the reference to Λ is clear) is the quotient of the category $\mathcal{D}_c(\mathcal{A})$ by the full subcategory of almost zero complexes.

Remark 3.2. Let X be a noetherian scheme. The condition that a sheaf of $\Lambda_\bullet$–modules M in $\mathrm{X}^{\mathbb{N}}_{\text{ét}}$ admits a morphism $\mathrm{N} \to \mathrm{M}$ with N adic is étale local on X. This follows from [3, V.3.2.3].

Furthermore, the category of almost adic $\Lambda_\bullet$-modules is an abelian subcategory closed under extensions (a Serre subcategory) of the category of all $\Lambda_\bullet$–modules in $\mathrm{X}^{\mathbb{N}}_{\text{ét}}$. From this it follows that for an algebraic stack $\mathcal{X}$, the category of almost adic $\Lambda_\bullet$–modules is a Serre subcategory of the category of all $\Lambda_\bullet$–modules in $\mathcal{X}^{\mathbb{N}}_{\text{lis-ét}}$.

In fact if M is almost adic on X, then the pair (N, u) of an adic sheaf N and an AR–isomorphism $u : \mathrm{N} \to \mathrm{M}$ is unique up to unique isomorphism. This follows from the description in [3, V.2.4.2 (ii)] of morphisms in the localization of the category of almost adic modules by the subcategory of AR-null modules. It follows that even when X is not quasi–compact, an almost adic sheaf M admits a morphism $\mathrm{N} \to \mathrm{M}$ with N adic whose kernel and cokernel are AR–null when restricted to an quasi–compact étale X–scheme.

As usual, we denote by Λ the image of $\Lambda_\bullet$ in $\mathbf{D}_c(\mathcal{X})$. By [3, exp V] the quotient of the subcategory of almost adic modules by the category of almost zero modules is abelian. By construction, a morphism $\mathrm{M} \to \mathrm{N}$ of $\mathcal{D}_c(\mathcal{A})$ is an isomorphism in $\mathbf{D}_c(\mathcal{X})$ if and only if its cone is almost zero. $\mathbf{D}_c(\mathcal{X})$ is a triangulated category and has a natural t-structure whose heart is the localization of the category of λ-modules by the full subcategory of almost zero systems (cf. [5]). Notice however that we do not know at this stage that in general $\mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M}, \mathrm{N})$ is a (small) set. In fact, this is equivalent to finding a left adjoint of the projection $\mathcal{D}_c(\mathcal{A}) \to \mathbf{D}_c(\mathcal{X})$ [15, section 7]. Therefore, we have to find a normalization functor $\mathrm{M} \to \hat{\mathrm{M}}$. We'll prove next that a suitably generalized version of Ekedahl's functor defined in [9] does the job. Note that by 2.6 the functor $\mathrm{R}\pi_* : \mathcal{D}_c(\mathcal{A}) \to \mathcal{D}_c(\mathcal{X})$ factors uniquely through a functor which we denote by the same symbols $\mathrm{R}\pi_* : \mathbf{D}_c(\mathcal{X}) \to \mathcal{D}_c(\mathcal{X})$.

Definition 3.3. We define the *normalization functor*

$$\mathbf{D}_c(\mathcal{X}) \to \mathcal{D}(\mathcal{A}), \quad \mathrm{M} \mapsto \hat{\mathrm{M}}$$

by the formula $\hat{\mathrm{M}} = \mathrm{L}\pi^*\mathrm{R}\pi_*\mathrm{M}$. A complex $\mathrm{M} \in \mathcal{D}(\mathcal{A})$ is *normalized* if the natural map $\hat{\mathrm{M}} \to \mathrm{M}$ is an isomorphism (where we write $\hat{\mathrm{M}}$ for the normalization functor applied to the image of M in $\mathbf{D}_c(\mathcal{X})$).

Notice that $\hat{\Lambda} = \Lambda$ (write $\Lambda_\bullet = \mathrm{L}\pi^*\Lambda$ and use 3.5 below for instance).

Remark 3.4. Because Λ is regular, the Tor dimension of π is $d = \dim(\Lambda) < \infty$ and therefore we do not have to use Spaltenstein's theory in order to define $\hat{\mathrm{M}}$.

Proposition 3.5 ([9, 2.2 (ii)]). *A complex* $\mathrm{M} \in \mathcal{D}(\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}, \Lambda_\bullet)$ *is normalized if and only if for all* n *the natural map*

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathrm{M}_{n+1} \to \mathrm{M}_n \tag{3.5.1}$$

is an isomorphism.

Proof. If $\mathrm{M} = \mathrm{L}\pi^*\mathrm{N}$ for some $\mathrm{N} \in \mathcal{D}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ then for all n we have $\mathrm{M}_n = \Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{N}$ so in this case the morphism 3.5.1 is equal to the natural isomorphism

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Lambda_{n+1} \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{N} \to \Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{N}.$$

This proves the "only if" direction.

For the "if" direction, note that since the functors e_n^* form a conservative set of functors, to verify that $\hat{\mathrm{M}} \to \mathrm{M}$ is an isomorphism it suffices to show that for every n the map $e_n^*\hat{\mathrm{M}} \to e_n^*\mathrm{M}$ is an isomorphism. Equivalently we must show that the natural map

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{R}\pi_*(\mathrm{M}) \to \mathrm{M}_n$$

is an isomorphism. As discussed in [9, bottom of page 198], the natural map $\mathrm{L}\pi^*\Lambda_n \to \pi^*\Lambda_n$ as AR–null cone. In the case when Λ is a discrete valuation ring with uniformizer λ, this can be seen as follows. A projective resolution of Λ_n is given by the complex

$$\Lambda \xrightarrow{\times\lambda^{n+1}} \Lambda.$$

¿From this it follows that $\mathrm{L}\pi^*(\Lambda_n)$ is represented by the complex

$$(\Lambda_m)_m \xrightarrow{\times\lambda^{n+1}} (\Lambda_m)_m.$$

Therefore the cone of $\mathrm{L}\pi^*(\Lambda_n) \to \pi^*\Lambda_n$ is in degrees $m \geq n$ and up to a shift equal to $\lambda^{m-n}\Lambda_m$ which is AR-null.

Returning to the case of general Λ, we obtain from the projection formula and 2.6

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{R}\pi_*(\mathrm{M}) \simeq e_n^{-1}\mathrm{R}\pi_*(\mathrm{L}\pi^*\Lambda_n \overset{\mathbf{L}}{\otimes} \mathrm{M}) \simeq e_n^{-1}\mathrm{R}\pi_*(\pi^*\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_\bullet} \mathrm{M}) = \mathrm{R}\pi_*(\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_\bullet} \mathrm{M}).$$

The proposition then follows from 2.9. □

We have a localization result analogous to lemma 2.5. Let $\mathrm{M} \in \mathcal{D}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$.

Lemma 3.6. *One has* $\mathrm{L}\pi^*(\mathrm{M}_\mathrm{U}) = (\mathrm{L}\pi^*\mathrm{M})_\mathrm{U}$ *in* $\mathcal{D}(\mathrm{U}^{\mathbf{N}}_{\text{ét}}, \Lambda_\bullet)$.

Proof. We use the notations of the proof of lemma 2.5. First, $j^* = \mathrm{L}j^*$ commutes with $\mathrm{L}\pi^*$ due to the commutative diagram

$$\begin{tikzcd}
\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "j"] \arrow[d, "\pi"] & \mathcal{X}^{\mathbf{N}}_{\text{lis-ét}} \arrow[d, "\pi"] \\
\mathcal{X}_{\text{lis-ét}|\mathrm{U}} \arrow[r, "j"] & \mathcal{X}_{\text{lis-ét}}
\end{tikzcd}.$$

One is therefore reduced to prove that $\epsilon_* = \mathrm{R}\epsilon_*$ commutes with $\mathrm{L}\pi^*$. We have certainly, with a slight abuse of notation,

$$\epsilon^{-1}\Lambda_\bullet = \Lambda_\bullet.$$

Therefore, if N denotes the restriction of M to $\mathcal{X}_{|\mathrm{U}}$ we get

$$\begin{aligned}
\epsilon_*\mathrm{L}\pi^*\mathrm{N} &= \epsilon_*(\Lambda_\bullet \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\Lambda} \pi^{-1}\mathrm{N}) \\
&= \epsilon_*(\epsilon^{-1}\Lambda_\bullet \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\Lambda} \pi^{-1}\mathrm{N}) \\
&= \Lambda_\bullet \overset{\mathbf{L}}{\otimes}_{\epsilon_*\pi^{-1}\Lambda} \epsilon_*\pi^{-1}\mathrm{N} \text{ by the projection formula} \\
&= \Lambda_\bullet \overset{\mathbf{L}}{\otimes}_{\pi^{-1}\Lambda} \pi^{-1}\epsilon_*\mathrm{N} \text{ because } \epsilon_* \text{ commutes with } \pi^{-1} \\
&= \mathrm{L}\pi^*\epsilon_*\mathrm{N}.
\end{aligned}$$

□

Remark 3.7. The same arguments used in the proof shows that if $\mathrm{M} \in \mathcal{D}_c(\mathcal{X}_{\text{lis-ét}}, \Lambda_\bullet)$ and M_U is bounded for $\mathrm{U} \in \text{Lisse-ét}(\mathcal{X})$, then $\hat{\mathrm{M}}_\mathrm{U}$ is also bounded. In particular, all $\hat{\mathrm{M}}_{\mathrm{U},n}$ are of finite tor-dimension.

Corollary 3.8. *Let* $\mathrm{M} \in \mathcal{D}(\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}, \Lambda_\bullet)$ *and* $\mathrm{U} \to \mathcal{X}$ *in* Lisse-ét$(\mathcal{X})$*. Then, the adjunction morphism*

$$\hat{\mathrm{M}} \to \mathrm{M}$$

restricts on $\mathrm{U}_{\text{ét}}$ *to the adjunction morphism* $\mathrm{L}\pi^*\mathrm{R}\pi_*\mathrm{M}_\mathrm{U} \to \mathrm{M}_\mathrm{U}$.

Proof. It is an immediate consequence of lemmas 3.6 and 2.5. □

We assume now that Λ is a discrete valuation ring with uniformizing parameter λ. Let us prove the analogue of [9, Proposition 2.2].

Theorem 3.9. *Let* M *be a* λ*-complex. Then,* $\hat{\mathrm{M}}$ *is constructible and* $\hat{\mathrm{M}} \to \mathrm{M}$ *has an almost zero cone.*

Proof. Let $\mathrm{U} \to \mathcal{X}$ in Lisse-ét($\mathcal{X}$) and

$$\mathrm{N} = \mathrm{M}_{\mathrm{U}} \in \mathcal{D}_c(\mathrm{U}_{\text{ét}}, \Lambda_\bullet).$$

Let us prove first that $(\hat{\mathrm{M}})_{\mathrm{U}} \in \mathcal{D}(\mathrm{U}_{\text{ét}})$ is constructible and that the cone of $(\hat{\mathrm{M}})_{\mathrm{U}} \to \mathrm{M}_{\mathrm{U}}$ is AR-null. We proceed by successive reductions.

(1) Let $d_{\mathrm{U}} = \mathrm{cd}_\ell(\mathrm{U}_{\text{ét}})$ be the ℓ-cohomological dimension of $\mathrm{U}_{\text{ét}}$. By an argument similar to the one used in the proof of 2.7 using 2.1.1, the cohomological dimension of $\mathrm{R}\pi_*$ is $\leq 1 + d_{\mathrm{U}}$. Therefore, $\mathrm{R}\pi_*$ maps $\mathcal{D}^{\pm,b}(\mathrm{U}_{\text{ét}}^{\mathbf{N}})$ to $\mathcal{D}^{\pm,b}(\mathrm{U}_{\text{ét}})$. Because $\mathrm{L}\pi^*$ is of finite cohomological dimension, the same is true for the normalization functor. More preciseley, there exists an integer d (depending only on $\mathrm{U} \to \mathcal{X}$ and Λ) such that for every a

$$\mathrm{N} \in \mathcal{D}^{\geq a}(\mathrm{U}_{\text{ét}}^{\mathbf{N}}) \Rightarrow \hat{\mathrm{N}} \in \mathcal{D}^{\geq a-d}(\mathrm{U}_{\text{ét}}^{\mathbf{N}}) \text{ and } \mathrm{N} \in \mathcal{D}^{\leq a}(\mathrm{U}_{\text{ét}}^{\mathbf{N}}) \Rightarrow \hat{\mathrm{N}} \in \mathcal{D}^{\leq a+d}(\mathrm{U}_{\text{ét}}^{\mathbf{N}}).$$

(2) One can assume $\mathrm{N} \in \mathrm{Mod}(\mathrm{U}_{\text{ét}}, \Lambda_\bullet)$. Indeed, one has by the previous observations

$$\mathcal{H}^i(\hat{\mathrm{N}}) = \mathcal{H}^i(\widehat{\mathrm{N}_i})$$

where $\mathrm{N}_i = \tau_{\geq i-d}\tau_{\leq i+d}\mathrm{N}$. Therefore one can assume N bounded. By induction, one can assume N is a λ-module.

(3) One can assume N adic. Indeed, there exists a morphism $\mathrm{A} \to \mathrm{N}$ with AR-null kernel and cokernel with A adic. In particular the cone of $\mathrm{A} \to \mathrm{N}$ is AR-null. It is therefore enough to observe that $\hat{\mathrm{A}} = \hat{\mathrm{N}}$, which is a consequence of 2.6.

(4) We use without further comments basic facts about the abelian category of λ-modules (cf. [3, exp. V] and [2, Rapport sur la formule des traces]).

In the category of λ-modules, there exists n_0 such that $\mathrm{N}/\ker(\lambda^{n_0})$ is torsion free (namely the action of λ has no kernel). Because $\mathbf{D}_c(\mathrm{U}_{\text{ét}})$ is triangulated, we just have to prove that the normalization of both $\mathrm{N}/\ker(\lambda^{n_0})$ and $\ker(\lambda^{n_0})$ are constructible and the corresponding cone is AR-null.

(5) The case of $\bar{\mathrm{N}} = \mathrm{N}/\ker(\lambda^{n_0})$. An adic representative L of $\bar{\mathrm{N}}$ has flat components L_n, in other words

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathrm{L}_{n+1} \to \mathrm{L}_n$$

is an isomorphism. By 3.5, L is normalized and therefore $\hat{\bar{\mathrm{N}}} = \hat{\mathrm{L}} = \mathrm{L}$ is constructible (even adic) and the cone $\mathrm{L} = \hat{\bar{\mathrm{N}}} \to \bar{\mathrm{N}}$ is AR-null because the kernel and cokernel of $\mathrm{L} \to \bar{\mathrm{N}}$ are AR-null.

(6) We can therefore assume $\lambda^{n_0}\mathrm{N} = 0$ (in the categories of λ-modules up to AR-isomorphisms) and even $\lambda\mathrm{N} = 0$ (look at the λ-adic filtration). The morphism

$$(\mathrm{N}_n)_{n\in\mathbf{N}} \to (\mathrm{N}_n/\lambda\mathrm{N}_n)_{n\in\mathbf{N}} \tag{3.9.1}$$

has AR-zero kernel and the normalization of both are therefore the same. But, N being adic, one has $\mathrm{N}_n/\lambda\mathrm{N}_n = \mathrm{N}_0$ for $n \geq 0$. In particular, the morphism 3.9.1 is nothing but

$$\mathrm{N} \to \pi^*\mathrm{N}_0 \tag{3.9.2}$$

and is an AR-isomorphism and

$$\hat{\mathrm{N}} = \widehat{\pi^*\mathrm{N}_0} = \mathrm{L}\pi^*\mathrm{N}_0$$

(2.7). One therefore has to show that the cone C of $\mathrm{L}\pi^*\mathrm{N}_0 \to \pi^*\mathrm{N}_0$ is almost zero. As before, one can assume replace $\mathcal{X}_{\text{lis-ét}}$ by $\mathrm{U}_{\text{ét}}$ for some affine scheme U. On U, there exists a *finite* stratification on which N_0 is smooth. Therefore, one can even assume that N_0 is constant and finally equal to Λ_0. In this case the cone of $\mathrm{L}\pi^*\Lambda_0 \to \pi^*\Lambda_0$ is AR-null by the same argument used in the proof of 3.5, and therefore this proves the first point.

We now have to prove that $\hat{\mathrm{M}}$ is cartesian. By 3.8 again, one is reduced to the following statement :

Let $f : \mathrm{V} \to \mathrm{U}$ be an $\mathcal{X}$-morphism in Lisse-ét$(\mathcal{X})$ which is smooth. Then,

$$f^*\widehat{\mathrm{M}_\mathrm{U}} = \widehat{\mathrm{M}_\mathrm{V}} = \widehat{f^*\mathrm{M}_\mathrm{U}}^1.$$

The same reductions as above allows to assume that M_U is concentrated in degree 0, and that we have a distinguished triangle

$$\mathrm{L} \to \mathrm{M}_\mathrm{U} \to \mathrm{C}$$

with C AR-null and L either equal to Λ_0 or adic with flat components. Using the exactness of f^* and the fact that M is cartesian, one gets a distinguished triangle

$$f^*\mathrm{L} \to \mathrm{M}_\mathrm{V} \to f^*\mathrm{C}$$

with $f^*\mathrm{C}$ AR-null. We get therefore $f^*\hat{\mathrm{M}}_\mathrm{U} = f^*\hat{\mathrm{L}}$ and $\widehat{f^*\mathrm{M}_\mathrm{U}} = \widehat{f^*\mathrm{L}}$: one can assume $\mathrm{M}_\mathrm{U} = \mathrm{L}$ and $\mathrm{M}_\mathrm{V} = f^*\mathrm{L}$. In both cases, namely L adic with flat components or $\mathrm{L} = \Lambda_0$, the computations above shows $f^*\hat{\mathrm{L}} = \widehat{f^*\mathrm{L}}$ proving that $\hat{\mathrm{M}}$ is cartesian. □

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[1] By 3.8, the notations there is no ambiguity in the notation.

Remark 3.10. The last part of the proof of the first point is proved in a greater generality in [9, lemma 3.2].

Remark 3.11. In general the functor $\mathrm{R}\pi_*$ does not take cartesian sheaves to cartesian sheaves. An example suggested by J. Riou is the following: Let $\mathrm{Y} = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field and $f : \mathrm{X} \to \mathrm{Y}$ a smooth k–variety. Let ℓ be a prime invertible in k and let $\mathrm{M} = (\mathrm{M}_n)$ be the projective system $\mathbb{Z}/\ell^{n+1}$ on Y. Then $\mathrm{R}\pi_*\mathrm{M}$ is the constant sheaf $\mathbb{Z}_\ell$, and so the claim $\mathrm{R}^i\Gamma(f^*\mathrm{R}\pi_*\mathrm{M})$ is the cohomology of X with values in the constant sheaf $\mathbb{Z}_\ell$. On the other hand, $\mathrm{R}^i\Gamma(\mathrm{R}\pi_*(f^*\mathrm{M}))$ is the usual ℓ–adic cohomology of X which in general does not agree with the cohomology with coefficients in $\mathbb{Z}_\ell$.

Corollary 3.12. *Let* $\mathrm{M} \in \mathbf{D}_c(\mathcal{X})$. *Then for any* $n \geq 0$, *one has*

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{R}\pi_*\mathrm{M} \in \mathcal{D}_c(\mathcal{X}, \Lambda_n).$$

Proof. Indeed, one has $e_n^{-1}\hat{\mathrm{M}} = \Lambda_n \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{R}\pi_*\mathrm{M}$ which is constructible by 3.9. ☐

We are now able to prove the existence of our adjoint.

Proposition 3.13. *The normalization functor is a left adjoint of the projection* $\mathcal{D}_c(\mathcal{X}^{\mathbf{N}}) \to \mathbf{D}_c(\mathcal{X})$. *In particular,* $\mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M}, \mathrm{N})$ *is small for any* $\mathrm{M}, \mathrm{N} \in \mathbf{D}_c(\mathcal{X})$.

Proof. With a slight abuse of notations, this means $\mathrm{Hom}_{\mathcal{D}_c}(\hat{\mathrm{M}}, \mathrm{N}) = \mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M}, \mathrm{N})$. If we start with a morphism $\hat{\mathrm{M}} \to \mathrm{N}$, we get a diagram

$$\begin{tikzcd}
& \hat{\mathrm{M}} \arrow[dl] \arrow[dr] & \\
\mathrm{M} & & \mathrm{N}
\end{tikzcd}$$

where $\hat{\mathrm{M}} \to \mathrm{M}$ is an isomorphism in $\mathbf{D}_c(\mathcal{X})$ by 3.8 and 3.9 which defines a morphism in $\mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M}, \mathrm{N})$. Conversely, starting from a diagram

$$\begin{tikzcd}
& \mathrm{L} \arrow[dl] \arrow[dr] & \\
\mathrm{M} & & \mathrm{N}
\end{tikzcd}$$

where $\mathrm{L} \to \mathrm{M}$ is an isomorphism in $\mathbf{D}_c(\mathcal{X})$. Therefore one has $\hat{\mathrm{M}} = \hat{\mathrm{L}}$ (2.6), and we get a morphism $\hat{\mathrm{M}} \to \hat{\mathrm{N}}$ in $\mathcal{D}_c$ and therefore, by composition, a morphism $\hat{\mathrm{M}} \to \mathrm{N}$. One checks that these construction are inverse each other. ☐

3.14. **Comparison with Deligne's approach.** Let $\mathrm{M}, \mathrm{N} \in \mathcal{D}_c(\mathcal{X}^{\mathbf{N}}, \Lambda_\bullet)$ and assume M is normalized. Then there is a sequence of morphisms

$$\mathrm{Rhom}(\mathrm{M}_n, \mathrm{N}_n) \to \mathrm{Rhom}(\mathrm{M}_n, \mathrm{N}_{n-1}) = \mathrm{Rhom}(\Lambda_{n-1} \overset{\mathbf{L}}{\otimes}_{\Lambda_n} \mathrm{M}_n, \mathrm{N}_{n-1}) = \mathrm{Rhom}(\mathrm{M}_{n-1}, \mathrm{N}_{n-1}).$$

Therefore, we get for each i a projective system $(\mathrm{Ext}^i(\mathrm{M}_n, \mathrm{N}_n))_{n \geq 0}$.

Proposition 3.15. *Let* $\mathrm{M}, \mathrm{N} \in \mathcal{D}_c(\mathcal{X}^{\mathbf{N}})$ *and assume* M *is normalized. Then there is an exact sequence*

$$0 \to \varprojlim{}^1 \mathrm{Ext}^{-1}(\mathrm{M}_n, \mathrm{N}_n) \to \mathrm{Hom}_{\mathcal{D}_c(\mathcal{X})}(\mathrm{M}, \mathrm{N}) \to \varprojlim \mathrm{Hom}_{\mathcal{D}(\mathcal{X}, \Lambda_n)}(\mathrm{M}_n, \mathrm{N}_n) \to 0.$$

Proof. Let $\mathcal{X}^{\leq n}_{\text{lis-ét}}$ be the $[0 \cdots n]$-simplicial topos of projective systems $(\mathrm{F}_m)_{m \leq n}$ on $\mathcal{X}_{\text{lis-ét}}$. Notice that the inclusion $[0 \cdots n] \to \mathbf{N}$ induces an open immersion of the corresponding topos and accordingly an open immersion

$$j_n : \mathcal{X}^{\leq n}_{\text{lis-ét}} \hookrightarrow \mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}. \tag{3.15.1}$$

The inverse image functor is just the truncation $\mathrm{F} = (\mathrm{F}_m)_{m \leq 0} \mapsto \mathrm{F}^{\leq n} = (\mathrm{F}_m)_{m \leq n}$. We get therefore an inductive system of open sub-topos of $\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}$:

$$\mathcal{X}^{\leq 0}_{\text{lis-ét}} \hookrightarrow \mathcal{X}^{\leq 1}_{\text{lis-ét}} \hookrightarrow \cdots \mathcal{X}^{\leq n}_{\text{lis-ét}} \hookrightarrow \cdots \mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}.$$

Fixing M, let

$$\mathrm{F} : \mathrm{Comp}^+(\mathcal{X}^{\mathbf{N}}, \Lambda_\bullet) \to \mathrm{Ab}^{\mathbf{N}}$$

be the functor

$$\mathrm{N} \mapsto \mathrm{Hom}(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}))_n.$$

Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Comp}^+(\mathcal{X}^{\mathbf{N}}, \Lambda_\bullet) \arrow[rr, "\mathrm{F}"] \arrow[dr, "\mathrm{Hom}(\mathrm{M},-)"'] & & \mathrm{Ab}^{\mathbf{N}} \arrow[dl, "\varprojlim"] \\
& \mathrm{Ab} &
\end{tikzcd}$$

which yields the equality

$$\mathrm{Rhom}(\mathrm{M}, \mathrm{N}) = \mathrm{R}\varprojlim \circ \mathrm{RF}(\mathrm{N}).$$

By the definition of F we have $\mathrm{R}^q\mathrm{F}(\mathrm{N}) = \mathrm{Ext}^q(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}))_n$. Because $\varprojlim$ is of cohomological dimension 1, there is an equality of functors $\tau_{\geq 0}\mathrm{R}\varprojlim = \tau_{\geq 0}\mathrm{R}\varprojlim \tau_{\geq -1}$.

Using the distinguished triangles

$$(\mathcal{H}^{-d}\mathrm{RF}(\mathrm{N}))[d] \to \tau_{\geq -d}\mathrm{RF}(\mathrm{N}) \to \tau_{\geq -d+1}\mathrm{RF}(\mathrm{N})$$

we get for $d = 1$ an exact sequence

$$0 \to \lim^1 \mathrm{Ext}^{-1}(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}) \to \mathrm{Hom}(\mathrm{M}, \mathrm{N}) \to \mathrm{R}^0 \varprojlim \tau_{\geq 0}\mathrm{RF}(\mathrm{N}) \to 0,$$

and for $d = 0$

$$\varprojlim \mathrm{Hom}(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}) = \mathrm{R}^0 \varprojlim \tau_{\geq 0}\mathrm{RF}(\mathrm{N})^{\cdot}$$

Therefore we just have to show the formula

$$\mathrm{Ext}^q(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}) = \mathrm{Ext}^q(\mathrm{M}_n, \mathrm{N}_n)$$

which follows from the following lemma which will also be useful below. □

Lemma 3.16. *Let* $\mathrm{M}, \mathrm{N} \in \mathcal{D}(\mathcal{X}^{\mathbf{N}})$ *and assume* M *is normalized. Then, one has*

1. $\mathrm{Rhom}(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}) = \mathrm{Rhom}(\mathrm{M}_n, \mathrm{N}_n)$.
2. $e_n^{-1} \mathcal{R}hom(\mathrm{M}, \mathrm{N}) = \mathcal{R}hom(\mathrm{M}_n, \mathrm{N}_n)$.

Proof. Let $\pi_n : \mathcal{X}^{\leq n}_{\text{lis-ét}} \to \mathcal{X}_{\text{lis-ét}}$ the restriction of π. It is a morphism of ringed topos ($\mathcal{X}$ is ringed by Λ_n and $\mathcal{X}^{\leq n}_{\text{lis-ét}}$ by $j_n^{-1}(\Lambda_\bullet) = (\Lambda_m)_{m \leq n}$). The morphisms $e_i : \mathcal{X} \to \mathcal{X}^{\mathbf{N}}, i \leq n$ can be localized in $\tilde{e}_i : \mathcal{X} \to \mathcal{X}^{\leq n}$, characterized by $e_i^{-1}\mathrm{M}^{\leq n} = \mathrm{M}_i$ for any object $\mathrm{M}^{\leq n}$ of $\mathcal{X}^{\leq n}_{\text{lis-ét}}$.They form a conservative sets of functors satisfying

$$e_i = j_n \circ \tilde{e}_i \tag{3.16.1}$$

One has

$$\pi_{n*}(\mathrm{M}^{\leq n}) = \varprojlim_{m \leq n} \mathrm{M}_m = \mathrm{M}_n = \tilde{e}_n^{-1}(\mathrm{M}).$$

It follows that π_{n*} is exact and therefore

$$\mathrm{R}\pi_{n*} = \pi_{n*} = \tilde{e}_n^{-1}. \tag{3.16.2}$$

The isomorphism $\mathrm{M}_n \to \mathrm{R}\pi_{n*}\mathrm{M}^{\leq n}$ defines by adjunction a morphism $\mathrm{L}\pi^*\mathrm{M}_n \to \mathrm{M}^{\leq n}$ whose pull back by $\tilde{e}_i$ is $\Lambda_i \overset{\mathbf{L}}{\otimes}_{\Lambda_n} \mathrm{M}_n \to \mathrm{M}_i$. Therefore, one gets

$$\mathrm{L}\pi_n^*\mathrm{M}_n = \mathrm{M}^{\leq n} \tag{3.16.3}$$

because M is normalized. Let us prove the first point. One has

$$\mathrm{Rhom}(\mathrm{M}^{\leq n}, \mathrm{N}^{\leq n}) \overset{3.16.3}{=} \mathrm{Rhom}(\mathrm{L}\pi_n^*\mathrm{M}_n, \mathrm{N}^{\leq n}) \overset{\text{adjunction}}{=} \mathrm{Rhom}(\mathrm{M}_n, \mathrm{R}\pi_{n*}\mathrm{N}^{\leq n}) \overset{3.16.2}{=} \mathrm{Rhom}(\mathrm{M}_n, \mathrm{N}_n).$$

proving the first point. The second point is analogous :

$$\begin{aligned} e_n^{-1}\,\mathcal{R}hom(\mathrm{M},\mathrm{N}) &= \tilde{e}_n^{-1}j_n^{-1}\,\mathcal{R}hom(\mathrm{M},\mathrm{N}) \text{ (3.16.1)} \\ &= \tilde{e}_n^{-1}\,\mathcal{R}hom(\mathrm{M}^{\leq n},\mathrm{N}^{\leq n}) \text{ } (j_n \text{ open immersion}) \\ &= \mathrm{R}\pi_{n*}\,\mathcal{R}hom(\mathrm{M}^{\leq n},\mathrm{N}^{\leq n}) \text{ (3.16.2)} \\ &= \mathrm{R}\pi_{n*}\,\mathcal{R}hom(\mathrm{L}\pi_n^*\mathrm{M}_n,\mathrm{N}^{\leq n}) \text{ (3.16.3)} \\ &= \mathcal{R}hom(\mathrm{M}_n,\mathrm{R}\pi_{n*}\mathrm{N}^{\leq n}) \text{ (projection formula)} \\ &= \mathcal{R}hom(\mathrm{M}_n,\mathrm{N}_n) \text{ (3.16.2)} \end{aligned}$$

□

Corollary 3.17. *Let* $\mathrm{M},\mathrm{N}\in\mathcal{D}_c(\mathcal{X}^{\mathbf{N}})$ *be normalized complexes. Then, one has an exact sequence*

$$0\to\varprojlim{}^1\operatorname{Ext}^{-1}(\mathrm{M}_n,\mathrm{N}_n)\to\operatorname{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M},\mathrm{N})\to\varprojlim\operatorname{Hom}_{\mathcal{D}(\mathcal{X},\Lambda_n)}(\mathrm{M}_n,\mathrm{N}_n)\to 0.$$

Remark 3.18. Using similar arguments (more precisely using Grothendieck spectral sequence of composite functors rather than truncations as above), one can show that for any adic constructible sheaf N, the cohomology group $\mathrm{H}^*(\mathcal{X},\mathrm{N})\overset{\text{def}}{=}\operatorname{Ext}^*_{\mathbf{D}_c(\mathcal{X})}(\Lambda,\mathrm{N})$ coincides with the continuous cohomology group of [18] (defined as the derived functor of $\mathrm{N}\mapsto\varprojlim\mathrm{H}^0(\mathcal{X},\mathrm{N}_n)$).

Now let k be either a finite field or an algebraically closed field, set $\mathrm{S}=\operatorname{Spec}(k)$, and let X be a k–variety. In this case Deligne defined in [7, 1.1.2] another triangulated category which we shall denote by $\mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X},\Lambda)$. This triangulated category is defined as follows. First let $\mathcal{D}^-_{\mathrm{Del}}(\mathrm{X},\Lambda)$ be the 2–categorial projective limit of the categories $\mathcal{D}^-(\mathrm{X},\Lambda_n)$ with respect to the transition morphisms

$$\overset{\mathbf{L}}{\otimes}_{\Lambda_n}\Lambda_{n-1}:\mathcal{D}^-(\mathrm{X},\Lambda_n)\to\mathcal{D}^-(\mathrm{X},\Lambda_{n-1}).$$

So an object K of $\mathcal{D}^-_{\mathrm{Del}}(\mathrm{X},\Lambda)$ is a projective system $(\mathrm{K}_n)_n$ with each $\mathrm{K}_n\in\mathrm{D}^-(\mathrm{X},\Lambda_n)$ and isomorphisms $\mathrm{K}_n\overset{\mathbf{L}}{\otimes}_{\Lambda_n}\Lambda_{n-1}\to\mathrm{K}_{n-1}$. The category $\mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X},\Lambda)$ is defined to be the full subcategory of $\mathcal{D}^-_{\mathrm{Del}}(\mathrm{X},\Lambda)$ consisting of objects $\mathrm{K}=(\mathrm{K}_n)$ with each $\mathrm{K}_n\in\mathcal{D}^b_c(\mathrm{X},\Lambda_n)$. By [7, 1.1.2 (e)] the category $\mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X},\Lambda)$ is triangulated with distinguished triangles defined to be those triangles inducing distinguished triangles in each $\mathcal{D}^b_c(\mathrm{X},\Lambda_n)$.

By 3.5, there is a natural triangluated functor

$$\mathrm{F}:\mathbf{D}_c(\mathrm{X},\Lambda)\to\mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X},\Lambda),\quad \mathrm{M}\mapsto\hat{\mathrm{M}}. \tag{3.18.1}$$

Lemma 3.19. *Let* $\mathrm{K} = (\mathrm{K}_n) \in \mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X}, \Lambda)$ *be an object.*

(i) For any integer i*, the projective system* $(\mathcal{H}^i(\mathrm{K}_n))_n$ *is almost adic.*

(ii) If $\mathrm{K}_0 \in \mathcal{D}^{[a,b]}_c(\mathrm{X}, \Lambda_0)$*, then for* $i < a$ *the system* $(\mathcal{H}^i(\mathrm{K}_n))_n$ *is AR-null.*

Proof. By the same argument used in [7, 1.1.2 (a)] it suffices to consider the case when $\mathrm{X} = \mathrm{Spec}(k)$. In this case, there exists by [3, XV, page 474 Lemme 1 and following remark] a bounded above complex of finite type flat Λ–modules $\mathrm{P}^{\cdot}$ such that K is the system obtained from the reductions $\mathrm{P}^{\cdot} \otimes \Lambda_n$. For a Λ–module M and an integer k let $\mathrm{M}[\lambda^k]$ denote the submodule of M of elements annihilated by λ^k. Then from the exact sequence

$$0 \longrightarrow \mathrm{P}^{\cdot} \xrightarrow{\lambda^k} \mathrm{P}^{\cdot} \longrightarrow \mathrm{P}^{\cdot}/\lambda^k \longrightarrow 0$$

one obtains for every n a short exact sequence

$$0 \to \mathrm{H}^i(\mathrm{P}^{\cdot}) \otimes \Lambda_n \to \mathrm{H}^i(\mathrm{K}_n) \to \mathrm{H}^{i+1}(\mathrm{P}^{\cdot})[\lambda^n] \to 0.$$

These short exact sequences fit together to form a short exact sequence of projective systems, where the transition maps

$$\mathrm{H}^{i+1}(\mathrm{P}^{\cdot})[\lambda^{n+1}] \to \mathrm{H}^{i+1}(\mathrm{P}^{\cdot})[\lambda^n]$$

are given by multiplication by λ. Since $\mathrm{H}^{i+1}(\mathrm{P}^{\cdot})$ is of finite type and in particular has bounded λ–torsion, it follows that the map of projective systems

$$\mathrm{H}^i(\mathrm{P}^{\cdot}) \otimes \Lambda_n \to \mathrm{H}^i(\mathrm{K}_n)$$

has AR-null kernel and cokernel. This proves (i).

For (ii), note that if $z \in \mathrm{P}^i$ is a closed element then modulo λ the element z is a boundary. Write $z = \lambda z' + d(a)$ for some $z' \in \mathrm{P}^i$ and $a \in \mathrm{P}^{i-1}$. Since P^{i+1} is flat over Λ the element z' is closed. It follows that $\mathrm{H}^i(\mathrm{P}^{\cdot}) = \lambda \mathrm{H}^i(\mathrm{P}^{\cdot})$. Since $\mathrm{H}^i(\mathrm{P}^{\cdot})$ is a finitely generated Λ–module, Nakayama's lemma implies that $\mathrm{H}^i(\mathrm{P}^{\cdot}) = 0$. Thus by (i) the system $\mathrm{H}^i(\mathrm{K}_n)$ is AR–isomorphic to 0 which implies (ii). □

Theorem 3.20. *The functor* F *in 3.18.1 is an equivalence of triangulated categories.*

Proof. Since the Ext^{-1}'s involved in 3.17 are finite dimensional for bounded constructible complexes, the full faithfulness follows from 3.17.

For the essential surjectivity, note first that any object $\mathrm{K} \in \mathcal{D}^b_{c,\mathrm{Del}}(\mathrm{X}, \Lambda)$ is induced by a complex $\mathrm{M} \in \mathcal{D}_c(\mathrm{X}^{\mathbf{N}}, \Lambda_\bullet)$ by restriction. For example represent each K_n by a homotopically injective complex I_n in which case the morphisms $\mathrm{K}_{n+1} \to \mathrm{K}_n$ defined in the derived category can be represented by actual maps of complexes $\mathrm{I}_{n+1} \to \mathrm{I}_n$. By 3.5 the complex M is normalized and by the preceding lemma the corresponding object of $\mathbf{D}_c(\mathrm{X},\Lambda)$ lies in $\mathbf{D}^b_c(\mathrm{X},\Lambda)$. It follows that if $\overline{\mathrm{M}}\in\mathbf{D}^b_c(\mathrm{X},\Lambda)$ denotes the image of M then K is isomorphic to $\mathrm{F}(\overline{\mathrm{M}})$. □

Remark 3.21. One can also define categories $\mathbf{D}_c(\mathcal{X},\mathbb{Q}_l)$. There are several different possible generalizations of the classical definition of this category for bounded complexes on noetherian schemes. The most useful generalizations seems to be to consider the full subcategory $\mathcal{T}$ of $\mathbf{D}_c(\mathcal{X},\mathbb{Z}_l)$ consisting of complexes K such that for every i there exists an integer $n\geq 1$ such that $\mathcal{H}^i(\mathrm{K})$ is annihilated by l^n. Note that if K is an unbounded complex there may not exist an integer n such that l^n annihilates all $\mathcal{H}^i(\mathrm{K})$. Furthermore, when $\mathcal{X}$ is not quasi–compact the condition is *not* local on $\mathcal{X}$. Nonetheless, by [15, 2.1] we can form the quotient of $\mathbf{D}_c(\mathcal{X},\mathbb{Z}_l)$ by the subcategory $\mathcal{T}$ and we denote the resulting triangulated category with t–structure (induced by the one on $\mathbf{D}_c(\mathcal{X},\mathbb{Z}_l)$) by $\mathbf{D}_c(\mathcal{X},\mathbb{Q}_l)$. If $\mathcal{X}$ is quasi–compact and $\mathrm{F},\mathrm{G}\in\mathbf{D}^b(\mathcal{X},\mathbb{Z}_l)$ one has

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{X},\mathbb{Z}_l)}(\mathrm{F},\mathrm{G})\otimes\mathbb{Q}\simeq\mathrm{Hom}_{\mathbf{D}^b(\mathcal{X},\mathbb{Q}_l)}(\mathrm{F},\mathrm{G}).$$

Using a similar 2–categorical limit method as in [7, 1.1.3] one can also define a triangulated category $\mathbf{D}_c(\mathcal{X},\overline{\mathbb{Q}}_l)$.

4. $\mathcal{R}hom$

We define the bifunctor

$$\mathcal{R}hom:\mathbf{D}_c(\mathcal{X})^{\mathrm{opp}}\times\mathbf{D}_c(\mathcal{X})\to\mathcal{D}(\mathcal{X})$$

by the formula

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})=\mathcal{R}hom_{\Lambda_\bullet}(\hat{\mathrm{M}},\hat{\mathrm{N}}).$$

Recall that $\mathcal{D}_c(\mathcal{X},\Lambda_n)$ denotes the usual derived category of complexes of Λ_n-modules with constructible cohomology.

Proposition 4.1. *Let* $\mathrm{M}\in\mathbf{D}^-_c(\mathcal{X})$ *and* $\mathrm{N}\in\mathbf{D}^+_c(\mathcal{X})$, *then* $\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})$ *has constructible cohomology and is normalized. Therefore, it defines an additive functor*

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}:\mathbf{D}^-_c(\mathcal{X})^{\mathrm{opp}}\times\mathbf{D}^+_c(\mathcal{X})\to\mathbf{D}^+_c(\mathcal{X}).$$

Proof. One can assume M, N normalized. By 3.16, one has the formula

$$e_n^{-1}\,\mathcal{R}hom(\mathrm{M},\mathrm{N})=\mathcal{R}hom(e_n^{-1}\mathrm{M},e_n^{-1}\mathrm{N}). \tag{4.1.1}$$

¿From this it follows that $\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})$ has constructible cohomology.

By 4.1.1 and 3.5, to prove that $\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})$ is normalized we have to show that

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_{n+1} \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{M}, \Lambda_{n+1} \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{N}) \to \mathcal{R}hom_{\Lambda_n}(\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{M}, \Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{N})$$

is an isomorphism. By 3.12, both $\mathrm{M}_{n+1} = \Lambda_{n+1} \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{M}$ and $\mathrm{N}_{n+1} = \Lambda_{n+1} \overset{\mathbf{L}}{\otimes}_{\Lambda} \mathrm{R}\pi_*\mathrm{N}$ are constructible complexees of Λ_{n+1} sheaves on $\mathrm{U}_{\text{ét}}$. One is reduced to the formula

$$\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathcal{R}hom_{\Lambda_{n+1}}(\mathrm{M}_{n+1}, \mathrm{N}_{n+1}) \to \mathcal{R}hom_{\Lambda_n}(\Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathrm{M}_{n+1}, \Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \mathrm{N}_{n+1})$$

for our constructible complexes M,N on $\mathrm{U}_{\text{ét}}$. This assertion is well-known (and is easy to prove), cf. [3, lemma II.7.1, II.7.2]. □

Remark 4.2. Using almost the same proof, one can define a functor

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}} : \mathbf{D}_c^b(\mathcal{X})^{\text{opp}} \times \mathbf{D}_c(\mathcal{X}) \to \mathbf{D}_c(\mathcal{X}).$$

5. $\mathbf{Rhom}_{\boldsymbol{\Lambda}}$

Let M,N in $\mathbf{D}_c^-(\mathcal{X}), \mathbf{D}_c^+(\mathcal{X})$ respectively. We define the functor

$$\mathbf{Rhom}_{\boldsymbol{\Lambda}} : \mathbf{D}_c^{-\text{opp}}(\mathcal{X}) \times \mathbf{D}_c^+(\mathcal{X}) \to \mathrm{Ab}$$

by the formula

$$\mathbf{Rhom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N}) = \mathrm{Rhom}_{\Lambda_\bullet}(\hat{\mathrm{M}}, \hat{\mathrm{N}}). \tag{5.0.1}$$

By 3.13, one has

$$\mathrm{H}^0\, \mathbf{Rhom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N}) = \mathrm{Hom}_{\mathcal{D}_c(\mathcal{X}^{\mathbf{N}})}(\hat{\mathrm{M}}, \hat{\mathrm{N}}) = \mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M},\mathrm{N}).$$

One has

$$\mathrm{Rhom}_{\Lambda_\bullet}(\hat{\mathrm{M}}, \hat{\mathrm{N}}) = \mathrm{Rhom}_{\Lambda_\bullet}(\Lambda_\bullet, \mathcal{R}hom(\hat{\mathrm{M}}, \hat{\mathrm{N}})).$$

By 4.1, $\mathcal{R}hom(\hat{\mathrm{M}}, \hat{\mathrm{N}})$ is constructible and normalized. Taking H^0, we get the formula

$$\mathrm{Hom}_{\mathcal{D}_c(\mathcal{X}^{\mathbf{N}})}(\hat{\mathrm{M}}, \hat{\mathrm{N}}) = \mathrm{Hom}_{\mathcal{D}_c(\mathcal{X}^{\mathbf{N}})}(\Lambda_\bullet, \mathcal{R}hom(\hat{\mathrm{M}}, \hat{\mathrm{N}})).$$

By 3.13, we get therefore the formula

$$\mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M},\mathrm{N}) = \mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\Lambda, \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})) \tag{5.0.2}$$

In summary, we have gotten the following result.

Proposition 5.1. *Let* M,N *in* $\mathbf{D}_c^-(\mathcal{X}), \mathbf{D}_c^+(\mathcal{X})$ *respectively.One has*

$$\mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M},\mathrm{N}) = \mathrm{H}^0\, \mathbf{Rhom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N}) = \mathrm{Hom}_{\mathbf{D}_c(\mathcal{X})}(\Lambda, \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M},\mathrm{N})).$$

Remark 5.2. Accordingly, one defines

$$\boldsymbol{\mathcal{E}xt_\Lambda}^*(\mathrm{M},\mathrm{N}) = \mathcal{H}^*(\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{M},\mathrm{N})) \text{ and } \mathbf{Ext_\Lambda}^*(\mathrm{M},\mathrm{N}) = \mathcal{H}^*(\mathbf{Rhom_\Lambda}(\mathrm{M},\mathrm{N}))$$

and

$$\mathbf{Hom_\Lambda}(\mathrm{M},\mathrm{N}) = \mathrm{H}om_{\mathbf{D}_c(\mathcal{X})}(\mathrm{M},\mathrm{N}) = \mathcal{H}^0(\mathbf{Rhom_\Lambda}(\mathrm{M},\mathrm{N})).$$

6. Tensor product

Let $\mathrm{M},\mathrm{N} \in \mathbf{D}_c(\mathcal{X})$. We define the total tensor product

$$\mathrm{M} \overset{\mathbf{L}}{\otimes}_\Lambda \mathrm{N} = \hat{\mathrm{M}} \overset{\mathbf{L}}{\otimes}_{\Lambda_\bullet} \hat{\mathrm{N}}.$$

It defines a bifunctor

$$\mathbf{D}_c(\mathcal{X}) \times \mathbf{D}_c(\mathcal{X}) \to \mathcal{D}_{\mathrm{cart}}(\mathcal{X}).$$

Proposition 6.1. *For any* $\mathrm{L},\mathrm{N},\mathrm{M} \in \mathbf{D}_c(\mathcal{X},\Lambda)$ *we have*

$$\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L} \overset{\mathbf{L}}{\otimes} \mathrm{N},\mathrm{M}) \simeq \boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L},\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{N},\mathrm{M})).$$

Proof. By definition this amounts to the usual adjunction formula

$$\mathcal{R}hom(\hat{\mathrm{L}} \overset{\mathbf{L}}{\otimes} \hat{\mathrm{N}},\hat{\mathrm{M}}) \simeq \mathcal{R}hom(\hat{\mathrm{L}},\mathcal{R}hom(\hat{\mathrm{N}},\hat{\mathrm{M}})).$$

□

Corollary 6.2. *For any* $\mathrm{L},\mathrm{M} \in \mathbf{D}_c(\mathcal{X},\Lambda)$ *there is a canonical evaluation morphism*

$$\mathrm{ev} : \boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L},\mathrm{M}) \overset{\mathbf{L}}{\otimes} \mathrm{L} \to \mathrm{M}.$$

Proof. The morphism ev is defined to be the image of the identity map under the isomorphism

$$\boldsymbol{\mathcal{R}hom_\Lambda}(\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L},\mathrm{M}),\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L},\mathrm{M})) \simeq \boldsymbol{\mathcal{R}hom_\Lambda}(\boldsymbol{\mathcal{R}hom_\Lambda}(\mathrm{L},\mathrm{M}) \overset{\mathbf{L}}{\otimes} \mathrm{L},\mathrm{M})$$

provided by 6.1. □

7. Duality

Let's denote by $f : \mathcal{X} \to \mathrm{S}$ the structural morphism. Let

$$\Omega_n = f^!\Lambda_n(\dim(\mathrm{S}))[2\dim(\mathrm{S})] \tag{7.0.1}$$

be the (relative) dualizing complex of $\mathcal{X}$ (ringed by Λ_n). Notice that $f^*\Lambda_n = \Lambda_n$, with a slight abuse of notation.

7.1. **Construction of the dualizing complex.**

Proposition 7.2. *One has* $\Omega_n = \Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Omega_{n+1}$.

Proof. The key point is the following lemma:

Lemma 7.3. *Let* M *be a complex of sheaves of* Λ_{n+1}*–modules of finite injective dimension. Then there is a canonical isomorphism*

$$\mathrm{M} \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Lambda_n \simeq \mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_n, \mathrm{M}).$$

Proof. Let S denote the acyclic complex on $\mathcal{X}$

$$\cdots \Lambda_{n+1} \xrightarrow{l^{n+1}} \Lambda_{n+1} \xrightarrow{l} \Lambda_{n+1} \xrightarrow{l^{n+1}} \Lambda_{n+1} \rightarrow \cdots,$$

where the map $\mathrm{S}^i \rightarrow \mathrm{S}^{i+1}$ is given by multipliciation by l if i is even and multiplication by l^{n+1} if i is odd. Let P denote the truncation $\sigma_{\leq 0}\mathrm{S}$ (the terms of S in degrees ≤ 0). Then P is a projective resolution of Λ_n viewed as a Λ_{n+1}–module and $\widehat{\mathrm{P}} := \mathrm{Hom}^{\cdot}(\mathrm{P}, \Lambda_{n+1})$ is isomorphic to $\sigma_{\geq 1}\mathrm{S}$ and is also a resolution of Λ_n. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Lambda_{n+1} \arrow[r, "l^{n+1}"] \arrow[d] & \Lambda_{n+1} \arrow[r] \arrow[d, "\times l"'] & 0 \arrow[d] \\
& 0 \arrow[r] & \Lambda_{n+1} \arrow[r, "l^{n+1}"] & \Lambda_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

defines a morphism of complexes $\mathrm{P} \rightarrow \widehat{\mathrm{P}}[1]$ whose cone is quasi–isomorphic to S[1]. Let M be a boundex complex of injectives. We then obtain a morphism

$$\mathrm{M} \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Lambda_n \simeq \mathrm{M} \otimes \mathrm{P} \simeq \mathrm{Hom}^{\cdot}(\widehat{\mathrm{P}}, \mathrm{M}) \rightarrow \mathrm{Hom}^{\cdot}(\mathrm{P}, \mathrm{M}) \simeq \mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_n, \mathrm{M}). \tag{7.3.1}$$

The cone of this morphism is isomorphic to $\mathcal{R}hom(\mathrm{S}, \mathrm{M})$ which is zero since S is acyclic. It follows that 7.3.1 is an isomorphism. □

In particular

$$\mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_n, f^!\Lambda_{n+1}) = \Lambda_n \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} f^!(\Lambda_{n+1}).$$

On the other hand, by [12, 4.4.3], one has

$$\mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_n, f^!\Lambda_{n+1}) = f^!\,\mathcal{R}hom_{\Lambda_{n+1}}(\Lambda_n, \Lambda_{n+1}) = f^!\Lambda_n.$$

Twisting and shifting, one gets 7.2. □

7.4. Let $\mathrm{U} \to \mathcal{X}$ be an object of Lisse-ét$(\mathcal{X})$ and let $\epsilon : \mathcal{X}|_{\mathrm{U}} \to \mathrm{U}_{\text{ét}}$ be the natural morphism of topos. Let us describe more explicitly the morphism ϵ. Let Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$ denote the category of morphisms $\mathrm{V} \to \mathrm{U}$ in Lisse-ét$(\mathcal{X})$. The category Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$ has a Grothendieck topology induced by the topology on Lisse-ét$(\mathcal{X})$, and the resulting topos is canonically isomorphic to the localized topos $\mathcal{X}_{\text{lis-ét}|\mathrm{U}}$. Note that there is a natural inclusion Lisse-ét$(\mathrm{U}) \hookrightarrow$ Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$ but this is not an equivalence of categories since for an object $(\mathrm{V} \to \mathrm{U}) \in$ Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$ the morphism $\mathrm{V} \to \mathrm{U}$ need not be smooth. Viewing $\mathcal{X}_{\text{lis-ét}|\mathrm{U}}$ in this way, the functor ϵ^{-1} maps F on $\mathrm{U}_{\text{ét}}$ to $\mathrm{F}_{\mathrm{V}} = \pi^{-1}\mathrm{F} \in \mathrm{V}_{\text{ét}}$ where $\pi : \mathrm{V} \to \mathrm{U} \in$ Lisse-ét$(\mathcal{X})_{|\mathrm{U}}$. For a sheaf $\mathrm{F} \in \mathcal{X}_{\text{lis-ét}|\mathrm{U}}$ corresponding to a collection of sheaves F_{V}, the sheaf $\epsilon_*\mathrm{F}$ is simply the sheaf F_{U}.

In particular, the functor ϵ_* is exact and, accordingly $\mathrm{H}^*(\mathrm{U}, \mathrm{F}) = \mathrm{H}^*(\mathrm{U}_{\text{ét}}, \mathrm{F}_{\mathrm{U}})$ for any sheaf of Λ modules of $\mathcal{X}$.

Theorem 7.5. *There exists a normalized complex $\Omega_\bullet \in \mathcal{D}_c(\mathcal{X}^{\mathbf{N}})$, unique up to canonical isomorphism, inducing the Ω_n.*

Proof. The topos $\mathcal{X}^{\mathbf{N}}_{\text{lis-ét}}$ can be described by the site $\mathcal{S}$ whose objects are pairs $(n, u : \mathrm{U} \to \mathcal{X})$ where u is a lisse-étale open and $n \in \mathbf{N}$. We want to use the gluing theorem [12, 2.3.3].

- Let us describe the localization morphisms explicitly. Let (U, n) be in $\mathcal{S}$. An object of the localized topos $\mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)}$ is equivalent to giving for every U–scheme of finite type $\mathrm{V} \to \mathrm{U}$, such that the composite $\alpha : \mathrm{V} \to \mathrm{U} \to \mathcal{X}$ is smooth of relative dimension d_α, a projective system

  $$\mathrm{F}_{\mathrm{V}} = (\mathrm{F}_{\mathrm{V},m}, m \leq n)$$

  where $\mathrm{F}_{\mathrm{V},m} \in \mathrm{V}_{\text{ét}}$ together with morphisms $f^{-1}\mathrm{F}_{\mathrm{V}} \to \mathrm{F}_{\mathrm{V}'}$ for U–morphisms $f : \mathrm{V}' \to \mathrm{V}$. The localization morphism

  $$j_n : \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)} \to \mathcal{X}^{\mathbf{N}}$$

  is defined by the truncation

  $$({j_n}^{-1}\mathrm{F}_\bullet)_{\mathrm{V}} = (\mathrm{F}_{m,\mathrm{V}})_{m \leq n}.$$

  We still denote ${j_n}^{-1}\Lambda_\bullet = (\Lambda_m)_{m \leq n}$ by $\Lambda_\bullet$ and we ring $\mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)}$ by $\Lambda_\bullet$.
- Notice that $\pi : \mathcal{X}^{\mathbf{N}} \to \mathcal{X}$ induces

  $$\pi_n : \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)} \to \mathcal{X}_{|\mathrm{U}}$$

defined by $\pi_n^{-1}(\mathrm{F}) = (\mathrm{F})_{m\leq n}$ (the constant projective system). One has

$$\pi_{n*}(\mathrm{F}_m)_{m\leq n} = \varprojlim_{m\leq n} \mathrm{F}_m = \mathrm{F}_n.$$

- As in the proof of 3.16, the morphisms $e_i : \mathcal{X} \to \mathcal{X}^{\mathbf{N}}, i \leq n$ can be localized in $\tilde{e}_i : \mathcal{X}_{|\mathrm{U}} \to \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)}$, characterized by $\tilde{e}_i^{-1}(\mathrm{F}_m)_{m\leq n} = \mathrm{F}_i$.They form a conservative sets of functors.
- One has a commutative diagram of topos

$$\begin{tikzcd}
\mathcal{X}_{|\mathrm{U}} \arrow[r, "\tilde{e}_n"] \arrow[ddr, "\epsilon"'] & \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)} \arrow[r, "j_n"] \arrow[d, "\pi_n"] \arrow[dd, bend left, "p_n"] & \mathcal{X}^{\mathbf{N}} \\
 & \mathcal{X}_{|\mathrm{U}} \arrow[d, "\epsilon"] & \\
 & \mathrm{U}_{\text{ét}} &
\end{tikzcd} \tag{7.5.1}$$

One has $\pi_n^{-1}(\Lambda_n) = (\Lambda_n)_{m\leq n}$ -the constant projective system with value Λ_n- which maps to $(\Lambda_m)_{m\leq n}$: we will ring $\mathcal{X}_{|\mathrm{U}}$ (and also both $\mathcal{X}$ and $\mathrm{U}_{\text{ét}}$) by Λ_n and therefore the previous diagram is a diagram of ringed topos. Notice that $e_n^{-1} = e_n^*$ implying the exactness of e_n^*.

- Let us define

$$\Omega_{\mathrm{U},n} = \mathrm{L}\pi_n^*\Omega_{n|\mathrm{U}} = \mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}\langle -d_\alpha\rangle \tag{7.5.2}$$

where $\mathrm{K}_{\mathrm{U},n} \in \mathcal{D}_c(\mathrm{U}_{\text{ét}}, \Lambda_n)$ is the dualizing complex.

- Let $f : (\mathrm{V}, m) \to (\mathrm{U}, n)$ be a morphism in $\mathcal{S}$. It induces a commutative diagram of ringed topos

$$\begin{tikzcd}
\mathcal{X}^{\mathbf{N}}_{|(\mathrm{V},m)} \arrow[r, "f"] \arrow[d, "p_m"] & \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)} \arrow[d, "p_n"] \\
\mathrm{V}_{\text{ét}} \arrow[r, "f"] & \mathrm{U}_{\text{ét}}
\end{tikzcd}.$$

By the construction of the dualizing complex in [12] and 7.2, one has therefore

$$\mathrm{L}f^*\Omega_{\mathrm{U},n} = \mathrm{L}\pi_m^*(\Lambda_m \overset{\mathbf{L}}{\otimes}_{\Lambda_n} \Omega_{n|\mathrm{V}}) = \mathrm{L}\pi_m^*\Omega_{m|\mathrm{V}} = \Omega_{\mathrm{V},m}. \tag{7.5.3}$$

Therefore, $\Omega_{\mathrm{U},n}$ defines locally an object $\mathcal{D}_c(\mathcal{S}, \Lambda_\bullet)$. Let's turn to the $\mathcal{E}xt$'s.

- The morphism of topos $\pi_n : \mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)} \to \mathcal{X}_{|\mathrm{U}}$ is defined by $\pi_n^{-1}\mathrm{F} = (\mathrm{F})_{m\leq n}$. One has therefore

$$\pi_{n*}\mathrm{F} = \mathrm{F}_n \text{ and } p_{n*}\mathrm{F} = \mathrm{F}_{n,\mathrm{U}}.$$

In particular, one gets the exactness of p_{n*} and the formulas

$$\mathrm{R}p_{n*} = p_{n*} \text{ and } \pi_{n*} = \tilde{e}_n^*. \tag{7.5.4}$$

Using (7.5.1) we get the formula

$$p_{n*}\mathrm{L}p_n^* = \epsilon_*\tilde{e}_n^*\mathrm{L}p_n^* = \epsilon_*\epsilon^* = \mathrm{Id}\,. \tag{7.5.5}$$

Therefore one has

$$\begin{aligned}\mathrm{Ext}^i(\mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}, \mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}) &= \mathrm{Ext}^i(\mathrm{K}_{\mathrm{U},n}, p_{n*}\mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}) \\ &= \mathrm{Ext}^i(\mathrm{K}_{\mathrm{U},n}, \mathrm{K}_{\mathrm{U},n}) \quad \text{by 7.5.5} \\ &= \mathrm{H}^i(\mathrm{U}_{\text{ét}}, \Lambda_n) \quad \text{by duality.}\end{aligned}$$

By sheafification, one gets

$$\mathcal{E}xt^i(\mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}, \mathrm{L}p_n^*\mathrm{K}_{\mathrm{U},n}) = \Lambda_\bullet \text{ for } i \neq 0 \text{ and } \Lambda_\bullet \text{ else.}$$

Therefore, the local data $(\Omega_{\mathrm{U},n})$ has vanishing negative $\mathcal{E}xt$'s. By [12, 3.2.2], there exists a unique $\Omega_\bullet \in \mathcal{D}_c(\mathcal{X}, \Lambda)$ inducing $\Omega_{\mathrm{U},n}$ on each $\mathcal{X}^{\mathbf{N}}_{|(\mathrm{U},n)}$. By 7.5.2, one has

- Using the formula $j_n \circ e_n = \tilde{e}_n \circ j$ (7.5.1) and 7.5.2, one obtains

$$(e_n^*\Omega_\bullet)_{|\mathrm{U}} = e_n^*\Omega_{(n,\mathrm{U})} = \Omega_{n|\mathrm{U}}.$$

By [12, 3.2.2], the isomorphisms glue to define a functorial isomorphism

$$e_n^*\Omega_\bullet = \Omega_n.$$

By 7.2 and 3.5, $\Omega_\bullet$ is normalized with constructible cohomology.

- The uniqueness is a direct consequence of (3.15).

□

7.6. **The duality theorem.** Let M be a normalized complex. By 3.16, one has

$$e_n^{-1}\,\mathcal{R}hom(\mathrm{M}, \Omega) \to \mathcal{R}hom(e_n^{-1}\mathrm{M}, e_n^{-1}\Omega) \tag{7.6.1}$$

(3.16). The complex Ω is of locally finite quasi–injective dimension in the following sense. If $\mathcal{X}$ is quasi–compact, then each Ω_n is of finite quasi-injective dimension, bounded by some integer N depending only on $\mathcal{X}$ and Λ, but not n. Therefore in the quasi–compact case one has

$$\boldsymbol{\mathcal{E}xt}_{\boldsymbol{\Lambda}}{}^i(\mathrm{M}, \Omega) = 0 \text{ for any } \mathrm{M} \in \mathbf{D}_c^{\geq 0}(\mathcal{X}) \text{ and } i \geq \mathrm{N}.$$

Let's now prove the duality theorem.

Theorem 7.7. *Let* $\mathrm{D} : \mathbf{D}_c(\mathcal{X})^{\mathrm{opp}} \to \mathcal{D}(\mathcal{X})$ *be the functor defined by* $\mathrm{D}(\mathrm{M}) = \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M}, \Omega) = \mathcal{R}hom_{\Lambda_\bullet}(\hat{\mathrm{M}}, \hat{\Omega})$.

1. *The essential image of* D *lies in* $\mathcal{D}_c(\mathcal{A})$.
2. *If* $\mathrm{D} : \mathbf{D}_c(\mathcal{X})^{\mathrm{opp}} \to \mathbf{D}_c(\mathcal{X})$ *denotes the induced functor, then* D *is involutive and maps* $\mathbf{D}_c^-(\mathcal{X})$ *into* $\mathbf{D}_c^+(\mathcal{X})$.

Proof. Both assertions are local on $\mathcal{X}$ so we may assume that $\mathcal{X}$ is quasi–compact. Because Ω is of finite quasi–injective dimension, to prove the first point it suffices to prove (1) for bounded below complexes. In this case the result follows from 4.1.

For the second point, one can assume M normalized (because $\hat{\mathrm{M}}$ is constructible (3.9) and normalized. Because Ω is normalized (7.2), the tautological biduality morphism

$$\mathcal{R}hom(\mathrm{M}, \mathcal{R}hom(\mathrm{M}, \Omega), \Omega) \to \mathrm{M}$$

defines a morphism

$$\mathrm{DD}(\mathrm{M}) \to \mathrm{M}.$$

Using 7.6.1, one is reduced to the analogous formula

$$\mathrm{D}_n \mathrm{D}_n(e_n^{-1}\mathrm{M}) = e_n^{-1}\mathrm{M}$$

where D_n is the dualizing functor on $\mathcal{D}_c(\mathcal{X}_{\text{lis-ét}}, \Lambda_n)$, which is proven in *loc. cit.*. □

Corollary 7.8. *For any* $\mathrm{N}, \mathrm{M} \in \mathbf{D}_c(\mathcal{X}, \Lambda)$ *there is a canonical isomorphism*

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M}, \mathrm{N}) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{D}(\mathrm{N}), \mathrm{D}(\mathrm{M})).$$

Proof. Indeed by 6.1 we have

$$\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{D}(\mathrm{N}), \mathrm{D}(\mathrm{M})) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{D}(\mathrm{N}) \overset{\mathbf{L}}{\otimes} \mathrm{M}, \Omega) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M}, \mathrm{DD}(\mathrm{N})) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M}, \mathrm{N}).$$

□

8. THE FUNCTORS $\mathrm{R}f_*$ AND $\mathrm{L}f^*$

Lemma 8.1. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a morphism of finite type between* S*–stacks. Then for any integer* n *and* $\mathrm{M} \in \mathcal{D}_c^+(\mathcal{X}, \Lambda_{n+1})$ *the natural map*

$$\mathrm{R}f_*\mathrm{M} \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Lambda_n \to \mathrm{R}f_*(\mathrm{M} \overset{\mathbf{L}}{\otimes}_{\Lambda_{n+1}} \Lambda_n)$$

is an isomorphism.

Proof. The assertion is clearly local in the smooth topology on $\mathcal{Y}$ so we may assume that $\mathcal{Y}$ is a scheme. Furthermore, if $\mathrm{X}_\bullet \to \mathcal{X}$ is a smooth hypercover by schemes and $\mathrm{M}_\bullet \in \mathcal{D}_c(\mathrm{X}_\bullet, \Lambda_{n+1})$ is the complex corresponding to M under the equivalence of categories $\mathcal{D}_c(\mathrm{X}_{\bullet,\text{ét}}, \Lambda_{n+1}) \simeq \mathcal{D}_c(\mathcal{X}, \Lambda_{n+1})$ then by [16, 9.8] it suffices to show the analogous statement for the morphism of topos

$$f_\bullet : \mathrm{X}_{\bullet,\text{ét}} \to \mathcal{Y}_{\text{et}}.$$

Furthermore by a standard spectral sequence argument (using the sequence defined in [16, 9.8]) it suffices to prove the analogous result for each of the morphisms $f_n : \mathrm{X}_{n,\text{ét}} \to \mathcal{Y}_{\text{ét}}$, and hence it suffices to prove the lemma for a finite type morphism of schemes of finite type over S with the étale topology where it is standard. □

Proposition 8.2. *Let* $\mathrm{M} = (\mathrm{M}_n)_n$ *be a bounded below* λ*–complex on* $\mathcal{X}$*. Then for any integer* i *the system* $\mathrm{R}^i f_* \mathrm{M} = (\mathrm{R}^i f_* \mathrm{M}_n)_n$ *is almost adic.*

Proof. The assertion is clearly local on $\mathcal{Y}$, and hence we may assume that both $\mathcal{X}$ and $\mathcal{Y}$ are quasi–compact.

By the same argument proving [16, 9.10] and [2, Th. finitude], the sheaves $\mathrm{R}^i f_* \mathrm{M}_n$ are constructible. The result then follows from [3, V.5.3.1] applied to the category of constructible sheaves on $\mathcal{X}_{\text{lis-ét}}$. □

Now consider the morphism of topos $f_\bullet : \mathcal{X}^{\mathbb{N}} \to \mathcal{Y}^{\mathbb{N}}$ induced by the morphism f. By the above, if $\mathrm{M} \in \mathcal{D}^+(\mathcal{X}^{\mathbb{N}})$ is a λ–complex then $\mathrm{R}f_*\mathrm{M}$ is a λ–complex on $\mathcal{Y}$. We therefore obtain a functor

$$\mathrm{R}f_* : \mathbf{D}_c^+(\mathcal{X}, \Lambda) \to \mathbf{D}_c^+(\mathcal{Y}, \Lambda).$$

It follows immediately from the definitions that the pullback functor $\mathrm{L}f^* : \mathcal{D}_c(\mathcal{Y}^{\mathbb{N}}, \Lambda) \to \mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda)$ take λ–complexes to λ–complexes and AR–null complexes to AR–null complexes and therefore induces a functor

$$\mathrm{L}f^* : \mathbf{D}_c(\mathcal{Y}, \Lambda) \to \mathbf{D}_c(\mathcal{X}, \Lambda), \mathrm{M} \mapsto \mathrm{R}f_*\hat{\mathrm{M}}.$$

Proposition 8.3. *Let* $\mathrm{M} \in \mathbf{D}_c^+(\mathcal{X}, \Lambda)$ *and* $\mathrm{N} \in \mathbf{D}_c^-(\mathcal{Y}, \Lambda)$*. Then there is a canonical isomorphism*

$$\mathrm{R}f_* \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{L}f^*\mathrm{N}, \mathrm{M}) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{N}, \mathrm{R}f_*\mathrm{M}).$$

Proof. We can rewrite the formula as

$$\mathrm{R}f_* \mathcal{R}hom(\mathrm{L}f^*\hat{\mathrm{N}}, \hat{\mathrm{M}}) \simeq \mathcal{R}hom(\hat{\mathrm{N}}, \mathrm{R}f_*\hat{\mathrm{M}})$$

which follows from the usual adjunction between $\mathrm{R}f_*$ and $\mathrm{L}f^*$. ☐

9. The functors $\mathrm{R}f_!$ and $\mathrm{R}f^!$

9.1. **Definitions.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a finite type morphism of S–stacks and let $\Omega_{\mathcal{X}}$ (resp. $\Omega_{\dagger}$) denote the dualizing complex of $\mathcal{X}$ (resp. $\mathcal{Y}$). Let $\mathrm{D}_{\mathcal{X}} : \mathbf{D}_c(\mathcal{X}) \to \mathbf{D}_c(\mathcal{X})$ denote the functor $\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(-, \Omega_{\mathcal{X}}) : \mathbf{D}_c(\mathcal{X}, \Lambda) \to \mathbf{D}_c(\mathcal{X}, \Lambda)$ and let $\mathrm{D}_{\mathcal{Y}} : \mathbf{D}_c(\mathcal{Y}, \Lambda) \to \mathbf{D}_c(\mathcal{Y}, \Lambda)$ denote $\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(-, \Omega_{\mathcal{Y}})$. We then define

$$\mathrm{R}f_! := \mathrm{D}_{\mathcal{Y}} \circ \mathrm{R}f_* \circ \mathrm{D}_{\mathcal{X}} : \mathbf{D}_c^-(\mathcal{X}, \Lambda) \to \mathbf{D}_c^-(\mathcal{Y}, \Lambda)$$

and

$$\mathrm{R}f^! := \mathrm{D}_{\mathcal{X}} \circ \mathrm{L}f^* \circ \mathrm{D}_{\mathcal{Y}} : \mathbf{D}_c(\mathcal{Y}, \Lambda) \to \mathbf{D}_c(\mathcal{X}, \Lambda).$$

Lemma 9.2. *For any* $\mathrm{N} \in \mathbf{D}_c^-(\mathcal{X}, \Lambda)$ *and* $\mathrm{M} \in \mathbf{D}_c^+(\mathcal{Y}, \Lambda)$ *there is a canonical isomorphism*

$$\mathrm{R}f_* \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{N}, \mathrm{L}f^!\mathrm{M}) \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{R}f_!\mathrm{N}, \mathrm{M}).$$

Proof. Set $\mathrm{N}' = \mathrm{D}_{\mathcal{X}}(\mathrm{N})$ and $\mathrm{M}' := \mathrm{D}_{\mathcal{Y}}(\mathrm{M})$. Then by 7.8 the formula can be written as

$$\mathrm{R}f_* \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{L}f^*\mathrm{M}', \mathrm{N}') \simeq \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{M}', \mathrm{R}f_*\mathrm{N}')$$

which is 8.3. ☐

Lemma 9.3. *If f is a smooth morphism of relative dimension d, then there is a canonical isomorphism* $\mathrm{R}f^!(\mathrm{F}) \simeq f^*\mathrm{F}\langle d\rangle$.

Proof. By the construction of the dualizing complex and [12, 4.5.2] we have $\Omega_{\mathcal{X}} \simeq f^*\Omega_{\mathcal{Y}}\langle d\rangle$. From this and biduality 7.7 the lemma follows. ☐

If f is a closed immersion, then we can also define the functor of sections with support $\underline{\mathrm{H}}^0_{\mathcal{X}}$ on the category of $\Lambda_\bullet$–modules in $\mathcal{Y}^{\mathbb{N}}$. This functor is right adjoint to f_* and taking derived functors we obtain an adjoint pair of functors

$$\mathrm{R}f_* : \mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet) \to \mathcal{D}_c(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet)$$

and

$$\mathrm{R}\underline{\mathrm{H}}^0_{\mathcal{X}} : \mathcal{D}_c(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet) \to \mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet).$$

Both of these functors take AR–null complexes to AR–nul complexes and hence induce adjoint functors on the categories $\mathbf{D}_c(\mathcal{Y}, \Lambda)$ and $\mathbf{D}_c(\mathcal{X}, \Lambda)$.

Lemma 9.4. *If f is a closed immersion, then* $\Omega_{\mathcal{X}} = f^*\mathrm{R}\underline{\mathrm{H}}^0_{\mathcal{X}}\Omega_{\mathcal{Y}}$.

Proof. By the gluing lemma this is a local assertion in the topos $\mathcal{X}^{\mathbb{N}}$ and hence the result follows from [12, 4.6.1]. ☐

Proposition 9.5. *If f is a closed immersion, then $f^! = f^*\mathrm{R}\underline{\mathrm{H}}^0_{\mathcal{X}}$ and $\mathrm{R}f_* = \mathrm{R}f_!$.*

Proof. This follows from the same argument proving [12, 4.6.2]. ☐

Finally using the argument of [12, 4.7] one shows:

Proposition 9.6. *If f is a universal homeomorphism then $f^*\Omega_{\mathcal{X}} = \Omega_{\mathcal{Y}}$, $\mathrm{R}f^! = f^*$, and $\mathrm{R}f_! = \mathrm{R}f_*$.*

There is also a projection formula

$$\mathrm{R}f_!(\mathrm{A} \overset{\mathbf{L}}{\otimes} f^*\mathrm{B}) \simeq \mathrm{R}f_!\mathrm{A} \overset{\mathbf{L}}{\otimes} \mathrm{B} \tag{9.6.1}$$

for $\mathrm{B} \in \mathbf{D}_c^-(\mathcal{Y}, \Lambda)$ and $\mathrm{A} \in \mathbf{D}_c(\mathcal{X}, \Lambda)$. This is shown by the same argument used to prove [12, 4.4.2].

10. Computing cohomology using hypercovers

For this we first need some cohomological descent results.

Let $\mathcal{X}$ be a algebraic stack over S and $\mathrm{X}_\bullet \to \mathcal{X}$ a strictly simplicial smooth hypercover with the X_i also S–stacks. We can then also consider the topos of projective systems in $\mathrm{X}_{\bullet,\text{lis-ét}}$ which we denote by $\mathrm{X}_\bullet^{\mathbb{N}}$.

Definition 10.1. (i) A sheaf F of $\Lambda_\bullet$–modules in $\mathrm{X}_\bullet^{\mathbb{N}}$ is *almost adic* if it is cartesian and if for every n the restriction $\mathrm{F}|_{\mathrm{X}_{n,\text{lis-ét}}}$ is almost adic.

(ii) An object $\mathrm{C} \in \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ is a *λ–complex* if for all i the cohomology sheaf $\mathcal{H}^i(\mathrm{C})$ is almost adic.

(iii) An object $\mathrm{C} \in \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ is *almost zero* if for every n the restriction of C to X_n is almost zero.

(iv) Let $\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \subset \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ denoted the triangulated subcategory whose objects are the λ–complexes. The category $\mathbf{D}_c(\mathrm{X}_\bullet, \Lambda)$ is the quotient of $\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ by the full subcategory of almost zero complexes.

As in 2.1 we have the projection morphism

$$\pi : (\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \to (\mathrm{X}_{\bullet,\text{lis-ét}}, \Lambda)$$

restricting for every n to the morphism $(\mathrm{X}_n^{\mathbb{N}}, \Lambda_\bullet) \to (\mathrm{X}_{n,\text{lis-ét}}, \Lambda)$ discussed in 2.1. By 2.6 the functor $\mathrm{R}\pi_* : \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \to \mathcal{D}(\mathrm{X}_\bullet, \Lambda)$ takes almost zero complexes to 0. By the universal property of the quotient category it follows that there is an induced functor

$$\mathrm{R}\pi_* : \mathbf{D}_c(\mathrm{X}_\bullet, \Lambda) \to \mathcal{D}(\mathrm{X}_\bullet, \Lambda).$$

We also define a normalization functor

$$\mathbf{D}_c(\mathrm{X}_\bullet, \Lambda) \to \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet), \mathrm{M} \mapsto \hat{\mathrm{M}}$$

by setting $\hat{\mathrm{M}} := \mathrm{L}\pi^*\mathrm{R}\pi_*(\mathrm{M})$.

Proposition 10.2. *Let* $\mathrm{M} \in \mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ *be a* λ*–complex. Then* $\hat{\mathrm{M}}$ *is in* $\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ *and the canonical map* $\hat{\mathrm{M}} \to \mathrm{M}$ *has almost zero cone.*

Proof. For any integer n, there is a canonical commutative diagram of ringed topos

$$\begin{tikzcd}
\mathrm{X}_\bullet^{\mathbb{N}} \arrow[r, "r_n"] \arrow[d, "\pi"'] & \mathrm{X}_n^\bullet \arrow[d, "\pi"] \\
\mathrm{X}_{\bullet,\text{lis-ét}} \arrow[r, "r_n"] & \mathrm{X}_{n,\text{lis-ét}},
\end{tikzcd}$$

where r_n denotes the restriction morphisms. Furthermore, the functors r_{n*} are exact and take injectives to injectives. It follows that for any $\mathrm{M} \in \mathcal{D}(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ there is a canonical isomorphism

$$\mathrm{R}\pi_*(r_{n*}(\mathrm{M})) \simeq r_{n*}\mathrm{R}\pi_*(\mathrm{M}).$$

¿From the definition of π^* it also follows that $r_{n*}\mathrm{L}\pi^* = \mathrm{L}\pi^* r_{n*}$, and from this it follows that the restriction of $\hat{\mathrm{M}}$ to X_n is simply the normalization of $\mathrm{M}|_{\mathrm{X}_n}$. From this and 3.9 the statement that $\hat{\mathrm{M}} \to \mathrm{M}$ has almost zero cone follows.

To see that $\hat{\mathrm{M}} \in \mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$, note that by 3.9 we know what for any integers i and n the restriction $\mathcal{H}^i(\hat{\mathrm{M}})|_{\mathrm{X}_n}$ is a constructible (and in particular cartesian) sheaf on $\mathrm{X}_{n,\text{lis-ét}}$. We also know by 2.5 that for any n and smooth morphism $\mathrm{U} \to \mathrm{X}_n$, the restriction of $\mathcal{H}^i(\hat{\mathrm{M}})$ to $\mathrm{U}_{\text{ét}}$ is equal to $\mathcal{H}^i(\widehat{\mathrm{M}_\mathrm{U}})$. From this and 3.9 it follows that the sheaves $\mathcal{H}^i(\hat{\mathrm{M}})$ are cartesian. In fact, this shows that if $\mathcal{F} \in \mathcal{D}_c(\mathcal{X}^{\mathbf{N}}, \Lambda_\bullet)$ denotes the complex obtain from the equivalence of categories (cohomological descent as in [12, 2.2.3])

$$\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbf{N}}, \Lambda_\bullet) \simeq \mathcal{D}_c(\mathcal{X}^{\mathbf{N}}, \Lambda_\bullet),$$

then $\mathcal{H}^i(\hat{\mathrm{M}})$ is the restriction to $\mathrm{X}_\bullet^{\mathbf{N}}$ of the sheaf $\mathcal{H}^i(\hat{\mathrm{F}})$. □

As in 3.13 it follows that the normalization functor induces a left adjoint to the projection $\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \to \mathbf{D}_c(\mathrm{X}_\bullet, \Lambda)$.

Let $\epsilon : \mathrm{X}_{\bullet,\text{lis-ét}} \to \mathcal{X}_{\text{lis-ét}}$ denote the projection, and write also $\epsilon : \mathrm{X}_\bullet^{\mathbb{N}} \to \mathcal{X}^{\mathbb{N}}$ for the morphism on topos of projective systems. There is a natural commutative diagram of topos

$$\begin{tikzcd}
\mathrm{X}_\bullet^{\mathbb{N}} \arrow[r, "\pi"] \arrow[d, "\epsilon"'] & \mathrm{X}_{\bullet,\text{lis-ét}} \arrow[d, "\epsilon"] \\
\mathcal{X}^{\mathbb{N}} \arrow[r, "\pi"] & \mathcal{X}_{\text{lis-ét}}.
\end{tikzcd}$$

By [12, 2.2.6], the functors $\mathrm{R}\epsilon_*$ and ϵ^* induce an equivalence of categories

$$\mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \simeq \mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet),$$

and the subcategories of almost zero complexes coincide under this equivalence.

We therefore have obtained

Proposition 10.3. *Let $\mathcal{X}$ be an algebraic stack over* S *and* $\mathrm{X}_\bullet \to \mathcal{X}$ *a strictly simplicial smooth hypercover with the* X_i *also* S*–stacks. Then, the morphism*

$$\mathrm{R}\epsilon_* : \mathbf{D}_c(\mathrm{X}_\bullet, \Lambda) \xrightarrow{\sim} \mathbf{D}_c(\mathcal{X}, \Lambda)$$

is an equivalence with inverse ϵ^*.

Consider next a morphism of nice stacks $f : \mathcal{X} \to \mathcal{Y}$. Choose a commutative diagram

$$\begin{tikzcd}
\mathrm{X}_\bullet \arrow[r, "\tilde{f}"] \arrow[d, "\epsilon_{\mathrm{X}}"'] & \mathrm{Y}_\bullet \arrow[d, "\epsilon_{\mathrm{Y}}"] \\
\mathcal{X} \arrow[r, "f"] & \mathcal{Y},
\end{tikzcd}$$

where ϵ_{X} and ϵ_{Y} are smooth (strictly simplicial) hypercovers by nice S–stacks. The functors $\mathrm{R}f_* : \mathcal{D}_c(\mathcal{X}^{\mathbb{N}}, \Lambda_\bullet) \to \mathcal{D}_c(\mathcal{Y}^{\mathbb{N}}, \Lambda_\bullet)$ and $\mathrm{R}\tilde{f}_* : \mathcal{D}_c(\mathrm{X}_\bullet^{\mathbb{N}}, \Lambda_\bullet) \to \mathcal{D}_c(\mathrm{Y}_\bullet^{\mathbb{N}}, \Lambda_\bullet)$ evidently take almost zero complexes to almost zero complexes and therefore induce functors

$$\mathrm{R}f_* : \mathbf{D}_c(\mathcal{X}, \Lambda) \to \mathbf{D}_c(\mathcal{Y}, \Lambda), \quad \mathrm{R}\tilde{f}_* : \mathbf{D}_c(\mathrm{X}_\bullet, \Lambda) \to \mathbf{D}_c(\mathrm{Y}_\bullet, \Lambda).$$

It follows from the construction that the diagram

$$\begin{tikzcd}
\mathbf{D}_c(\mathcal{X}, \Lambda) \arrow[r, "10.3"] \arrow[d, "\mathrm{R}f_*"'] & \mathbf{D}_c(\mathrm{X}_\bullet, \Lambda) \arrow[d, "\mathrm{R}\tilde{f}_*"] \\
\mathbf{D}_c(\mathcal{Y}, \Lambda) \arrow[r, "10.3"] & \mathbf{D}_c(\mathrm{Y}_\bullet, \Lambda)
\end{tikzcd}$$

commutes.

Corollary 10.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of nice* S*–stacks, and let* $\mathrm{X}_\bullet \to \mathcal{X}$ *be a strictly simplicial smooth hypercover by nice* S*–stacks. For every n, let $f_n : \mathrm{X}_n \to \mathcal{Y}$ be the projection. Then for any* $\mathrm{F} \in \mathbf{D}_c^+(\mathcal{X}, \Lambda)$ *there is a canonical spectral sequence in the category of λ–modules*

$$\mathrm{E}_1^{pq} = \mathrm{R}^q f_{p*}(\mathrm{F}|_{\mathrm{X}_p}) \implies \mathrm{R}^{p+q} f_*(\mathrm{F}).$$

Proof. We take $\mathrm{Y}_\bullet \to \mathcal{Y}$ to be the constant simplicial topos associated to $\mathcal{Y}$. Let $\mathrm{F}_\bullet$ denote $\epsilon_{\mathrm{X}}^*\mathrm{F}$. We then have

$$\mathrm{R}f_*(\mathrm{F}) = \mathrm{R}f_*\mathrm{R}\epsilon_{\mathrm{X}*}(\mathrm{F}_\bullet) = \mathrm{R}\epsilon_{\mathrm{Y}*}\mathrm{R}\tilde{f}_*(\mathrm{F}_\bullet).$$

The functor $\mathrm{R}\epsilon_{\mathrm{Y}*}$ is just the total complex functor (which passes to $\mathbf{D}_c$), and hence we obtain the corollary from the standard spectral sequence associated to a bicomplex. □

Corollary 10.5. *With notation as in the preceding corollary, let* $\mathrm{F} \in \mathbf{D}_c^-(\mathcal{X}, \Lambda)$. *Then there is a canonical spectral sequence in the category of λ–modules*

$$\mathrm{E}_1^{pq} = \mathcal{H}^q(\mathrm{D}_{\mathcal{Y}}(\mathrm{R}f_{p!}(\mathrm{F}))) \implies \mathcal{H}^{p+q}(\mathrm{D}_{\mathcal{Y}}(\mathrm{R}f_!\mathrm{F})).$$

Proof. Apply the preceding corollary to $\mathrm{D}_{\mathcal{X}}(\mathrm{F})$. □

11. Kunneth formula

We prove the Kunneth formula using the method of [12, §5.7].

Lemma 11.1. *For any* $\mathrm{P}_1, \mathrm{P}_2, \mathrm{M}_1, \mathrm{M}_2 \in \mathbf{D}_c(\mathcal{X}, \Lambda)$ *there is a canonical morphism*

$$\boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_1, \mathrm{M}_1) \overset{\mathbf{L}}{\otimes} \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_2, \mathrm{M}_2) \to \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_1 \overset{\mathbf{L}}{\otimes} \mathrm{P}_2, \mathrm{M}_1 \overset{\mathbf{L}}{\otimes} \mathrm{M}_2).$$

Proof. By 6.1 it suffices to exhibit a morphism

$$\boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_1, \mathrm{M}_1) \overset{\mathbf{L}}{\otimes} \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_2, \mathrm{M}_2) \overset{\mathbf{L}}{\otimes} \mathrm{P}_1 \overset{\mathbf{L}}{\otimes} \mathrm{P}_2 \to \mathrm{M}_1 \overset{\mathbf{L}}{\otimes} \mathrm{M}_2$$

which we obtain from the two evaluation morphisms

$$\boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{P}_i, \mathrm{M}_i) \otimes \mathrm{P}_i \to \mathrm{M}_i.$$

□

Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be nice stacks, and set $\mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2$ with projections $p_i : \mathcal{Y} \to \mathcal{Y}_i$. For $\mathrm{L}_i \in \mathbf{D}_c(\mathcal{Y}_i, \Lambda)$ let $\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2 \in \mathbf{D}_c(\mathcal{Y}, \Lambda)$ denote $p_1^*\mathrm{L}_1 \overset{\mathbf{L}}{\otimes} p_2^*\mathrm{L}_2$.

Lemma 11.2. *There is a natural isomorphism* $\Omega_{\mathcal{Y}} \simeq \Omega_{\mathcal{Y}_1} \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \Omega_{\mathcal{Y}_2}$ *in* $\mathbf{D}_c(\mathcal{Y}, \Lambda)$.

Proof. This is reduced to [12, 5.7.1] by the same argument proving 7.5 using the gluing lemma. □

Lemma 11.3. *For* $\mathrm{L}_i \in \mathbf{D}_c^-(\mathcal{Y}_i)$ *(*$i = 1, 2$*) there is a canonical isomorphism*

$$\mathrm{D}_{\mathcal{Y}_1}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{D}_{\mathcal{Y}_2}(\mathrm{L}_2) \to \mathrm{D}_{\mathcal{Y}}(\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2). \tag{11.3.1}$$

Proof. Note first that there is a canonical map

$$p_i^* \mathrm{D}_{\mathcal{Y}_i}(\mathrm{L}_i) \to \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(p_i^* \mathrm{L}_i, p_i^* \Omega_{\mathcal{Y}_i}). \tag{11.3.2}$$

Indeed by adjunction giving such a morphism is equivalent to giving a morphism

$$\mathrm{D}_{\mathcal{Y}_i}(\mathrm{L}_i) \to \mathrm{R}p_{i*} \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(p_i^* \mathrm{L}_i, p_i^* \Omega_{\mathcal{Y}_i}),$$

and this in turn is by 8.3 equivalent to giving a morphism

$$\mathrm{D}_{\mathcal{Y}_i}(\mathrm{L}_i) \to \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{L}_i, \mathrm{R}p_{i*} p_i^* \Omega_{\mathcal{Y}_i}).$$

We therefore obtain the map 11.3.2 from the adjunction morphism $\Omega_{\mathcal{Y}_i} \to \mathrm{R}p_{i*} p_i^* \Omega_{\mathcal{Y}_i}$.

Combining this with 11.1 we obtain a morphism

$$\mathrm{D}_{\mathcal{Y}_1}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{D}_{\mathcal{Y}_2}(\mathrm{L}_2) \to \boldsymbol{\mathcal{R}hom}_{\mathbf{\Lambda}}(\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2, \Omega_{\mathcal{Y}_1} \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \Omega_{\mathcal{Y}_2}),$$

which by 11.2 defines the morphism 11.3.2.

To see that this morphism is an isomorphism, note that by the definition this morphism is given by the natural map

$$\mathcal{R}hom_{\Lambda_\bullet}(\hat{\mathrm{L}}_1, \Omega_{\mathcal{Y}_1}) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathcal{R}hom_{\Lambda_\bullet}(\hat{\mathrm{L}}_2, \Omega_{\mathcal{Y}_2}) \to \mathcal{R}hom_{\Lambda_\bullet}(\hat{\mathrm{L}}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \hat{\mathrm{L}}_2, \Omega_{\mathcal{Y}})$$

in the topos $\mathcal{Y}^{\mathbb{N}}$. That it is an isomorphism therefore follows from [12, 5.7.5]. □

Let $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ be morphisms of nice S–stacks, set $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$, and let $f := f_1 \times f_2 : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. Let $\mathrm{L}_i \in \mathbf{D}_c^-(\mathcal{X}_i, \Lambda)$ $(i = 1, 2)$.

Theorem 11.4. *There is a canonical isomorphism in* $\mathbf{D}_c(\mathcal{Y}, \Lambda)$

$$\mathrm{R}f_!(\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2) \to \mathrm{R}f_{1!}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{R}f_{2!}(\mathrm{L}_2). \tag{11.4.1}$$

Proof. As in [12, proof of 5.7.5] we define the morphism as the composite

$$\begin{aligned}
\mathrm{R}f_!(\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2) &\xrightarrow{\simeq} \mathrm{D}_{\mathcal{Y}}(f_* \mathrm{D}_{\mathcal{X}}(\mathrm{L}_1 \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{L}_2)) \\
&\xrightarrow{\simeq} \mathrm{D}_{\mathcal{Y}}(f_*(\mathrm{D}_{\mathcal{X}_1}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{D}_{\mathcal{X}_2}(\mathrm{L}_2))) \\
&\longrightarrow \mathrm{D}_{\mathcal{Y}}(f_{1*} \mathrm{D}_{\mathcal{X}_1}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} (f_{2*} \mathrm{D}_{\mathcal{X}_2}(\mathrm{L}_2))) \\
&\xrightarrow{\simeq} \mathrm{D}_{\mathcal{Y}_1}(f_{1*} \mathrm{D}_{\mathcal{X}_1}(\mathrm{L}_1)) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{D}_{\mathcal{Y}_2}(f_{2*} \mathrm{D}_{\mathcal{X}_2}(\mathrm{L}_2)) \\
&\xrightarrow{\simeq} \mathrm{R}f_{1!}(\mathrm{L}_1) \overset{\mathbf{L}}{\otimes}_{\mathrm{S}} \mathrm{R}f_{2!}(\mathrm{L}_2).
\end{aligned}$$

That this morphism is an isomorphism is reduced, as in the proof of 11.3, to loc. cit. □

12. Base change theorem

Theorem 12.1. *Let*

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "a"] \arrow[d, "f'"'] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{Y}' \arrow[r, "b"] & \mathcal{Y}
\end{tikzcd}$$

be a cartesian square of nice S*–stacks. Then the two functors*

$$b^*\mathrm{R}f_!, \mathrm{R}f'_!a^* : \mathbf{D}_c^-(\mathcal{X}, \Lambda) \to \mathbf{D}_c^-(\mathcal{Y}', \Lambda)$$

are canonically isomorphic.

The proof of 12.1 follows essentially the same outline as the proof of [12, 5.5.6].

Lemma 12.2. *For any* $\mathrm{A}, \mathrm{B}, \mathrm{C} \in \mathbf{D}_c(\mathcal{X}, \Lambda)$ *there is a canonical morphism*

$$\mathrm{A} \overset{\mathbf{L}}{\otimes} \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{B}, \mathrm{C}) \to \boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\boldsymbol{\mathcal{R}hom}_{\boldsymbol{\Lambda}}(\mathrm{A}, \mathrm{B}), \mathrm{C}).$$

Proof. This is shown by the same argument proving [12, 5.5.7]. □

By [12, 5.4.4], there exists a commutative diagram

$$\begin{tikzcd}
\mathrm{Y}'_\bullet \arrow[r, "j"] \arrow[d, "p"'] & \mathrm{Y}_\bullet \arrow[d, "q"] \\
\mathcal{Y}' \arrow[r, "b"] & \mathcal{Y},
\end{tikzcd} \tag{12.2.1}$$

where p and q are smooth hypercovers and j is a closed immersion.

Let $\mathcal{X}'_{\mathrm{Y}'_\bullet}$ denote the base change $\mathcal{X}' \times_{\mathcal{Y}'} \mathrm{Y}'_\bullet$ and $\mathcal{X}_{\mathrm{Y}_\bullet}$ the base change $\mathcal{X} \times_{\mathcal{Y}} \mathrm{Y}_\bullet$. Then there is a cartesian diagram

$$\begin{tikzcd}
\mathcal{X}'_{\mathrm{Y}'_\bullet} \arrow[r, "i"] \arrow[d, "g'"'] & \mathcal{X}_{\mathrm{Y}_\bullet} \arrow[d, "g"] \\
\mathrm{Y}'_\bullet \arrow[r, "j"] & \mathrm{Y}_\bullet,
\end{tikzcd}$$

where i and j are closed immersions.

As in [12, section 5] let $\omega_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}$ (resp. $\omega_{\mathcal{X}_{\mathrm{Y}_\bullet}}$, $\omega_{\mathrm{Y}'_\bullet}$, $\omega_{\mathrm{Y}_\bullet}$) denote the pullback of $\Omega_{\mathcal{X}'}$ (resp. $\Omega_{\mathcal{X}}$, $\Omega_{\mathcal{Y}'}$, $\Omega_{\mathcal{Y}}$) to $\mathcal{X}'_{\mathrm{Y}'_\bullet}$ (resp. $\mathcal{X}_{\mathrm{Y}_\bullet}$, $\mathrm{Y}'_\bullet$, $\mathrm{Y}_\bullet$), and let $\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}$ (resp. $\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}}$, $\mathrm{D}_{\mathrm{Y}'_\bullet}$, $\mathrm{D}_{\mathrm{Y}_\bullet}$) denote the functor $\mathcal{R}hom(-, \omega_{\mathrm{X}'_{\mathrm{Y}'_\bullet}})$ (resp. $\mathcal{R}hom(-, \omega_{\mathcal{X}_{\mathrm{Y}_\bullet}})$, $\mathcal{R}hom(-, \omega_{\mathrm{Y}'_\bullet})$, $\mathcal{R}hom(-, \omega_{\mathrm{Y}_\bullet})$). Note that these functors are defined already on the level of the derivated category of projective systems, though they also pass to the categories $\mathbf{D}_c$.

Let $\mathcal{F}$ denote the functor

$$\mathrm{D}_{\mathrm{Y}'_\bullet} j^* \mathrm{D}_{\mathrm{Y}_\bullet} \mathrm{R}g_* \mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}} i_* \mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}} : \mathcal{D}_c(\mathcal{X}'^{\mathbb{N}}_{\mathrm{Y}'_\bullet}, \Lambda_\bullet) \to \mathcal{D}_c(\mathrm{Y}'^{\mathbb{N}}_\bullet, \Lambda_\bullet).$$

By the same argument proving [12, 5.5.8] one sees that there is a canonical isomorphism $\mathcal{F} \simeq \mathrm{R}g'_*$ (define a morphism of functors as in the beginning of the proof of [12, 5.5.8] and then to check that it is an isomorphism it suffices to consider each Λ_n where the result is loc. cit.), and hence also an isomorphism $\mathbf{F} \simeq \mathrm{R}g'_* : \mathbf{D}_c(\mathcal{X}'_{\mathrm{Y}'_\bullet}, \Lambda) \to \mathbf{D}_c(\mathrm{Y}'_\bullet, \Lambda)$. This isomorphism induces a morphism of functors

$$\begin{aligned} j^*\mathrm{D}_{\mathrm{Y}_\bullet}\mathrm{R}g_*\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}} &\to j^*\mathrm{D}_{\mathrm{Y}_\bullet}\mathrm{R}g_*\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}}i_*i^* \quad (\mathrm{id} \to i_*i^*) \\ &\simeq \mathrm{D}_{\mathrm{Y}'_\bullet}\mathrm{D}_{\mathrm{Y}'_\bullet}j^*\mathrm{D}_{\mathrm{Y}_\bullet}\mathrm{R}g_*\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}}i_*\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}i^* \quad (7.7) \\ &\simeq \mathrm{D}_{\mathrm{Y}'_\bullet}\mathbf{F}\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}i^* \quad (\text{definition}) \\ &\simeq \mathrm{D}_{\mathrm{Y}'_\bullet}\mathrm{R}g'_*\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}i^*. \end{aligned} \tag{12.2.2}$$

If $\epsilon : \mathcal{X}_{\mathrm{Y}_\bullet} \to \mathcal{X}$ and $\epsilon' : \mathcal{X}'_{\mathrm{Y}_\bullet} \to \mathcal{X}'$ are the projections, we then obtain a morphism

$$\begin{aligned} b^*\mathrm{R}f_! &\simeq b^*\mathrm{R}q_*\mathrm{D}_{\mathrm{Y}_\bullet}\mathrm{R}g_*\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}}\epsilon^* \quad (\text{cohomological descent}) \\ &\to \mathrm{R}p_*j^*\mathrm{D}_{\mathrm{Y}_\bullet}\mathrm{R}g_*\mathrm{D}_{\mathcal{X}_{\mathrm{Y}_\bullet}}\epsilon^* \quad (\text{base change morphism}) \\ &\to \mathrm{R}p_*\mathrm{D}_{\mathrm{Y}'_\bullet}\mathrm{R}g'_*\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}i^*\epsilon^* \quad (12.2.2) \\ &\simeq \mathrm{R}p_*\mathrm{D}_{\mathrm{Y}'_\bullet}\mathrm{R}g'_*\mathrm{D}_{\mathcal{X}'_{\mathrm{Y}'_\bullet}}\epsilon'^*a^* \quad (i^*\epsilon^* = \epsilon'^*a^*) \\ &\simeq \mathrm{R}f'_!a^* \quad (\text{cohomological descent}). \end{aligned} \tag{12.2.3}$$

which we call the *base change morphism.* That it is an isomorphism is shown as in the proof of [12, 5.5.6] by reduction to the case of schemes. This completes the proof of 12.1. ☐

As in [12] for some special classes of morphisms one can describe the base change arrow more explicitly. The proofs that the following alternate definitions coincide with the one in 12.1 proceed as in [12] so we omit them.

By 7.7 to prove the formula $b^*\mathrm{R}f_! = \mathrm{R}f'_!a^*$ it suffices to prove the dual version $b^!\mathrm{R}f_* = \mathrm{R}f'_*a^!$ which can do directly in the following cases.

12.3. **Smooth base change.** By 9.3 the formula $b^!\mathrm{R}f_* = \mathrm{R}f'_*a^!$ is equivelent to the formula $b^*\mathrm{R}f_* = \mathrm{R}f'_*a^*$. We can therefore take the base change morphism $b^*\mathrm{R}f_* \to \mathrm{R}f'_*a^*$ (note that the construction of this arrow uses only adjunction for $(b^*, \mathrm{R}b_*)$ and $(a^*, \mathrm{R}a_*)$). To prove that this map is an isomorphism, note that it suffices to verify that it is an isomorphism locally in the topos $\mathcal{Y}'^{\mathbb{N}}$ where it follows from the case of finite coefficients [12, 5.1].

12.4. **Base change by a universal homeomorphism.** By 9.6 in this case $b^! = b^*$ and $a^! = a^*$. We then again take the base change arrow $b^*\mathrm{R}f_* \to \mathrm{R}f'_*a^*$ which as in the case of a smooth base change is an isomorphism by reduction to the case of finite coefficients [12, 5.4].

12.5. **Base change by an immersion.** In this case one can define the base change arrow using the projection formula 9.6.1 as in [12, 5.3].

Note first of all that by shrinking on $\mathcal{Y}$ it suffices to consider the case of a closed immersion. Let $\mathrm{A} \in \mathbf{D}_c(\mathcal{X}, \Lambda)$. Since b is a closed immersion, we have $b^*\mathrm{R}b_* = \mathrm{id}$. By the projection formulas for b and f we have

$$\mathrm{R}b_*b^*\mathrm{R}f_!\mathrm{A} = \mathrm{R}b_*(\Lambda) \overset{\mathbf{L}}{\otimes} \mathrm{R}f_!\mathrm{A} = \mathrm{R}f_!(\mathrm{A} \overset{\mathbf{L}}{\otimes} f^*\mathrm{R}b_*\Lambda).$$

Now clearly $f^*b_* = a_*f'^*$. We therefore have

$$\begin{aligned} \mathrm{R}f_!(\mathrm{A} \overset{\mathbf{L}}{\otimes} f^*\mathrm{R}b_*\Lambda) &\simeq \mathrm{R}f_!(\mathrm{A} \overset{\mathbf{L}}{\otimes} \mathrm{R}a_*f'^*\Lambda) \\ &\simeq \mathrm{R}f_!a_*(a^*\mathrm{A} \overset{\mathbf{L}}{\otimes} f'^*\Lambda) \\ &\simeq b_*\mathrm{R}f'_!(a^*\mathrm{A}) \end{aligned}$$

Applying b^* we obtain the base change isomorphism.

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École Polytechnique CMLS UMR 7640 CNRS F-91128 Palaiseau Cedex France
E-mail address: `laszlo@math.polytechnique.fr`

University of Texas at Austin Department of Mathematics 1 University Station C1200 Austin, TX 78712-0257, USA
E-mail address: `molsson@math.utexas.edu`