POINTS WITH LARGE STABILIZER GROUPS AND SECTIONS OF VECTOR BUNDLES

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Abstract. We discuss how to construct canonical sections of certain vector bundles on degenerating varieties using equivariant and stack-theoretic techniques. We discuss in detail the examples of hypersurfaces and $K3$ surfaces.

1. Introduction

1.1. This article is inspired by classical work on theta functions, especially that of Mumford in [7, 8, 9], and recent work of Gross, Hacking, Keel, and Siebert [6] and Alexeev, Engel, and Thompson [3].

The basic phenomenon we wish to investigate is how degenerations of algebraic varieties give rise to sections of certain vector bundles in tubular neighborhoods of boundary points. In this article we briefly discuss some general definitions and results, placing the questions in a general stack-theoretic context. After this, we discuss in detail hypersurfaces and $K3$ surfaces.

1.2. At the core of Mumford's construction of theta functions on moduli spaces of abelian varieties are certain representation theoretic observations. To motivate our later constructions we summarize briefly some salient aspects of Mumford's theory (for more details see [8]). Fix positive integers $\delta = (d_1, \dots, d_g)$, with $d_{i+1}|d_i$, and let $K(\delta)$ denote the group scheme

$$K(\delta) := (\prod_i \mu_{d_i}) \times (\prod_i \mathbb{Z}/(d_i)).$$

There is a natural pairing

$$\langle \cdot, \cdot \rangle : K(\delta) \times K(\delta) \to \mathbf{G}_m,$$

which one can use to define a "Heisenberg" group scheme $\mathscr{G}_\delta$ over $\mathrm{Spec}(\mathbf{Z})$. Following [11], let $\mathscr{T}_{g,\delta}$ denote the moduli stack over $\mathbf{Z}[1/d]$, where $d = \prod_i d_i$, whose fiber over a $\mathbf{Z}[1/d]$-scheme S is the groupoid of triples (A, P, L), where

(i) A is an abelian scheme of relative dimension g over S.
(ii) P is an A-torsor
(iii) L is a relatively ample invertible sheaf on P.
(iv) If $\mathscr{G}_{(P,L)}$ denotes the group scheme of automorphisms of the pair (P, L), then the quotient $\mathscr{G}_{(P,L)}/\mathbf{G}_m$, where $\mathbf{G}_m$ is embedded in $\mathscr{G}_{(P,L)}$ through its action on L, is étale locally isomorphic to $K(\delta)$.

It is shown in [11, 5.10.3] that $\mathscr{T}_{g,\delta}$ is an algebraic stack (in fact, it is shown in loc. cit. that a certain compactification is an algebraic stack), which is a $\mathbf{G}_m$-gerbe over a Deligne-Mumford stack $\mathscr{A}_{g,\delta}$ [11, 5.1.4].

If $(\mathscr{A}, \mathscr{P}, \mathscr{L})$ denotes the universal triple over $\mathscr{T}_{g,\delta}$ and $\pi : \mathscr{P} \to \mathscr{T}_{g,\delta}$ the structure morphism, then $\mathscr{E} := \pi_*\mathscr{L}$ is a vector bundle on $\mathscr{T}_{g,\delta}$ which comes equipped with an action of $\mathscr{G}_{(\mathscr{P},\mathscr{L})}$.

Let $\mathscr{S}_{g,\delta}$ denote the stack classifying quadruples (A, P, L, σ), where $(A, P, L) \in \mathscr{T}_{g,\delta}$ and $\sigma : \mathscr{G}_{(P,L)} \simeq \mathscr{G}_\delta$ is an isomorphism which induces the identity on $\mathbf{G}_m$. The forgetful map

$$\mathscr{S}_{g,\delta} \to \mathscr{T}_{g,\delta}, \quad (A, P, L, \sigma) \mapsto (A, P, L)$$

is finite and étale [11, 6.3.7].

Now the group scheme $\mathscr{G}_\delta$ has a unique irreducible representation V_δ on which $\mathbf{G}_m$ acts via scalars (see [7, Proposition 3]), and the only automorphisms of V_δ are multiplication by scalars. Let $\mathscr{V}_\delta$ denote the trivial vector bundle with $\mathscr{G}_\delta$-action on $\mathscr{S}_{g,\delta}$ defined by V_δ. Then the two vector bundles with $\mathscr{G}_\delta$-action $\mathscr{V}_\delta$ and $\mathscr{E}$ are locally isomorphic (as representations) and any two local isomorphisms differ by scalars. As a result we obtain a canonical isomorphism

$$\mathbf{P}(\mathscr{V}_\delta) \simeq \mathbf{P}(\mathscr{E})$$

over $\mathscr{S}_{g,\delta}$.

The key feature from the viewpoint of this article is the presence of an algebraic group acting on the vector bundle over the moduli space whose representation theory imparts strong structure on the vector bundle $\mathscr{E}$.

Remark 1.3. In Mumford's setting one gets functions on the moduli stack from the preceding discussion as follows. An explicit description of the representation V_δ can be found in [7, p. 297]. The underlying vector space is given by the space of functions from $K(\delta)$ to $\mathbf{G}_a$. In particular, with this description we get an isomorphism $\mathbf{P}(V_\delta) \simeq \mathbf{P}^{d^2-1}$. Let $(\mathscr{A}, \mathscr{P}, \mathscr{L})$ be the universal triple over $\mathscr{S}_{g,\delta}$ and let $\mathscr{U}_{g,\delta} \subset \mathscr{S}_{g,\delta}$ be the maximal open substack over which $\mathscr{L}$ is generated by global sections, and let $\mathscr{P}_{U_{g,\delta}}$ be the restriction of $\mathscr{P}$ to $\mathscr{U}_{g,\delta}$. We then get a morphism

$$\mathscr{P}_{U_{g,\delta}} \to \mathbf{P}^{d^2-1}.$$

Loosely speaking, we get functions on $\mathscr{P}_{U_{g,\delta}}$. This approach can be modified to give functions on certain variants of $\mathscr{A}_{g,\delta}$ involving symmetric polarizations; see [8].

Example 1.4. Let C be a genus 1 curve over a field K and let L be an ample invertible sheaf on C of degree d invertible in K. Let E be the Jacobian of C. The natural map $C \to \mathrm{Pic}^1_C$ is an isomorphism, so C has a natural structure of an E-torsor. The triple (E, C, L) is then an object of $\mathscr{T}_{1,d}$; note that δ in this case is simply the integer d. To get an isomorphism $\mathbf{P}\Gamma(C, L) \simeq \mathbf{P}(V_d)$ we need the additional information of an isomorphism $\mathscr{G}_d \simeq \mathscr{G}_{(C,L)}$.

In the case when K is the fraction field of a complete local ring R and the map

$$\mathrm{Spec}(K) \to \mathscr{T}_{1,d}$$

extends to a map

$$\mathrm{Spec}(R) \to \overline{\mathscr{T}}_{1,d},$$

where $\overline{\mathscr{T}}_{1,d}$ is the compactification of $\mathscr{T}_{1,d}$ described in [11, 5.10.3], this can be done as follows. The tautological theta group $\mathscr{G}_{(\mathscr{P},\mathscr{L})}$ over $\mathscr{T}_{1,d}$ extends to $\overline{\mathscr{T}}_{1,d}$, and therefore we get a model $\mathscr{G}_R$ over R of $\mathscr{G}_{(C,L)}$. Now the scheme of isomorphisms between $\mathscr{G}_R$ and $\mathscr{G}_d$, which induce the identity on $\mathbf{G}_m$, is étale over R. Therefore an isomorphism $\mathscr{G}_k \simeq \mathscr{G}_d$ over the residue field k of R, which induces the identity on $\mathbf{G}_m$, induces an isomorphism $\mathscr{G}_{(C,L)} \simeq \mathscr{G}_d$ over K.

In the case when the induced map

$$\mathrm{Spec}(k) \to \overline{\mathscr{T}}_{1,d}$$

lands in the boundary of $\overline{\mathscr{T}}_{1,d}$, an isomorphism $\mathscr{G}_k \simeq \mathscr{G}_d$ can be obtained from the geometry of the limit point. The limit point corresponds to a d-gon with action of $\mathbf{G}_m$. The theta group is then canonically identified with $\mathscr{G}_d$, the action of $\mathbf{Z}/(d)$ given by rotation and the action of $\boldsymbol{\mu}_d$ given by the $\mathbf{G}_m$-action. In this way, a suitable degeneration of (C, L) induces an isomorphism $\mathscr{G}_{(C,L)} \simeq \mathscr{G}_d$.

1.5. In other contexts one does not have an algebraic group acting globally over the moduli space. However, the main point of this article is to explain how one can use degenerations to highly degenerate objects, where one does have an algebraic group acting, and then use representation theory to describe the restriction of global vector bundles to a formal neighborhood of boundary points.

In this article we discuss this in two main examples. The first is the case of Calabi-Yau hypersurfaces discussed in section 3. The second is the case of polarized $K3$ surfaces discussed in sections 6-8. Prior to this, we develop in section 2 a formal framework for thinking about θ-functions. Sections 4 and 5 are devoted to background material needed for the $K3$-surface case.

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2. A formal framework

2.1. **Intuition.** This subsection is not intended as precise mathematics, but to aid the reader we provide some informal discussion to put the stack-theoretic considerations which follow in later subsections in context.

Let A be a complete local ring over a field k. Assume that the residue field of A is k and denote by K the field of fractions of A. Let X_K be a smooth projective variety over K equipped with an invertible sheaf L_K. The basic goal is to use geometric properties of (X_K, L_K) to construct, in certain situations, a canonical basis for $H^0(X_K, L_K)$ – what is meant by *canonical* here is part of the question we wish to address in this article.

If L_K is ample, which we assume henceforth, we can consider the algebraic stack $\mathcal{S}$ classifying polarized varieties (Y, M). This is an algebraic stack locally of finite type over k. It is highly non-separated and its global structure is complicated, but this is not an obstacle for the purposes of this article. The pair (X_K, L_K) corresponds to a morphism

$$f_K : \mathrm{Spec}(K) \to \mathcal{S}.$$

Suppose given an extension (X, L) of (X_K, L_K) to a projective flat family over $\mathrm{Spec}(A)$ giving an extension

$$f : \mathrm{Spec}(A) \to \mathcal{S}$$

of f_K. Suppose further that, after possibly replacing $\mathcal{S}$ by an open substack containing the image of f, the association $(Y, M) \mapsto H^0(Y, M)$ defines a locally free sheaf $\mathcal{E}$ on $\mathcal{S}$. So

$$H^0(X_K, L_K) = f_K^*\mathcal{E}.$$

Let

$$f_0 : \mathrm{Spec}(k) \to \mathcal{S}$$

be the restriction of f to the closed point of $\mathrm{Spec}(A)$, so f_0 corresponds to the reduction (X_k, L_k) of (X, L). Assume that the automorphism group scheme of (X_k, L_k) is a torus T (in the rest of the article we will also consider different groups, but for the purposes of this informal discussion this simplification is helpful). We then have an immersion $i : BT \hookrightarrow \mathcal{S}$ and the map f_0 factors through a morphism

$$\bar{f}_0 : \mathrm{Spec}(k) \to BT.$$

Let $\mathcal{E}_0$ denote $i^*\mathcal{E}$. Under the correspondence between quasi-coherent sheaves on BT and T-representations the sheaf $\mathcal{E}_0$ corresponds to $H^0(X_k, L_k)$ with its natural T-action. Assume further that all the characters of T occurring in $H^0(X_k, L_k)$ appear with multiplicity 1. Then the character decomposition of $H^0(X_k, L_k)$ defines a decomposition

$$\mathcal{E}_0 \simeq \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$$

of $\mathcal{E}_0$ into a sum of line bundles. Loosely speaking we have a basis for $H^0(X_k, L_k)$, well-defined up to scaling, and we would like to find a method to move this basis over the closed fiber to the generic fiber.

Assume for the moment that we have a notion of formal completion of an algebraic stack along a substack (such a theory has been developed in [E], but in what follows we will prefer to work simply with the system of infinitesimal neighborhoods) and let $\widehat{\mathcal{S}}$ denote the formal completion of $\mathcal{S}$ along $i : BT \hookrightarrow \mathcal{S}$. Let $\widehat{\mathcal{E}}$ denote the pullback of $\mathcal{E}$ to $\widehat{\mathcal{S}}$. Suppose we can find a pair (r, ρ), where r is a morphism as indicated in the following diagram

$$\begin{tikzcd} BT \arrow[r, hook, "i"] \arrow[dr, equal] & \widehat{\mathcal{S}} \arrow[d, dashed, "r"] \\ & BT, \end{tikzcd}$$

and $\rho : r^*\mathcal{E}_0 \simeq \widehat{\mathcal{E}}$ is an isomorphism reducing to the identity over BT. We then get a decomposition

$$\widehat{\mathcal{E}} \simeq r^*\mathcal{E}_0 \simeq \oplus_i r^*\mathcal{L}_i$$

of $\widehat{\mathcal{E}}$ into a direct sum of line bundles. If

$$\hat{f} : \mathrm{Spec}(A) \to \widehat{\mathcal{S}}$$

denotes the morphism defined by (X, L) we then also get a decomposition of $H^0(X, L)$ into a sum of 1-dimensional subspaces. Passing to the generic fiber we get the desired basis for $H^0(X_K, L_K)$, well-defined up to scaling.

Summary: The construction of (r, ρ) yields θ-functions.

For the rest of this section we make the preceding informal discussion precise, and address the questions of existence and uniqueness of the pair (r, ρ).

Remark 2.2. Note that giving a morphism $r : \widehat{\mathcal{S}} \to BT$ as above is equivalent to giving a T-torsor $\widehat{P} \to \widehat{\mathcal{S}}$ reducing to the trivial torsor over BT. Such a torsor $\widehat{P}$ is necessarily a formal scheme, since its reduction to BT is a scheme, and therefore gives a presentation $\widehat{\mathcal{S}} \simeq [\widehat{P}/T]$, in a suitable sense of formal stacks.

We will use this point of view to understand the existence of such maps r in what follows.

2.3. **Minimal presentations.** Throughout this section we work over a base scheme S.

2.4. Let $\mathcal{S}$ be an algebraic stack of finite type over S and let G/S be a linearly reductive group scheme. Let $\mathcal{G} \hookrightarrow \mathcal{S}$ be a closed substack, and suppose $\mathcal{G}$ is a G-gerbe over a closed subscheme $S_0 \subset S$.

For each integer n let $i_n : \mathcal{G} \hookrightarrow \mathcal{S}_n$ be the n-th order neighborhood of $\mathcal{G}$ in $\mathcal{S}$ defined by the quasi-coherent sheaf of ideals $J^{n+1} \subset \mathcal{O}_{\mathcal{S}}$, where $J \subset \mathcal{O}_{\mathcal{S}}$ is the ideal of $\mathcal{G}$ in $\mathcal{S}$.

Definition 2.5. An *n-th order minimal presentation of* $\mathcal{S}$ is a pair (r_n, σ), where $r_n : \mathcal{S}_n \to \mathcal{G}$ is a morphism of stacks and $\sigma : r_n \circ i_n \simeq \mathrm{id}_{\mathcal{G}}$ is an isomorphism of functors $\mathcal{G} \to \mathcal{G}$.

Remark 2.6. We will be most interested in the case when $\mathcal{G} = BG$ is the trivial gerbe over S_0. In this case giving an n-th order infinitesimal presentation is equivalent to giving a deformation $P_n \to \mathcal{S}_n$ of the trivial G-torsor $P_0 \to BG$. Observe that in this situation the stack P_n is an algebraic space, being a deformation of the algebraic space P_0, and therefore we get a presentation $[P_n/G] \simeq \mathcal{S}_n$. The terminology "minimal presentation" arises from this: a trivialization of the gerbe $\mathcal{G}$ identifies n-th order minimal presentations with presentations of the stack $\mathcal{S}_n$ as a quotient stack lifting the presentation of the trivial gerbe.

2.7. Morphisms of n-th order minimal presentations are defined to be morphisms of functors compatible with the identifications of the restrictions to $\mathcal{G}$. We denote the resulting category of n-th order minimal presentations by

$$\mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S}).$$

2.8. For $m < n$ there is a reduction functor

$$\pi_{n,m} : \mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S}) \to \mathrm{Pres}^m(\mathcal{G} \subset \mathcal{S}).$$

Definition 2.9. A *formal minimal presentation of* $\mathcal{S}$ is an object of the 2-categorical limit

$$\mathrm{Pres}^\infty(\mathcal{G} \subset \mathcal{S}) := \lim_n \mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S}).$$

2.10. Allowing n to be ∞ we also have reduction functors

$$\pi_{\infty,m} : \mathrm{Pres}^\infty(\mathcal{G} \subset \mathcal{S}) \to \mathrm{Pres}^m(\mathcal{G} \subset \mathcal{S}).$$

The following summarizes the basic features of the category of formal minimal presentations (loosely speaking, formal minimal presentations always exist and are unique up to *non-canonical* isomorphism).

Proposition 2.11. *The category* $\mathrm{Pres}^\infty(\mathcal{G} \subset \mathcal{S})$ *has one isomorphism class of objects, and the automorphism of any object in this class can be written as a projective limit*

$$\varprojlim_{n \geq 0} U_n,$$

where $U_0 = \{e\}$*,* $U_n \to U_{n-1}$ *is surjective for all* n*, and* $\mathrm{Ker}(U_n \to U_{n-1})$ *is the underlying abelian group of an* $\mathcal{O}_{S_0}$*-module.*

Proof. It suffices to show that each $\mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S})$ has one isomorphism class of objects and that for any object $(r_n, \sigma_n) \in \mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S})$ with image $(r_{n-1}, \sigma_{n-1}) \in \mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S})$ the group of automorphisms U_n of (r_n, σ_n) surjects onto $U_{n-1} := \mathrm{Aut}(r_{n-1}, \sigma_{n-1})$ with kernel given by the underlying abelian group of an $\mathcal{O}_{S_0}$-module.

Lemma 2.12. *Let $\mathcal{G}$ be a G-gerbe over S_0. Then the relative cotangent complex $L_{\mathcal{G}/S_0}$ is isomorphic to $\mathcal{L}[-1]$, where $\mathcal{L}$ is a locally free sheaf of finite rank.*

If $\mathcal{G} = BG$ then $\mathcal{L}$ is the locally free sheaf on $\mathcal{G}$ dual to the vector bundle associated to the Lie algebra $\mathrm{Lie}(G)$ with the adjoint action of G.

Proof. The assertion is étale local on S_0 so it suffices to consider the case when $\mathcal{G} = BG$, where the result follows from a basic calculation. □

By standard deformation theory the obstruction to lifting an object $(r_{n-1}, \sigma_{n-1}) \in \mathrm{Pres}^{n-1}(\mathcal{G} \subset \mathcal{S})$ to $\mathrm{Pres}^n(\mathcal{G} \subset \mathcal{S})$ is a class in

$$\mathrm{Ext}^1(L_{\mathcal{G}/S_0}, J^n/J^{n+1}) \simeq H^2(\mathcal{G}, \mathcal{L}^\vee \otimes J^n/J^{n+1}),$$

and the obstruction to lifting an element of U_{n-1} to U_n is an element of

$$\mathrm{Ext}^0(L_{\mathcal{G}/S_0}, J^n/J^{n+1}) \simeq H^1(\mathcal{G}, \mathcal{L}^\vee \otimes J^n/J^{n+1}).$$

Since S is affine we have

$$H^i(\mathcal{G}, \mathcal{L}^\vee \otimes J^n/J^{n+1}) \simeq H^0(S_0, R^i\pi_*(\mathcal{L}^\vee \otimes J^n/J^{n+1})),$$

where $\pi : \mathcal{G} \to S_0$ is the structure morphism. Now since G is linearly reductive, we have $R^i\pi_*(\mathcal{L}^\vee \otimes J^n/J^{n+1}) = 0$ for $i \neq 0$. Indeed, étale locally on S this is calculated by the cohomology of a quasi-coherent sheaf on BG; equivalently, by the group cohomology of a representation of G. We therefore find that

$$\mathrm{Ext}^i(L_{\mathcal{G}/S_0}, J^n/J^{n+1}) \simeq H^{i+1}(\mathcal{G}, \mathcal{L}^\vee \otimes J^n/J^{n+1}) = 0$$

for $i \neq -1$. In particular, the obstruction to lifting (r_{n-1}, σ_{n-1}) is zero.

Similarly the set of isomorphism classes of liftings is a torsor under $\mathrm{Ext}^0(L_{\mathcal{G}/S_0}, J^n/J^{n+1})$, which is zero, and the group of infinitesimal automorphisms is cannonically in bijection with $\mathrm{Ext}^{-1}(L_{\mathcal{G}/S_0}, J^n/J^{n+1})$. This completes the proof of 2.11. □

Note that the proof gives an explicit description of the automorphism groups of objects of $\mathrm{Pres}^\infty(\mathcal{G} \subset \mathcal{S})$. In particular:

Corollary 2.13. *Suppose that for every $n \geq 1$ we have $(\mathrm{Lie}(G) \otimes J^n/J^{n+1})^G = 0$. Then a formal minimal presentation exists and is unique up to unique isomorphism.*

Proof. Indeed in this case we have $R^0\pi_*(\mathcal{L}^\vee \otimes J^n/J^{n+1}) = 0$, and therefore

$$\mathrm{Ext}^{-1}(L_{\mathcal{G}/S_0}, J^n/J^{n+1}) = 0$$

for all n. With notation as in the above proof it follows that the maps $U_n \to U_{n-1}$ are isomorphisms for all n. □

2.14. **Deformations of vector bundles.**

2.15. We continue with the setup and notation of 2.4. Assume further that we have fixed a formal minimal presentation

$$\hat{r} := \{(r_n, \sigma_n)\}_{n\geq 0} \in \mathrm{Pres}^\infty(\mathcal{G} \subset \mathcal{S}).$$

Let $\mathcal{E}$ be a vector bundle on $\mathcal{S}$ and let $\mathcal{E}_n$ be its pullback to $\mathcal{S}_n$. Pulling back $\mathcal{E}_0$ on $\mathcal{G}$ along the maps r_n we obtain a compatible system of vector bundles $\mathcal{F}_n$ on the $\mathcal{S}_n$, with $\mathcal{F}_0 = \mathcal{E}_0$.

2.16. We want to understand the relationship between the systems $\{\mathcal{E}_n\}$ and $\{\mathcal{F}_n\}$.

To this end, note that if $\mathcal{R}_n$ is a vector bundle on $\mathcal{S}_n$ with reduction $\mathcal{R}_0$ to $\mathcal{G}$ then the deformations of $\mathcal{R}_n$ to $\mathcal{S}_{n+1}$ are controlled by the cohomology groups

$$H^*(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1})$$

in the following sense:

(i) There is an obstruction $o(\mathcal{R}_n) \in H^2(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1})$ whose vanishing is necessary and sufficient for the existence of a lifting of $\mathcal{R}_n$ to $\mathcal{S}_{n+1}$.
(ii) If $o(\mathcal{R}_n) = 0$ then the set of isomorphism classes of liftings of $\mathcal{R}_n$ to $\mathcal{S}_{n+1}$ is a torsor under $H^1(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1})$.
(iii) The group of infinitesimal automorphisms of a lifting to $\mathcal{S}_{n+1}$ is canonically isomorphic to $H^0(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1})$.

Now in our case, the linear reductivity of G combined with the assumption that S is affine implies that $H^i(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1}) = 0$ for $i > 0$, as in the proof of 2.11. We conclude:

Corollary 2.17. *If $\mathcal{R}_n$ is a vector bundle on $\mathcal{S}_n$ with reduction $\mathcal{R}_0$ to $\mathcal{G}$ then there exists a lifting $\mathcal{R}_{n+1}$ of $\mathcal{R}_n$ to $\mathcal{S}_{n+1}$, and such a lifting is unique up to isomorphism. The group of automorphisms of such a lifting $\mathcal{R}_{n+1}$ which reduce to the identity on $\mathcal{R}_n$ is canonically isomorphic to*

$$H^0(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1}).$$

Proof. This follows from the preceding discussion. □

Corollary 2.18. *(i) There exists an isomorphism of systems*

$$\rho : \{\mathcal{E}_n\} \to \{\mathcal{F}_n\} \tag{2.18.1}$$

reducing to the given isomorphism $\mathcal{E}_0 \simeq \mathcal{F}_0$.

(ii) The automorphism group of the system $\{\mathcal{E}_n\}$ can be written as a projective limit $\varprojlim_{n\geq 0} V_n$, where $V_0 = \{e\}$, $V_n \to V_{n-1}$ is surjective for all n, and $\mathrm{Ker}(V_n \to V_{n-1})$ *is the underlying abelian group of an $\mathcal{O}_{S_0}$-module.*

(iii) Suppose further that

$$H^0(\mathcal{G}, \mathcal{E}nd(\mathcal{R}_0) \otimes J^n/J^{n+1}) = 0$$

for all n. Then the isomorphism (2.18.1) *is unique.*

Proof. This is immediate from 2.17. □

3. Hypersurfaces

In this section we explain how to construct a formal minimal presentation in the example of a particular degeneration of hypersurfaces. In the following sections we discuss in more detail more complicated degenerations of $K3$ surfaces.

3.1. Let $(Y, \mathcal{O}_Y(1))$ be the polarized scheme over $\mathbf{Z}$ given by

$$\{X_0 \cdots X_n = 0\} \subset \mathbf{P}^n_{\mathbf{Z}}.$$

Let $R' \to R$ be a surjective map of rings with square-zero kernel I, and let $(Y_R, \mathcal{O}_{Y_R}(1))$ denote the base change of $(Y, \mathcal{O}_Y(1))$ to R.

Proposition 3.2. *(i) Any locally trivial deformation (in the étale topology) of* $(Y_R, \mathcal{O}_{Y_R}(1))$ *to* R' *is trivial in the sense that it is isomorphic, as a deformation, to the base change of* $(Y, \mathcal{O}_Y(1))$ *to* R'.

(ii) The automorphism group scheme of $(Y, \mathcal{O}_Y(1))$ *is isomorphic to* $\mathbf{G}_m^{n+1} \rtimes S_{n+1}$ *acting in the standard manner on the coordinates.*

Proof. The key input to the proof is the following observations:

Lemma 3.3. *Let* $m \geq 1$ *be an integer and consider the triple*

$$(\mathbf{P}^m_R, \mathcal{O}_{\mathbf{P}^m_R}(1), \cup_{i=0}^m V(X_i)), \tag{3.3.1}$$

consisting of the projective space over R*, the canonical ample invertible sheaf, and the divisor given by the union of the coordinate axes.*

(i) Any deformation of this triple to R' *which is locally trivial, in the sense that the deformation of the divisor is locally trivial, is isomorphic to*

$$(\mathbf{P}^m_{R'}, \mathcal{O}_{\mathbf{P}^m_{R'}}(1), \cup_{i=0}^m V(X_i))$$

(ii) The automorphism group scheme of the triple (3.3.1) *is isomorphic to the semi-direct product* $\mathbf{G}_m^{m+1} \rtimes S_{m+1}$ *acting in the standard way on the coordinates* $X_0, \ldots, X_m$.

Proof. Let $(T, \mathcal{O}_T(1), D)$ be a deformation of (3.3.1) to R'. The assumption that the deformation is locally trivial implies that $D = \cup_{i=0}^m D_i$, where D_i is a deformation of the divisor $V(X_i)$ to T. Furthermore, since $H^1(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}) = 0$ we have

$$\mathcal{O}_T(D_i) \simeq \mathcal{O}_T(1)$$

for all i.

Fix such isomorphisms reducing to the standard isomorphisms given by the X_i over R. The canonical sections $s_i \in \Gamma(T, \mathcal{O}_T(D_i))$ then define liftings of the X_i to $\Gamma(T, \mathcal{O}_T(1))$ which then induce an isomorphism

$$(T, \mathcal{O}_T(1), D) \simeq (\mathbf{P}^m_{R'}, \mathcal{O}_{\mathbf{P}^m_{R'}}(1), \cup_{i=0}^m V(X_i)).$$

This proves (i).

For (ii), note that any automorphism must preserve the lines in $\Gamma(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}^m}(1))$ spanned by the X_i and therefore is given by an element of $\mathbf{G}_m^{m+1} \rtimes S_{m+1}$ as claimed. $\square$

From this we can deduce the proposition. If $(Y', \mathscr{O}_{Y'}(1))/R'$ is a locally trivial deformation of $(Y, \mathscr{O}_Y(1))$, let $\mathscr{I}'$ denote the ideal defining the support of $\mathscr{E}xt^1(\Omega^1_{Y'/R'}, \mathscr{O}_{Y'})$. Étale locally, using the local triviality, the scheme Y' is isomorphic to

$$R'[z_1, \dots, z_n]/(z_1 \cdots z_r)$$

for some r, and the ideal sheaf $\mathscr{I}'$ is generated by the monomials $z_1 \cdots \hat{z}_i \cdots z_r$ for $1 \leq i \leq r$. In particular, the blowup of Y' along $\mathscr{I}'$ is a disjoint union $\coprod_i Y'_i$, where $Y'_i \hookrightarrow Y_i$ is a closed subscheme which is a flat deformation of $V(X_i) \subset Y$. Looking at the intersections of the components we find that for all subsets $I \subset \{0, \dots, n\}$, the scheme-theoretic intersection

$$Y'_I := \bigcap_{i \in I} Y'_i$$

together with the restriction $\mathscr{O}_{Y'_I}(1)$ of $\mathscr{O}_{Y'}(1)$ and the divisor

$$D'_I := \bigcup_{j \notin I} (Y_I \cap Y_j)$$

defines a locally trivial deformation

$$(Y'_I, \mathscr{O}_{Y'_I}(1), D'_I)$$

of $(V(\prod_{i \in I} X_i), \mathscr{O}_{V(\prod_{i\in I} X_i)}(1), D_I)$, where D_I is the divisor defined by the coordinate hyperplanes on $V(\prod_{i \in I})$, which is a projective space with coordinates indexed by elements of $\{0, \dots, n\}$ not in I.

By 3.3 (i) each $(Y'_I, \mathscr{O}_{Y'_I}(1), D'_I)$ is isomorphic to the trivial deformation $(Y_{I,R'}, \mathscr{O}_{Y_{I,R'}}(1), D_{I,R'})$. Fix an isomorphism

$$\sigma_i; (Y'_i, \mathscr{O}_{Y'_i}(1), D'_i) \simeq (Y_{i,R'}, \mathscr{O}_{Y_{i,R'}}(1), D_{i,R'})$$

for each $i = 0, \dots, n$. The deformation is then specified by the gluing data of the double intersections, which by 3.3 (ii) is given by sections of $\mathrm{Lie}(\mathbf{G}_{m,\{i,j\}}) \otimes I$, where $\mathbf{G}_{m,\{i,j\}}$ denotes the group scheme of functions from $\{0, \dots, n\} \backslash \{i, j\}$ to $\mathbf{G}_m$. The compatibility with triple intersections is the condition that for each $i < j < k$ we have

$$u^k_{ij} u^i_{jk} = u^k_{ij},$$

in $\mathbf{G}_{m,\{i,j,k\}}$, where we write u^k_{ij} (resp. u^i_{jk}, u^j_{ik}) for the image of u_{ij} (resp. u_{jk}, u_{ik}).

For $s \in \{0, \dots, n\}$ and $i, j \subset \{0, \dots, \hat{s}, \dots, n\}$ let $u^{(s)}_{ij} \in \mathrm{Lie}(\mathbf{G}_m) \otimes I \simeq I$ denote the s-component of u_{ij}. The compatibility on triple overlaps then gives that for triples $i < j < k$ of elements of $\{0, \dots, \hat{s}, \dots, n\}$ we have

$$u^{(s)}_{ij} + u^{(s)}_{jk} = u^{(s)}_{ik}.$$

Therefore the $\{u^{(s)}_{ij}\}$ define an element of the complex calculating the Cech cohomology of the point with respect to the cover given by n copies of the point. In particular, the H^1 of this complex is trivial. It follows that there exist elements $v_i \in I$ for $i \neq s$ such that $u_{ij} = v_i - v_j$. Applying the infinitesimal automorphism of $(Y_{i,R'}, \mathscr{O}_{Y_{i,R'}}(1), D_{i,R'})$ given by v_i for each $i \neq s$ we can then arrange that the $u^{(s)}_{ij}$ are all 1. Repeating this for each s we obtain part (i) of 3.2.

Part (ii) of 3.2 follows from part (ii) of 3.3 applied to each irreducible component. $\square$

Remark 3.4. Using the preceding techniques we can also understand the locally trivial deformations of the underlying scheme Y. If Y'/R' is such a deformation then again we get for each $J \subset \{0, \ldots, n\}$ a deformation (Y'_J, D'_J) of $(Y_{J,R}, D_{J,R})$. Since $H^j(Y_J, \mathcal{O}_{Y_J}) = 0$ for $j > 0$ the restriction of $\mathcal{O}_{Y_R}(1)$ to $Y_{J,R}$ lifts uniquely to Y'_J, and using this we see that

$$(Y'_J, D'_J) \simeq (Y_{J,R'}, D_{J,R'}).$$

From this we also see that the automorphism group of (Y'_J, D'_J) is given by

$$(\mathbf{G}_{m,J}/\mathbf{G}_m) \rtimes Aut(\{0, \ldots, n\} - J),$$

where

$$\mathbf{G}_{m,J} := \mathrm{Fun}(\{0, \ldots, n\} - J, \mathbf{G}_m).$$

Let $\mathscr{C}^\bullet$ denote the complex which in degree j is given by

$$\mathscr{C}^j := \oplus_{J \subset \{0,\ldots,n\}, \# J = j} \mathrm{Lie}(\mathbf{G}_{m,J}) \otimes I,$$

and transition maps given by the restriction maps. Then the proof of 3.2 shows that the locally trivial deformations of $(Y_R, \mathcal{O}_{Y_R}(1))$ are given by $H^1(\mathscr{C}^\bullet)$. Let $\mathscr{K}^\bullet \subset \mathscr{C}^\bullet$ denote the subcomplex which in degree j is given by

$$\mathscr{K}^j := \oplus_{J \subset \{0,\ldots,n\}, \# J = j} \mathrm{Lie}(\mathbf{G}_m) \otimes I.$$

Then $\mathscr{K}^j$ calculates the cohomology $H^*(Y_R, \mathcal{O}_{Y_R})$ and from the preceding discussion it follows that if $\overline{\mathscr{C}}^\bullet$ denotes the quotient complex $\mathscr{C}^\bullet/\mathscr{K}^\bullet$ then the locally trivial deformations of Y_R are classified by $H^1(\overline{\mathscr{C}}^\bullet)$. The boundary map

$$H^1(\overline{\mathscr{C}}^\bullet) \rightarrow H^2(\mathscr{K}^\bullet) \simeq H^2(Y_R, \mathcal{O}_{Y_R}) \tag{3.4.1}$$

sends the class of a locally trival deformation Y'/R' to the obstruction to deforming $\mathcal{O}_{Y_R}(1)$ to Y'. Since we showed in the proof of 3.2 that $H^1(\mathscr{C}^\bullet) = 0$ we conclude that this obstruction map is injective.

3.5. Let $\mathscr{M}$ denote the fibered category over $\mathbf{Z}[1/n!]$ whose fiber over a scheme S are pairs (X, L), where:

(i) $f : X \rightarrow S$ is a flat proper S-scheme.
(ii) L is a relatively ample invertible sheaf on X.
(iii) L is base point free in all fibers, the sheaf f_*L is locally free of rank $n+1$, the formation of f_*L commutes with arbitrary base change on S, and the natural map

$$X \rightarrow \mathbf{P}(f_*L)$$

is a closed immersion which étale locally on S identifies X with a hypersurface in $\mathbf{P}(f_*L)$ of degree $n+1$.

Lemma 3.6. *The stack $\mathscr{M}$ is a smooth algebraic stack over $\mathbf{Z}[1/n!]$ of dimension*

$$\binom{2n+1}{n} - (n+1)^2 - 1.$$

Proof. Let $\widetilde{\mathscr{M}}$ denote the fibered category over $\mathbf{Z}[1/n!]$ whose fiber over a scheme S is the groupoid of triples (X, L, σ), where $(X, L) \in \mathscr{M}(S)$ and $\sigma : \mathcal{O}_S^{n+1} \rightarrow f_*L$ is an isomorphism of vector bundles. There is a projection map

$$\pi : \widetilde{\mathscr{M}} \rightarrow \mathscr{M}$$

realizing $\widetilde{\mathscr{M}}$ as a GL_{n+1}-torsor over $\mathscr{M}$.

The stack $\widetilde{\mathscr{M}}$ is in fact a scheme. Indeed the isomorphism σ gives a closed immersion

$$X \hookrightarrow \mathbf{P}(f_*L) \simeq \mathbf{P}^{n+1}_S$$

realizing X as a hypersurface of degree $n+1$ in projective space. Such hypersurfaces are themselves classified by a projective space of dimension $\binom{2n+1}{n} - 1$ (with coordinates the coefficients of the defining polynomial). This gives an identification of $\widetilde{\mathscr{M}}$ with this projective space. This therefore realizes $\mathscr{M}$ as a quotient stack

$$\mathscr{M} = [\widetilde{\mathscr{M}}/GL_{n+1}],$$

which also gives the formula for the dimension. □

Remark 3.7. By [12, 1.9] the stack $\mathscr{M}$ is generically a scheme.

3.8. The object $(Y, \mathscr{O}_Y(1))$ defined in 3.1, restricted to $\mathbf{Z}[1/n!]$, defines by 3.2 a closed immersion

$$B_{\mathbf{Z}[1/n!]}\mathscr{G} \hookrightarrow \mathscr{M},$$

where $\mathscr{G} := \mathbf{G}_m^{n+1} \rtimes S_{n+1}$. To understand the infinitesimal neighborhoods of this closed immersion we need to understand the deformation theory of $(Y, \mathscr{O}_Y(1))$.

In preparation for this we need a few calculations.

Lemma 3.9. *For any ring R We have $H^i(Y_R, \mathscr{O}_{Y_R}) = 0$ for $i \neq 0, n$ and $H^0(Y_R, \mathscr{O}_{Y_R}) \simeq H^n(Y_R, \mathscr{O}_{Y_R}) \simeq R$.*

Proof. The ordering of the components of Y gives a resolution

$$\mathscr{O}_Y \to \oplus_i \mathscr{O}_{Y_i} \to \oplus_{i<j} \mathscr{O}_{Y_{ij}} \to \oplus_{i<j<s} \mathscr{O}_{Y_{ijs}} \to \cdots,$$

where the differentials are obtained by taking alternating sums of the restriction maps. Since each of the components occuring in this resolution are projective spaces, and hence their structure sheaves have no higher cohomology, we find that the cohomology of Y is given by the cohomology of the complex

$$\oplus_i R \to \oplus_{i<j} R \to \oplus_{i<j<s} R \to \cdots.$$

This complex is computing the cohomology of the boundary of the standard n-simplex with coefficients in R. The result follows. □

Lemma 3.10. *For any ring R the R-module $H^0(D_R, \mathscr{O}_{D_R}(n+1))$ is a free R-module of rank*

$$\binom{2n+1}{n} - 1 - n^2 - n.$$

Proof. The R-module $H^0(D, \mathscr{O}_D(n+1))$ can be characterized as the subset of

$$\oplus_{i \neq j} H^0(D_{ij}, \mathscr{O}_{D_{ij}}(n+1)) \simeq \oplus_{i \neq j} k[X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_n]_{n+1}$$

of polynomials which agree on the triple overlaps. From this description we see that $H^0(D, \mathscr{O}_D(n+1))$ has a basis given by the monomials in $X_0, \ldots, X_n$ of degree $n+1$ in which at least two variables do not occur. An elementary calculation shows that the number of such monomials is as in the lemma. □

3.11. Note that it also follows from the proof that the restriction map

$$H^0(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(n+1)) \to H^0(D, \mathscr{O}_D(n+1))$$

is surjective.

3.12. For a ring R and R-module I we can consider the ring of dual numbers $R[I] = R \oplus I$, and the set of isomorphism classes of deformations of $(Y_R, \mathscr{O}_{Y_R}(1))$ to $R[I]$. We denote this set by $\mathbf{T}_{(Y_R, \mathscr{O}_{Y_R}(1))}$. Standard deformation theory gives the set an R-module structure.

The deformations of the underlying scheme Y_R is given by

$$\mathrm{Ext}^1(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R}) \otimes_R I.$$

The local to global spectral sequence for Ext places this module in an exact sequence

$$0 \to H^1(Y_R, \mathscr{E}xt^0(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R})) \otimes I \to \mathrm{Ext}^1(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R}) \otimes_R I \to H^0(D_R, \mathscr{O}_{D_R}(n+1)) \otimes I,$$

where we use the isomorphism [5, 2.3]

$$H^0(Y_R, \mathscr{E}xt^1(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R})) \simeq H^0(D_R, \mathscr{O}_{D_R}(n+1)).$$

Furthermore the subspace $H^1(Y_R, \mathscr{E}xt^0(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R})) \otimes I$ corresponds to locally trivial deformations.

Remark 3.13. It will be useful to have a variant of the preceding result in the presence of a group action. Assume that I comes equipped with a $\mathscr{G}$-action, and let $\mathscr{Y}_R$ denote the stack quotient Y_R/R. The $\mathscr{G}$-action on I induces a $\mathscr{G}$-action on $R[I]$ and we have a diagram

$$\begin{tikzcd} \mathscr{Y}_R \arrow[d] & \\ B_R\mathscr{G} \arrow[r, hook] & {[\mathrm{Spec}(R[I])/\mathscr{G}].} \end{tikzcd}$$

The preceding discussion then generalizes to show that the isomorphism classes of deformations of $\mathscr{Y}_R$ to $[\mathrm{Spec}(R[I])/\mathscr{G}]$ are given by the $\mathscr{G}$-invariants

$$(\mathrm{Ext}^1(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R}) \otimes I)^{\mathscr{G}},$$

the locally trivial deformations are given by

$$(H^0(Y_R, \mathscr{E}xt^0(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R})) \otimes I)^{\mathscr{G}},$$

and so on.

Lemma 3.14. *The map*

$$\mathrm{Ext}^1(\Omega^1_{Y_R/R}, \mathscr{O}_{Y_R}) \otimes_R I \to H^0(D_R, \mathscr{O}_{D_R}(n+1)) \otimes I \tag{3.14.1}$$

is surjective.

Proof. It suffices to consider the case when $I = R$. To ease notation we suppress the reference to R in the remainder of the proof.

Recall that the isomorphism

$$\mathscr{O}_D(n+1) \simeq \mathscr{E}xt^1(\Omega^1_Y, \mathscr{O}_Y)$$

is induced from the exact sequence

$$0 \to \mathcal{O}_Y(-n-1) \to \Omega^1_{\mathbf{P}^n}|_Y \to \Omega^1_Y \to 0 \tag{3.14.2}$$

by applying $\mathscr{R}Hom(-, \mathcal{O}_Y)$. Applying global $\mathrm{Ext}(-, \mathcal{O}_Y)$ one obtains a commutative diagram

$$\begin{tikzcd} H^0(Y, \mathcal{O}_Y(n+1)) \arrow[r] \arrow[d] & H^0(D, \mathcal{O}_D(n+1)) \arrow[d, "\simeq"] \\ \mathrm{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) \arrow[r] & H^0(Y, \mathscr{E}xt^1(\Omega^1_Y, \mathcal{O}_Y)), \end{tikzcd}$$

where the horizontal morphisms are restriction maps and the vertical morphisms are induced by (3.14.2). As noted in 3.11 the top horizontal map is surjective, which then implies the lemma. ☐

3.15. The group $H^1(Y_R, \mathscr{E}xt^0(\Omega^1_{Y_R/R}, \mathcal{O}_{Y_R}) \otimes I$ classifies locally trivial deformations. The obstruction to lifting $\mathcal{O}_{Y_R}(1)$ to a deformation is a class in $H^2(Y, \mathcal{O}_Y)$. By 3.9 this group is zero if $n > 3$ and therefore in this case the maps

$$\mathbf{T}_{(Y_R, \mathcal{O}_{Y_R}(1))} \otimes I \to \mathrm{Ext}^1(\Omega^1_{Y_R}, \mathcal{O}_{Y_R}) \otimes I \to H^0(D, \mathcal{O}_D(n+1)) \otimes I$$

are all isomorphisms.

If $n = 3$ we have $H^2(Y_R, \mathcal{O}_{Y_R}) \simeq R$, and by 3.2 (i) (see also 3.4) we have an injective map

$$H^1(Y_R, \mathscr{E}xt^0(\Omega^1_{Y_R/R}, \mathcal{O}_{Y_R})) \hookrightarrow H^2(Y, \mathcal{O}_{Y_R}) \simeq R. \tag{3.15.1}$$

Proposition 3.16 (Case $n = 3$)**.** *(i) The map* (3.15.1) *is surjective.*

(ii) The map

$$\mathbf{T}_{(Y_R, \mathcal{O}_{Y_R}(1))} \otimes I \to H^0(D, \mathcal{O}_D(4)) \otimes I$$

is an isomorphism.

Proof. Statement (ii) follows from (i), whose demonstration occupies the remainder of the proof.

It suffices to consider the case when R is artinian local.

We will exhibit a locally trivial deformation $\mathscr{Y}/R[\epsilon]$ of Y_R such that for any field k and ring homomorphism $\rho : R \to k$ the obstruction to lifting $\mathcal{O}_Y(1)$ to $\mathscr{Y}_\rho$, defined to be the deformation of Y_k to $k[\epsilon]$ obtained by base change along ρ, is nonzero. This suffices to prove the proposition, for if $S \subset R$ denotes the ideal which is the image of (3.15.1) and if $S \neq R$ then R/S is a nonzero ring and therefore admits a morphism to a field $\rho : R \to k$ such that the base change of (3.15.1) to k is zero. On the other hand, the existence of the family implies that the image of (3.15.1) remains nonzero after composition with all $\rho : R \to k$. We therefore must have $S = R$.

To construct the family let us begin by calculating the Picard group of Y_R. To ease notation we suppress the reference to R for the remainder of the proof. For $i \in \{0, 1, 2, 3\}$ let $Y_i \subset Y$ be the component defined by $X_i = 0$. We then have an exact sequence

$$0 \to \mathcal{O}^*_Y \to \oplus_i \mathcal{O}^*_{Y_i} \to \oplus_{(i,j)} \mathcal{O}^*_{D_{ij}} \to \oplus_{(i,j,h)} \mathcal{O}^*_{D_{ijh}} \to 0,$$

where for a subset $\{i, j, h\} \subset \{0, 1, 2, 3\}$ we denote by D_{ijh} the point given by $X_i = X_j = X_h = 0$.

The spectral sequence of a filtered complex associated to this resolution then has E_1^{*0}-row given by the complex

$$\oplus_i R^* \to \oplus_{(i,j)} R^* \to \oplus_{(i,j,k)} R^*,$$

whose cohomology is R^* in degrees 0 and 2 and 0 otherwise, and E_1^{*1}-row given by

$$\oplus_i \mathbf{Z} \to \oplus_{(i,j)} \mathbf{Z}.$$

Looking at the E_2-page we find that $\mathrm{Pic}(Y)$ is a subgroup of $\mathbf{Z}$, and since $\mathcal{O}_Y(1)$ maps to a generator in $\mathbf{Z}$ we conclude that in fact $\mathrm{Pic}(Y) = \mathbf{Z}$ with generator $\mathcal{O}_Y(1)$.

Let $Y' \subset Y$ be the union of the components Y_i for $i = 1, 2, 3$ so that $Y = Y' \cup Y_0$ with $Y' \cap Y_0$ a cycle of three rational curves. We denote this cycle of rational curves by Σ and the three components by Σ_i $(i = 1, 2, 3)$. Denote the point $\Sigma_i \cap \Sigma_j$ by P_{ij}. Then we have a resolution

$$\mathcal{O}_\Sigma^* \to \oplus_{i=1}^3 \mathcal{O}_{\Sigma_i}^* \to \oplus_{i,j} \mathcal{O}_{P_{ij}}^*$$

from which it follows that

$$\mathrm{Pic}(\Sigma) \simeq \mathbf{G}_m \times \mathbf{Z}^3.$$

The restriction map

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(\Sigma)$$

is given by the diagonal embedding $\mathbf{Z} \hookrightarrow \mathbf{Z}^3$.

There is an action of $\mathbf{G}_m$ on Σ given by multiplication on Σ_1 and the identity on the other components. Chasing through the above description of $\mathrm{Pic}(\Sigma)$ we see that the induced action of $\mathbf{G}_m$ on $\mathrm{Pic}(\Sigma)$ is given by

$$(v, (a, b, c)) \mapsto (u^a v, (a, b, c)).$$

From this we get a 1-parameter family $\mathcal{Y}$ over $\mathbf{G}_m$ of locally trivial deformations of Y by associating to unit $u \in \mathbf{G}_m$ (scheme-theoretic point) the space obtained by gluing Y' and Y_0 along the map $u : \Sigma \to \Sigma$. Equivalently, by forming the pushout in the category of algebraic spaces of the diagram

$$\begin{tikzcd} \Sigma \arrow[r, hook, "u"] \arrow[d, hook] & Y' \\ Y_0. & \end{tikzcd}$$

See [4, 6.1] for the existence of this pushout.

Let $u \in \mathbf{G}_m(R)$ be a unit and denote by $\mathcal{Y}_u$ the scheme over R obtained by base change from $\mathcal{Y}$. A calculation as above shows that $\mathrm{Pic}(Y'_u) = \mathbf{Z}$ and $\mathrm{Pic}(Y_{0,u}) = \mathbf{Z}$. Furthermore the two restriction maps to Σ send a generator $1 \in \mathbf{Z}$ to $(u, 1, 1, 1)$ and $(1, 1, 1, 1)$ respectively in $\mathbf{G}_m(R) \times \mathbf{Z}^3$. It follows that unless $u = 1$ we have $\mathrm{Pic}(\mathcal{Y}_u) = 0$, and the same remains true after arbitrary base change $R \to k$. The proposition follows. $\square$

Remark 3.17. In the setting of 3.13, the action of $\mathcal{G}$ on $\mathbf{T}_{(Y_R, \mathcal{O}_{Y_R}(1))} \otimes I$ induces via the isomorphism

$$\mathbf{T}_{(Y_R, \mathcal{O}_{Y_R}(1))} \simeq H^0(D, \mathcal{O}_D(n+1))$$

an action ρ' of $\mathscr{G}$ on $H^0(D, \mathscr{O}_D(n+1))$. On the other hand, the $\mathscr{G}$-action on $(Y, \mathscr{O}_Y(1))$ also induces by restriction a $\mathscr{G}$-action ρ on $H^0(D, \mathscr{O}_D(n+1))$. We claim that

$$\rho' = \rho \otimes \chi,$$

where $\chi : \mathscr{G} \to \mathbf{G}_m$ is the 1-dimensional character sending $((u_0, \dots, u_n), \sigma)$ to $u_0 \cdots u_n$. This can be seen by noting that the map

$$\mathscr{O}_Y(-n-1) \to \Omega^1_{\mathbf{P}^n}|_Y$$

given by differentiating the equation for Y is not $\mathscr{G}$-equivariant, but rather should be viewed as a $\mathscr{G}$-equivariant morphism

$$\mathscr{O}_Y(-n-1) \otimes \chi^{-1} \to \Omega^1_{\mathbf{P}^n}|_Y.$$

We therefore get a $\mathscr{G}$-equivariant isomorphism

$$\mathscr{O}_D(n+1) \otimes \chi \simeq \mathscr{E}xt^1(\Omega^1_Y, \mathscr{O}_Y),$$

which implies the result.

Now by a similar argument we find that in the setting of 3.13 the isomorphism classes of equivariant deformations are given by the $\mathscr{G}$-invariants

$$(H^0(D, \mathscr{O}_D(n+1)) \otimes \chi \otimes I)^{\mathscr{G}}. \tag{3.17.1}$$

This can be interpreted as follows. Let $\mathscr{M}_1 \subset \mathscr{M}$ denote the first infinitesimal neighborhood of $B\mathscr{G} \hookrightarrow \mathscr{M}$ and let J denote the $\mathscr{G}$-representation over $\mathbf{Z}[1/n!]$ corresponding to the ideal of the imbedding $B\mathscr{G} \hookrightarrow \mathscr{M}_1$. Then, as in the case of schemes, the set of morphisms filling in the diagram

$$\begin{tikzcd} B_R\mathscr{G} \arrow[r, hook] \arrow[d] & {[\mathrm{Spec}(R[I])/\mathscr{G}]} \arrow[d, dashed] \\ B\mathscr{G} \arrow[r, hook] & \mathscr{M}_1, \end{tikzcd}$$

is in bijection with

$$\mathrm{Hom}_R(J \otimes R, I)^{\mathscr{G}}.$$

On the other hand, the description (3.17.1) of this set, combined with the Yoneda lemma, shows that

$$J \simeq H^0(D, \mathscr{O}_D(n+1) \otimes \chi)^\vee.$$

3.18. Let M denote the dual $\mathbf{Z}[1/n!]$-module of $H^0(D, \mathscr{O}_D(n+1)) \otimes \chi$, viewed as a $\mathscr{G}$-representation, and let

$$U := \mathrm{Spec}(\mathrm{Sym}^\bullet M)$$

denote the affine scheme associated to the symmetric algebra on M. Let $U_m \subset U$ denote the closed subscheme defined by the ideal generated by monomials of degree $m+1$. So we have inclusions

$$\mathrm{Spec}(\mathbf{Z}[1/n!]) = U_0 \subset U_1 \subset \cdots \subset U.$$

The $\mathscr{G}$-action on $H^0(D, \mathscr{O}_D(n+1)) \otimes \chi$ induces a $\mathscr{G}$-action on U, and the U_n.

There is a canonical $\mathscr{G}$-invariant section of

$$H^0(D, \mathscr{O}_D(n+1)) \otimes \chi \otimes M \simeq \mathrm{Hom}(H^0(D, \mathscr{O}_D(n+1)), H^0(D, \mathscr{O}_D(n+1)))$$

given by the identity map on $H^0(D, \mathcal{O}_D(n+1))$. By 3.17 this section defines a lifting of $[Y/\mathcal{G}]$ to $[U_1/\mathcal{G}]$. We think of this as a lifting

$$r_1 : [U_1/\mathcal{G}] \to \mathcal{M}_1 \subset \mathcal{M}.$$

By 3.15.1 (ii) this map is an isomorphism $[U_1/\mathcal{G}] \simeq \mathcal{M}_1$. We claim that we can extend r_1 to a compatible system of maps

$$r_m : [U_m/\mathcal{G}] \to \mathcal{M}_m.$$

Indeed given an extension r_{m-1} to obstruction to finding r_m reducing r_{m-1} is a class in

$$\mathrm{Ext}^1(r_{m-1}^* L_{\mathcal{M}/\mathbf{Z}[1/n!]}, \mathcal{J}_m), \tag{3.18.1}$$

where $\mathcal{J}_m$ is the ideal of $[U_{m-1}/\mathbf{G}] \hookrightarrow [U_m/\mathbf{G}]$. Since $\mathcal{M}$ is smooth over $\mathbf{Z}[1/n!]$ the complex $r_{m-1}^* L_{\mathcal{M}/\mathbf{Z}[1/n!]}$ is concentrated in degrees 0 and 1, and since $\mathcal{G}$ is linearly reductive and U_m is affine it follows that

$$\mathrm{RHom}(r_{m-1}^* L_{\mathcal{M}/\mathbf{Z}[1/n!]}, \mathcal{J}_m)$$

is concentrated in degrees -1 and 0. In particular (3.18.1) is zero and r_{m-1} extends to a map r_m.

Since both $[U/\mathcal{G}]$ and $\mathcal{M}$ are smooth and r_1 is an isomorphism, it follows that each r_m is an isomorphism as well.

3.19. Using this we can make explicit the results of sections 2 and 2.14.

We have $\mathrm{Lie}(\mathcal{G}) \simeq \mathbf{G}_a^{n+1}$, with adjoint action given by the permutation action of S_{n+1} on the factors.

Let $J_m \subset \mathcal{O}_{U_m}$ be the ideal of U_{m-1}. Then as a $\mathcal{G}$-representation we have

$$J_m \simeq \mathrm{Sym}^m(H^0(D, \mathcal{O}_D(n+1)) \otimes \chi).$$

Now recall that M has a basis monomials in $X_0, \ldots, X_n$ of degree $n+1$ for which at least two of the variables do not occur. From this we can see explicitly that there exists m for which $J_m^{\mathcal{G}} \neq 0$ and the conditions in 2.13 (ii) do not hold. For example, in J_{n+1} we have the product of the monomials X_i^{n+1}, which is $\mathcal{G}$-invariant.

3.20. We can remedy this situation by using a bit more information about the moduli interpretation of the stack $\mathcal{M}$.

Let V_0 denote the $\mathcal{G}$-representation $H^0(Y, \mathcal{O}_Y(1))$. We can then consider the projective space

$$W := \mathbf{P}(\mathrm{Sym}^{n+1}(V_0)^\vee),$$

equipped with $\mathcal{G}$-action. The kernel of the map

$$\mathrm{Sym}^{n+1} V_0 \to H^0(Y, \mathcal{O}_Y(n+1))$$

defines a $\mathcal{G}$-invariant point $P_0 \in W$.

Let W_n denote the n-th infinitesimal neighborhood of P_0 in W. By the universal property of projective space, there is a compatible collection of rank 1 submodules

$$K_n \hookrightarrow (\mathrm{Sym}^{n+1})_{W_n}$$

which defines a deformation

$$(Y_n, \mathcal{O}_{Y_n}(1))$$

of $(Y, \mathscr{O}_Y(1))$ to W_n. Moreover, the $\mathscr{G}$-action on V_0 induces a $\mathscr{G}$-action on W, and consequently also a $\mathscr{G}$-action on the families $(Y_n, \mathscr{O}_{Y_n}(1))$ over the action on W_n. We therefore get morphisms

$$g_n : [W_n/\mathscr{G}] \to \mathscr{M}_n.$$

Lemma 3.21. *The morphisms g_n are smooth.*

Proof. It suffices to show that the composition

$$\tilde{g}_n : W_n \to \mathscr{M}_n$$

of g_n with the projection $W_n \to [W_n/\mathscr{G}]$ is smooth. Let $s : T \to \mathscr{M}_n$ be a morphism from a scheme and let $T_0 \subset T$ be the fiber product $B\mathscr{G} \times_{\mathscr{M}_n} T$, a closed subscheme of T defined by a nilpotent ideal. The morphism s corresponds to a polarized scheme $(h : X_T \to T, \mathscr{O}_{X_T}(1))$ together with an isomorphism between its reduction to T_0 and $(Y_{T_0}, \mathscr{O}_{Y_{T_0}}(1))$. Then the data of a dotted arrow filling in the diagram

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, hook] & W_n \arrow[d, "\tilde{g}_n"] \\ T \arrow[r, "s"] \arrow[ur, dashed] & \mathscr{M}_n \end{tikzcd}$$

is equivalent to the data of an isomorphism $V_{0,T} \simeq h_*\mathscr{O}_{X_T}(1)$, reducing to the given isomorphism over T_0. From this interpretation it follows that $\tilde{g}_n$ is formally smooth, and therefore smooth. $\square$

3.22. The morphisms $\{\tilde{g}_n\}$ induces a morphism of tangent spaces for the resulting deformation functors

$$\mathbf{T}_W(P_0) \to \mathbf{T}_{(Y_k, \mathscr{O}_{Y_k}(1))}, \tag{3.22.1}$$

which is a morphism of $\mathscr{G}$-representations. The vector space $\mathbf{T}_{(Y_k, \mathscr{O}_{Y_k}(1))}$ has the following interpretation in terms of the stack $\mathscr{M}$. It classifies the set of isomorphism classes of dotted arrows filling in the diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, hook] \arrow[d, "y_0"] & \mathrm{Spec}(k[\epsilon]) \arrow[d, dashed, "y"] \\ B\mathscr{G} \arrow[r, hook, "i"] & \mathscr{M}, \end{tikzcd}$$

where y_0 is the morphism corresponding to $(Y_k, \mathscr{O}_{Y_k}(1))$. This vector space is given by

$$\mathrm{Ext}^1(y_0^* i^* L_{\mathscr{M}/\mathbf{Z}[1/n!]}, k \cdot \epsilon) \simeq y_0^* \mathscr{H}^{-1}(i^* L_{\mathscr{M}/\mathbf{Z}[1/n!]})^\vee.$$

Let $\mathscr{I}$ be the ideal of $i : B\mathscr{G} \hookrightarrow \mathscr{M}$. Then since $B\mathscr{G}$ and $\mathscr{M}$ are smooth we have a distinguished triangle

$$i^* L_{\mathscr{M}/\mathbf{Z}[1/n!]} \to L_{B\mathscr{G}/\mathbf{Z}[1/n!]} \to i^*\mathscr{I}[1] \to i^* L_{\mathscr{M}/\mathbf{Z}[1/n!]}[1].$$

The complex $L_{B\mathscr{G}/\mathbf{Z}[1/n!]}$ is isomorphic to a sheaf placed in degree 1, and therefore we conclude that

$$y_0^* \mathscr{H}^{-1}(i^* L_{\mathscr{M}/\mathbf{Z}[1/n!]})^\vee \simeq y_0^* i^* \mathscr{I}^\vee.$$

If

$$L_0 \subset \mathrm{Sym}^{n+1}(V_0)$$

is the line corresponding to P_0 then the map (3.22.1) is the natural projection map

$$\mathrm{Sym}^{n+1}(V_0)/L_0 \simeq H^0(Y, \mathcal{O}_Y(n+1)) \to H^0(D, \mathcal{O}_D(n+1)),$$

with $\mathcal{G}$-action the standard action twisted by χ. Looking at the weights of the $\mathcal{G}$-action we find that there is a unique splitting of the map of $\mathcal{G}$-representations

$$\mathrm{Sym}^{n+1}(V_0) \to \mathbf{T}_{(Y_k, \mathcal{O}_{Y_k}(1))}$$

induced by (3.22.1). The image of this splitting and L_0 generate a $\mathcal{G}$-invariant subspace $T \subset \mathrm{Sym}^{n+1} V_0^\vee$ such that

$$\mathbf{P}(T) \subset W$$

is $\mathcal{G}$-invariant and contains P_0, and the induced map

$$\mathbf{T}_{\mathbf{P}(T)}(P_0) \to \mathbf{T}_{(Y_k, \mathcal{O}_{Y_k}(1))}$$

is an isomorphism. By the same argument as in 3.18 it follows that if $Z_n \subset \mathbf{P}(T)$ denotes the n-th infinitesimal neighborhood of P_0 then the induced maps

$$\gamma_n : [Z_n/\mathcal{G}] \to \mathcal{M}_n$$

are isomorphisms.

These isomorphisms γ_n induce a formal minimal presentation

$$\{r_n : \mathcal{M}_n \to B\mathcal{G}\}.$$

Furthermore, if $\mathcal{X}_n \to \mathcal{M}_n$ denotes the universal family then the description of $\mathcal{M}_n$ as $[Z_n/\mathcal{G}]$ provides a canonical imbedding

$$\mathcal{X}_n \hookrightarrow \mathbf{P}(V_0)_{\mathcal{M}_n},$$

and therefore a compatible collection of trivializations

$$\sigma_n : E_{\mathcal{M}_n} \simeq r_n^* V_0.$$

Summary 3.23. *We have constructed, in a canonical way from the starting data $(Y, \mathcal{O}_Y(1))$, a formal minimal presentation $\{r_n : \mathcal{M}_n \to B\mathcal{G}\}$ and a compatible collection of trivializations $\rho_n : r_n^* V_0 \simeq E_{\mathcal{M}_n}$.*

Example 3.24. Let R be a complete local ring with field of fractions K, and let $(X_R, \mathcal{O}_{X_R}(1))$ be a polarized scheme over R defining a morphism

$$\mathrm{Spec}(R) \to \mathcal{M}$$

whose closed fiber is isomorphic to $(Y, \mathcal{O}_Y(1))$. We then get a basis for $\Gamma(X_K, \mathcal{O}_{X_K}(1))$, well-defined up to scalars and permutation, in the following sense: There is an S_{n+1}-torsor $\pi : Q \to \mathrm{Spec}(K)$ and a decomposition

$$\pi^*\Gamma(X_K, \mathcal{O}_{X_K}(1)) \simeq L_0 \oplus L_1 \oplus \cdots \oplus L_n$$

of $\pi^*\Gamma(X_K, \mathcal{O}_{X_K}(1))$ into rank 1 submodules.

Indeed by the Grothendieck existence theorem it suffices to construct such torsors and decompositions over each of the reductions of $\mathrm{Spec}(R)$, and since the morphisms from these reductions to $\mathcal{M}$ factor through the neighborhoods $\mathcal{M}_n$, it suffices to construct such torsors and decompositions over the $\mathcal{M}_n$. For this in turn it suffices to construct the torsor and decomposition for V_0 over $B\mathcal{G}$. The torsor in question the pullback of the tautological torsor

$$\mathrm{Spec}(k) \to BS_{n+1}$$

pulled back along the projection $B\mathscr{G} \to BS_{n+1}$, and the decomposition is obtained by noting that we have a cartesian diagram

$$\begin{tikzcd} B\mathbf{G}_m^{n+1} \arrow[r] \arrow[d] & \mathrm{Spec}(k) \arrow[d] \\ B\mathscr{G} \arrow[r] & BS_{n+1}. \end{tikzcd}$$

4. Interlude: slicing of smooth presentations

In this section we discuss some general stack-theoretic results which will be used in the next section. Throughout we work over a field k.

4.1. Let

$$\pi : Z \to \mathscr{M}$$

be a smooth surjective morphism from a scheme Z to an algebraic stack $\mathscr{M}$ over k, and assume Z is the spectrum of an artinian local ring with residue field k. Let $x \in Z(k)$ denote the closed point.

Assume given a closed immersion of schemes

$$i : Z \hookrightarrow A,$$

where A is smooth over k.

Remark 4.2. We will be particularly interested in the case when A is the affine space associated to a finite dimensional k-vector space V (so $A = \mathrm{Spec}(\mathrm{Sym}^\bullet V)$), such that x is sent to the zero section of A.

Let $A' \hookrightarrow A$ be a closed immersion of smooth k-schemes defined by an ideal $I \subset \mathscr{O}_A$, and form the fiber product diagram

$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, hook] & A' \arrow[d, hook] \\ Z \arrow[r, hook] & A. \end{tikzcd}$$

Assume that $x \in A'$ so that $x \in Z'(k)$.

Proposition 4.3. *Assume that the map of k-vector spaces*

$$I(x) \to \Omega^1_{Z/\mathscr{M}}(x) \tag{4.3.1}$$

induced by the composition

$$I(x) \to \Omega^1_{A/k}(x) \to \Omega^1_{Z/k}(x) \to \Omega^1_{Z/\mathscr{M}}(x)$$

is injective. Then the composition

$$Z' \hookrightarrow Z \to \mathscr{M}$$

is smooth and

$$\Omega^1_{Z'/\mathscr{M}}(x) \simeq \Omega^1_{Z/\mathscr{M}}(x)/I(x).$$

Proof. We verify the infinitesimal lifting criterion for $Z'/\mathscr{M}$. So consider a commutative diagram of solid arrows

$$\begin{tikzcd} T_0 \arrow[r, "t_0"] \arrow[dd, hook] & Z' \arrow[r, hook] \arrow[d, hook] & A' \arrow[d, hook] \\ & Z \arrow[r, hook] \arrow[d] & A \\ T \arrow[r, "\bar{t}"] \arrow[ur, dashed, "\rho"] & \mathscr{M} & \end{tikzcd},$$

where T is the spectrum of an artinian local ring R, $J \subset R$ is an ideal annihilated by the maximal ideal, and T_0 is the spectrum of R/J. Let L denote the residue field of R, so L is a field extension of k.

Since $Z \to \mathscr{M}$ is smooth, there exists a morphism $\rho : T \to Z$ over $\mathscr{M}$ as indicated in the diagram. The morphism ρ factors through Z' if and only if the composition $T \to Z \to A$ factors through A', and this is, in turn, equivalent to the vanishing of the map

$$\lambda_\rho : I(x) \otimes_k L \to J \tag{4.3.2}$$

induced by the composition

$$I \subset \mathscr{O}_A \to \mathscr{O}_Z \to \mathscr{O}_T.$$

Now the set of ρ filling in the diagram form a torsor under the group

$$\mathrm{Hom}(\Omega^1_{Z/\mathscr{M}}(x) \otimes_k L, J).$$

Changing the choice of ρ by an element ∂ in this group has the effect of changing λ_ρ to $\lambda_\rho + \partial|_{I(x)}$, where $\partial|_{I(x)}$ denote the composition

$$I(x) \otimes_k L \longrightarrow \Omega^1_{Z/\mathscr{M}}(x) \otimes_k L \xrightarrow{\partial} J.$$

Our assumption that (4.3.1) is injective therefore ensures that we can always modify ρ so that it factors through Z'.

The remaining statements in the proposition also follow from this discussion. $\square$

Remark 4.4. In the setting of 4.2 consider a subspace $K \subset V$ with associated quotient $V' := V/K$. Let $A' \hookrightarrow A$ be the inclusion of affine spaces induced by the surjection $V \to V'$. Then $I(x) \simeq K$, and in 4.3 we are asking that the map

$$K \to \Omega^1_{Z/\mathscr{M}}(x)$$

is injective.

5. Interlude: Some representation theory

5.1. Throughout this section we work over a field k of characteristic 0. Let $\mathscr{G}$ be a group scheme over k, which is an extension

$$1 \to \mathbf{T} \to \mathscr{G} \to H \to 1$$

of a finite group H by a torus $\mathbf{T}$. For every $h \in H$ this extension gives a homomorphism

$$c_h : \mathbf{T} \to \mathbf{T}, \quad u \mapsto \tilde{h}^{-1} u \tilde{h},$$

where $\tilde{h} \in \mathscr{G}$ is any lift of h.

For a character χ of $\mathbf{T}$ and $h \in H$ let χ^h be the character $\chi \circ c_h$. If V is a representation of $\mathscr{G}$ and $V_\chi \subset V$ is the maximal subspace on which $\mathbf{T}$ acts through χ then for any $\tilde{h} \in \mathscr{G}$ with image $h \in H$ we have

$$\tilde{h}(V_\chi) = V_{\chi^h}.$$

5.2. Let W be a representation of $\mathscr{G}$, and suppose given a decomposition

$$W = \oplus_{i \in I} L_i$$

of W into one-dimensional subspaces L_i, which are $\mathbf{T}$-stable. We assume that the L_i are stable under $\mathscr{G}$ in the sense that for any $\tilde{h} \in \mathscr{G}$ we have $\tilde{h}(L_i)$ equal to L_j for some $j \in I$. Starting with this setup, we claim that any inclusion

$$j : E \hookrightarrow W$$

of an irreducible $\mathscr{G}$-representation E, for which the characters of $\mathbf{T}$ acting on E all occur with multiplicity one, has a canonical splitting

$$r : W \to E.$$

5.3. For a character χ of $\mathbf{T}$ let $I_\chi \subset I$ denote the subset of those indices $i \in I$ for which $\mathbf{T}$ acts on L_i through the character χ, and let $X(E)$ denote the set of those characters of $\mathbf{T}$ occurring in E. Note that since we assume that E is irreducible the group H acts transitively on $X(E)$.

For $\chi \in X(E)$ and $i \in I_\chi$ let

$$j_i : E_\chi \to L_i$$

be the composition of $j|_{E_\chi}$ with the projection

$$W = \oplus_{i \in I} L_i \to L_i.$$

Define $\tilde{r}_i : L_i \to E_\chi$ to be zero if j_i is zero, and j_i^{-1} otherwise (note that j_i is a map of 1-dimensional $\mathbf{T}$-representations). Let

$$\tilde{r}_\chi : W_\chi = \oplus_{i \in I_\chi} L_i \to E_\chi$$

be the sum of the $\tilde{r}_i$. Then define $\tilde{r} : W \to E$ to be the composition

$$W \xrightarrow{\text{projection}} \oplus_{\chi \in X(E)} W_\chi \xrightarrow{\sum \tilde{r}_\chi} E.$$

For $\chi \in X(E)$ let N_χ denote the number of $i \in I_\chi$ for which j_i is nonzero.

Proposition 5.4. *(i) The number N_χ is independent of the choice of χ.*

(ii) The map $\tilde{r}$ is a map of $\mathscr{G}$-representations.

(iii) The composition $\tilde{r} \circ j : E \to E$ is equal to multiplication by N_χ (defined as in (i)).

Proof. For $\tilde{h} \in \mathscr{G}$ mapping to $h \in H$ the action of $\tilde{h}$ induces maps

$$\tilde{h} : E_\chi \to E_{\chi^h}.$$

Furthermore, by our assumptions the action of $\tilde{h}$ sends L_i ($i \in I_\chi$) to $L_{h(i)}$ for some uniquely determined $i \in I_{\chi^h}$. Furthermore, since j is a map of $\mathscr{G}$-representations we have a commutative diagram

$$\begin{CD} E_\chi @>{j_i}>> L_i \\ @V{\tilde{h}}VV @VV{\tilde{h}}V \\ E_{\chi^h} @>{j_{h(i)}}>> L_{h(i)}. \end{CD}$$

It follows that j_i is nonzero if and only if $j_{h(i)}$ is nonzero. This implies (i). Furthermore, from the commutativity of these diagrams it follows that the map $\tilde{r}$ is a map of $\mathscr{G}$-representations. Finally (iii) follows immediately from the definition. □

5.5. So far we have not used the assumption that k has characteristic 0. However, if N is the integer denoted N_χ in (i), then to get a splitting $r : W \to E$ as desired we take

$$r := \frac{1}{N}\tilde{r} : W \to E.$$

This requires the integer N to be invertible in k. In particular, if k has characteristic 0 we can define this map.

Remark 5.6. Similarly if W and E are as above and $\pi : W \to E$ is a surjective map of $\mathscr{G}$-representations, then we have a canonical section $s : E \to W$ of π. This follows by applying the preceding discussion to the duals $E^\vee \hookrightarrow W^\vee$.

6. $K3$ SURFACES

6.1. Throughout this section we work over a field k.

Let $(Y, \mathcal{O}_Y(1))$ be a projective scheme with a very ample invertible sheaf. We will impose various assumptions (i)-(vii) indicated below.

First of all, we assume that $(Y, \mathcal{O}_Y(1))$ is obtained by gluing toric varieties in the following sense:

(i) Y is seminormal.
(ii) Let $\{Y_i\}_{i\in I}$ be the irreducible components of Y, and let $\mathcal{O}_{Y_i}(1)$ denote the restriction of $\mathcal{O}_Y(1)$ to Y_i. We assume that $(Y_i, \mathcal{O}_{Y_i}(1))$ comes equipped with an action of a torus $\mathbf{T}_i$ giving $(Y_i, \mathcal{O}_{Y_i}(1))$ the structure of a projective normal toric variety associated to an integral polytope $\Delta_i \subset X_{i,\mathbf{R}}$, where $X_i := \mathrm{Hom}(\mathbf{T}_i, \mathbf{G}_m)$.
(iii) For two irreducible components Y_i and Y_j the intersection Y_{ij} with the restriction $\mathcal{O}_{Y_{ij}}(1)$ of $\mathcal{O}_Y(1)$ is a union of toric curves corresponding to faces Δ_{ij} in Δ_i and Δ_j. Similarly for higher intersections.

Gluing together the polytopes Δ_i along these intersections Δ_{ij} we get a topological space Δ. We assume:

(iv) The topological space Δ is homeomorphic to the 2-sphere.

6.2. The seminormality will be used as follows. Taking the coskeleton of the surjection

$$\pi : \coprod_{i\in I} Y_i \to Y$$

we get a simplicial scheme $\widetilde{Y}_\bullet$, with each $\widetilde{Y}_n$ the disjoint union of n-fold intersections of components Y_i (possibly repeated). There is an augmentation

$$a_\bullet : \widetilde{Y}_\bullet \to Y$$

with each $a_n : \widetilde{Y}_n \to Y$ a coproduct of closed immersions. In particular, each a_n is finite. For a quasi-coherent sheaf $\mathscr{F}$ on Y we can then form the sheaf $a_\bullet^*\mathscr{F}$ in the simplicial topos $\widetilde{Y}_{\bullet,\text{ét}}$ and there is an augmentation

$$\mathscr{F} \to Ra_{\bullet *}a_\bullet^*\mathscr{F}.$$

The seminormality assumption implies the following:

Lemma 6.3. *For every invertible sheaf $\mathscr{L}$ on Y the map*

$$\mathscr{L} \to Ra_{\bullet *}a_\bullet^*\mathscr{L}$$

is an isomorphism in the derived category of Y.

Proof. The assertion is local on Y so it suffices to consider the case when $\mathscr{L} = \mathscr{O}_Y$.

Since the morphisms a_n are all finite, the complex $Ra_{\bullet *}\mathscr{O}_{\widetilde{Y}_\bullet}$ is represented by the complex obtained by taking the normalized complex of the simplicial sheaf given by

$$[n] \mapsto \oplus_{\underline{i}=(i_0,\dots,i_n)}\mathscr{O}_{Y_{\underline{i}}}.$$

where $Y_{\underline{i}}$ denotes $Y_{i_0} \cap \cdots Y_{i_n}$.

Therefore $R^0a_{\bullet *}\mathscr{O}_{\widetilde{Y}_\bullet}$ is a finite algebra over $\mathscr{O}_Y$ such that the map on spectra induces an isomorphism on residue fields. From this, and the definition of seminormality, it follows that the map $\mathscr{O}_Y \to R^0a_{\bullet *}$ is an isomorphism.

To see that the complex $Ra_{\bullet *}\mathscr{O}_{\widetilde{Y}}$ has no higher cohomology, it suffices to calculate locally around a point x in the singular locus of Y, where the complex is the same as the complex for a broken toric variety as in [2, 2.5.4]. □

6.4. Let f (resp. e, v) denote the number of faces (resp. edges, vertices) in Δ. Condition (iii) implies that

$$f - e + v = 2.$$

Proposition 6.5. *(i) $h^0(Y, \mathscr{O}_Y) = h^2(Y, \mathscr{O}_Y) = 2$ and $h^i(Y, \mathscr{O}_Y) = 0$ for $i \neq 0, 2$.*

(ii) $h^0(Y, \mathscr{O}_Y(1)) = v$.

(iii) For all $m, i > 0$ we have $h^i(Y, \mathscr{O}_Y(m)) = 0$.

Proof. By 6.3 it suffices to calculate $H^*(\widetilde{Y}_\bullet, \mathscr{O}_{\widetilde{Y}_\bullet}(m))$ for $m \geq 0$. Since $H^i(\widetilde{Y}_n, \mathscr{O}_{\widetilde{Y}_n}(m)) = 0$ for all $i > 0$ and $m \geq 0$ (see [10, 2.7]), we find that $H^i(Y, \mathscr{O}_Y(m))$ is computed by the cohomology of the simplicial k-vector space

$$[n] \mapsto \Gamma(\widetilde{Y}_n, \mathscr{O}_{\widetilde{Y}_n}(m)).$$

For $m = 0$ this is just the simplicial module computing the cohomology of the Δ with coefficients in k, so (i) follows from assumption (iv). For $m > 0$ let $\mathscr{V}_m \subset \Delta$ denote the points $p \in \Delta$ for which mp is integral, and let

$$j : \mathscr{V}_m \hookrightarrow \Delta$$

be the inclusion. Then it follows from the above that $H^*(Y, \mathscr{O}_Y(m))$ is computed by the cohomology with coefficients in k of the simplicial set obtained by taking the coskeleton of the surjection

$$\coprod_i j^{-1}(\Delta_i) \to \mathscr{V}_m.$$

From this statements (ii) and (iii) follow. $\square$

6.6. Let $(P, \mathscr{O}_P(1))$ be a polarized toric variety associated to an integral polytope and with torus T and boundary divisor $\Sigma \subset P$.

Let R be a k-algebra with $\mathrm{Spec}(R)$ connected and let α be an automorphism of $(P_R, \mathscr{O}_{P_R}(1))$ which preserves the boundary divisor Σ (scheme-theoretically).

For any vertex v of the polytope the corresponding rank 1 submodule of $\Gamma(P_R, \mathscr{O}_{P_R}(1))$ can be characterized as the intersection of the kernels of the maps from $\Gamma(P_R, \mathscr{O}_{P_R}(1))$ to the fibers of $\mathscr{O}_{P_R}(1)$ at the points of P corresponding to vertices $v' \neq v$.

From this and the assumption that α preserves Σ it follows that α induces an automorphism of the polytope defining $(P, \mathscr{O}_P(1))$. Furthermore, if α induces the identity on the polytope associated to $(P, \mathscr{O}_P(1))$ then α preserves the standard basis for $\Gamma(P, \mathscr{O}_P(1))$. It follows that α is induced by an element of $T(R)$.

From this the following result follows:

Corollary 6.7. *Let $\mathbf{Aut}'(P, \mathscr{O}_P(1))$ be the functor of automorphisms of $(P, \mathscr{O}_P(1))$ which preserve Σ. Then $\mathbf{Aut}'(P, \mathscr{O}_P(1))$ is represented by an extension of a finite group by $T \times \mathbf{G}_m$, where $\mathbf{G}_m$ acts trivially on P and via the standard action on $\mathscr{O}_P(1)$.*

Proof. Note that $\mathbf{Aut}'(P, \mathscr{O}_P(1))$ is a group scheme. In fact, is a closed subgroup scheme of $GL(\Gamma(P, \mathscr{O}_P(1)))$. The statement of the corollary follows from this and the preceding discussion. $\square$

Lemma 6.8. *Let $\mathscr{G}$ denote the automorphism group scheme $\mathbf{Aut}(Y, \mathscr{O}_Y(1))$ of the pair $(Y, \mathscr{O}_Y(1))$. Then $\mathscr{G}$ is an extension of a diagonalizable group scheme $\mathbf{T}$ by a finite group H:*

$$1 \to \mathbf{T} \to \mathscr{G} \to H \to 1.$$

Proof. There is a natural map

$$\mathscr{G} \to \mathbf{S}$$

from $\mathscr{G}$ to the group of permutations of the vertices of Δ. It suffices to show that the kernel of this map is an extension of a finite group by a diagonalizable group scheme. An element of this kernel necessarily preserves the irreducible components of Y and their intersections, and therefore is given by an element of

$$\mathrm{Eq}\Big(\prod_i \mathbf{Aut}'(Y_i, \mathscr{O}_{Y_i}(1)) \rightrightarrows \prod_{i,j} \mathbf{Aut}'(Y_{ij}, \mathscr{O}_{Y_{ij}})\Big), \tag{6.8.1}$$

where $\mathbf{Aut}'(Y_i, \mathscr{O}_{Y_i}(1))$ is defined as above. Write $\mathbf{Aut}'(Y_i, \mathscr{O}_{Y_i}(1))$ as an extension

$$1 \to \mathbf{T}_i \to \mathbf{Aut}'(Y_i, \mathscr{O}_{Y_i}(1)) \to H_i \to 1,$$

with H_i finite and $\mathbf{T}_i$ a torus, and similarly for the double intersections. We then see that the connected component of the equalizer (6.8.1) is given by an equalizer

$$\mathrm{Eq}(\prod_i \mathbf{T}_i \rightrightarrows \prod_{i,j} \mathbf{T}_{ij})$$

of maps of tori, and therefore is diagonalizable. $\square$

We make the following additional assumptions on $\mathscr{G}$:

(v) For all i the map $\mathbf{T} \to \mathbf{T}_i$ is surjective.
(vi) The order of H is invertible in k.
(vii) For each vertex $w \in \Delta$ we get a corresponding character $\chi_w : \mathbf{T} \to \mathbf{G}_m$ obtained by choosing any Δ_i containing w and setting χ_w to be the composition $\mathbf{T} \to \mathbf{T}_i \xrightarrow{w} \mathbf{G}_m$ (this is independent of the choice of Δ_i). We assume that the characters in the set $\{\chi_w\}_{w\in\Delta}$ are all distinct.

Remark 6.9. The assumption (vi) implies that $\mathscr{G}$ is linearly reductive (see for example [1, 2.7 (c)]).

6.10. Let E_0 denote the $\mathscr{G}$-representation over k given by $\Gamma(Y, \mathscr{O}_Y(1))$, and let S_d denote $\mathrm{Sym}^d E_0$. Let M_d denote $\Gamma(Y, \mathscr{O}_Y(d))$ and let M_d^i denote $\Gamma(Y_i, \mathscr{O}_{Y_i}(d))$. Note that each character of $\mathbf{T}_i$ appearing in M_d^i occurs with multiplicity 1, and therefore each irreducible representation of $\mathscr{G}$ appearing in M_d^i also occurs with multiplicity one. In fact, if $m\Delta_i \subset X_{i,\mathbf{R}}$ denotes the dilation of Δ_i obtained by multiplying by m then M_d^i has basis the integral points $m\Delta_i \cap X_i \subset X_{i,\mathbf{R}}$ of $m\Delta_i$. By taking the sum of the inclusions

$$j_i : M_d^i \hookrightarrow M_d$$

we get a description of M_d as a cokernel

$$\oplus_{i,j} M_d^{i,j} \to \oplus_i M_d^i \to M_d \to 0,$$

where $M_d^{i,j}$ denotes $\Gamma(Y_i \cap Y_j, \mathscr{O}_{Y_i \cap Y_j}(d))$ embedded diagonally. Note that in fact this description of M_d is compatible with the $\mathscr{G}$-action, where H acts on $\oplus_i M_d^i$ by permuting the factors.

From this it also follows that the canonical map

$$S_d \to M_d$$

is a surjective map of $\mathscr{G}$-representations.

6.11. Let $\mathscr{M}$ denote the stack which to any scheme S associates the groupoid of pairs $(f : X \to S, \mathscr{O}_X(1))$, where

(i) $f : X \to S$ is a proper flat morphism of schemes with connected geometric fibers of pure dimension 2.
(ii) $\mathscr{O}_X(1)$ is a relatively ample invertible sheaf on X such that the formation of the sheaf $f_*\mathscr{O}_X(1)$ commutes with arbitrary base change on S, and this sheaf is locally free of rank $n+1$.
(iii) We have $R^i f_*\mathscr{O}_X(d) = 0$ for all $i, d > 0$.

The polarized scheme $(Y, \mathcal{O}_Y(1))$ defines a closed immersion

$$B\mathcal{G} \hookrightarrow \mathcal{M}.$$

We can describe the formal neighborhood of this closed immersion as follows.

6.12. For a finite dimensional vector space W and integer $r \leq \dim(W)$, let $\mathrm{Gr}(r, W)$ denote the Grassmanian of r-dimensional subspaces in W.

Let $\Sigma_d \subset S_d$ denote the kernel of the surjection

$$S_d \to \Gamma(Y, \mathcal{O}_Y(d)),$$

and let M_d denote $\Gamma(Y, \mathcal{O}_Y(d))$, which is isomorphic to the quotient S_d/Σ_d. Because the kernel of the map

$$S := \oplus_{d\geq 0} S_d \to \Gamma_*(Y, \mathcal{O}_Y(1))$$

is a finitely generated ideal, there exists a finite number of integers $d_1, \ldots, d_r$ for which the quotient

$$\Sigma_d/E_0 \cdot \Sigma_{d-1}$$

is nonzero.

Lemma 6.13. *Let A be an artinian local ring with residue field k and let $(X, \mathcal{O}_X(1))$ be a flat deformation of $(Y, \mathcal{O}_Y(1))$ to A. Then $E := \Gamma(X, \mathcal{O}_X(1))$ is a flat A-module with $E\otimes_A k \simeq E_0$, the maps*

$$\mathrm{Sym}^d E \to \Gamma(X, \mathcal{O}_X(d)) \tag{6.13.1}$$

are surjective for all $d \geq 1$, and the kernel K_d of this map is a flat deformation of Σ_d to A.

Proof. By a standard devissage the flatness of E and the surjectivity of (6.13.1) follows from the vanishing of $H^1(Y, \mathcal{O}_Y(d))$ for $d > 0$ explained in 6.5 (iii). The flatness of K_d follows from consideration of the exact sequence

$$0 \to K_d \to \mathrm{Sym}^d E \to \Gamma(X, \mathcal{O}_X(d)) \to 0,$$

and the flatness of $\Gamma(X, \mathcal{O}_X(d))$. $\square$

6.14. For $i = 1, \ldots, r$ let r_i be the rank of K_d, and consider the product of Grassmanians

$$\mathbf{G} := \prod_{i=1}^{r} \mathrm{Gr}(r_i, S_{d_i}).$$

6.15. Let U denote algebraic space which to any scheme T associates the set of isomorphism classes of triples

$$(f : X \to T, \mathcal{O}_X(1), \sigma), \tag{6.15.1}$$

where $(X, \mathcal{O}_X(1)) \in \mathcal{M}(T)$ and $\sigma : E_0 \to f_*\mathcal{O}_X(1)$ is an isomorphism of locally free sheaves. So U is a $GL(E_0)$-torsor over $\mathcal{M}$. There is a tautological isomorphism $\sigma : E_0 \simeq H^0(Y, \mathcal{O}_Y(1))$ (the identity map) defining a point $x \in U$.

Let $\mathcal{M}' \subset \mathcal{M}$ denote the maximal open substack classifying pairs $(f : X \to T, \mathcal{O}_X(1))$ satisfying the additional condition that for all geometric points $\bar{t} \to T$ we have $H^i(X_{\bar{t}}, \mathcal{O}_{X_{\bar{t}}}(m)) = 0$ for all $i, m > 0$. Let $U' \subset U$ be the preimage of $\mathcal{M}'$.

As in the proof of 6.13 if $(f : X \to T, \mathcal{O}_X(1)) \in \mathcal{M}'(T)$ then the sheaves $f_*\mathcal{O}_X(m)$ are locally free on T for all m and their formation commutes with arbitrary base change. Furthermore, the same is true of the kernels

$$K_d := \mathrm{Ker}(\mathrm{Sym}^d(f_*\mathcal{O}_X(1)) \to f_*\mathcal{O}_X(d)).$$

Let $\mathcal{N} \subset \mathcal{M}'$ be the maximal open substack over which the rank of K_{d_i} equals the rank of Σ_{d_i} for $i = 1, \ldots, r$ and where the K_{d_i} generate all the kernels K_d. Let $W \subset U'$ be the preimage of $\mathcal{N}$.

Then $x \in W$ and we get a map

$$g : W \to \mathbf{Gr},$$

which is an immersion since the K_{d_i} generate all the kernels.

6.16. For $i = 1, \ldots, r$ choose a splitting of $\mathcal{G}$-representations

$$s_i : M_{d_i} \to S_{d_i}$$

of the projection $S_{d_i} \to M_{d_i}$. Such a choice defines a decomposition

$$S_{d_i} \simeq \Sigma_{d_i} \oplus M_{d_i}.$$

This decomposition defines an open subset

$$A_i := \mathrm{Hom}(\Sigma_{d_i}, M_{d_i}) \subset \mathbf{Gr}(r_i, S_{d_i}),$$

give by the spectrum of the symmetric algebra on $M_{d_i}^\vee \otimes \Sigma_{d_i}$. Let $\mathbf{A}$ denote the product

$$\mathbf{A} := \prod_{i=1}^{r} A_i \subset \mathbf{Gr}(r_i, S_{d_i}).$$

If V denotes the direct sum

$$V := \oplus_{i=1}^{r} M_{d_i}^\vee \otimes \Sigma_{d_i}$$

then $\mathbf{A}$ is the spectrum of the symmetric algebra on V.

Let $Z \subset W$ denote $g^{-1}(\mathbf{A})$. We then have

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r, "x"] \arrow[d] & Z \arrow[r, hook, "i"] \arrow[d, "\pi"] & \mathbf{A} \\
B\mathcal{G} \arrow[r, hook] & \mathcal{M},
\end{tikzcd}$$

where $\mathcal{G}$ acts compatibly on all the schemes on the top line over the stacks on the bottom line, and π is smooth.

Remark 6.17. If the characteristic of k is 0 then there are canonical splittings s_i. Indeed by 6.10 it suffices to construct compatible maps

$$M_{d_i}^j \to S_{d_i},$$

and since the multiplicity of each character occurring in $M_{d_i}^i$ is 1 this follows from 5.6.

6.18. Let us describe $T_{Z/\mathscr{M}}(x)$. This vector space is isomorphic to $T_{Z_0/B\mathscr{G}}(x)$. Let $\mathscr{E}_0$ denote the sheaf on $B\mathscr{G}$ corresponding to the $\mathscr{G}$-representation E_0. Then Z_0 is the scheme over $B\mathscr{G}$ given by $\underline{\mathrm{Isom}}(E_0, \mathscr{E}_0)$. It follows that Z_0 is isomorphic to the quotient $GL(E_0)/\mathscr{G}$ (quotient by right translation) with $\mathscr{G}$-action induced by left-translation.

It follows that $T_{Z_0/B\mathscr{G}}(x)$ is isomorphic to $\mathrm{End}(E_0)$ with $\mathscr{G}$-action induced by the adjoint action. Furthermore, the vector space $T_{Z_0}(x)$ is the quotient of $T_{Z_0/B\mathscr{G}}(x)$ given by

$$T_{Z_0}(x) \simeq \mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G}),$$

where $\mathrm{Lie}(\mathscr{G}) \hookrightarrow \mathrm{End}(E_0)$ is obtained by taking Lie algebras of the representation $\mathscr{G} \to GL(E_0)$. In this way the diagram of $\mathscr{G}$-representations

$$T_{Z/\mathscr{M}}(x) \to T_{Z_0}(x) \to T_{\mathbf{A}}$$

is identified with the diagram

$$\mathrm{End}(E_0) \twoheadrightarrow \mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G}) \xhookrightarrow{j} V^\vee.$$

6.19. Choose a splitting

$$t : V^\vee \to \mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G})$$

of the inclusion j, and let $V \to V'$ be the quotient of V dual to the inclusion

$$\mathrm{Ker}(t) \hookrightarrow V^\vee.$$

Let $\mathbf{A}' \hookrightarrow \mathbf{A}$ be the closed immersion defined by $V \to V'$, and let

$$Z' \to \mathscr{M}$$

be the fiber product $Z \times_{\mathbf{A}} \mathbf{A}'$.

Then by 4.4 the scheme Z' is smooth over $\mathscr{G}$ and comes equipped with a $\mathscr{G}$-action.

Remark 6.20. The decomposition of E_0 into distinct eigenspaces for the $\mathbf{T}$-action induces a decomposition of $\mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G})$ into a direct sum of irreducible representations. Similarly V has a canonical decomposition as a $\mathbf{T}$-representation into 1-dimensional eigenspaces which are preserved under the H-action. We can therefore apply the results of section 5 to the projection from V onto each irreducible factor of $\mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G})$ to get a canonical splitting t, when k has characteristic 0.

6.21. Consider the reduction

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r, hook] \arrow[dr] & Z_0' \arrow[d] \\ & B\mathscr{G}. \end{tikzcd}$$

Both maps to $B\mathscr{G}$ are smooth and the map

$$\Omega^1_{Z_0'/B\mathscr{G}}(x) \to \Omega^1_{\mathrm{Spec}(k)/B\mathscr{G}}$$

is an isomorphism. It follows that the map $\mathrm{Spec}(k) \to Z_0'$ is an open immersion.

Let $Z_n' \to \mathscr{M}_n$ be the pullback of Z' to $\mathscr{M}_n$, and let $U_n' \subset Z_n'$ be the unique open subset lifting $\mathrm{Spec}(k) \subset Z_0'$. Then U_n' is $\mathscr{G}$-invariant, since this is true of its reduction. Similarly U_n' is a $\mathscr{G}$-torsor over $\mathscr{M}_n$ since this is true of its reduction. In summary:

Corollary 6.22. *The collection*

$$\{U'_n \to \mathscr{M}_n\}$$

defines a formal minimal presentation for $\mathscr{M}$ at $B\mathscr{G}$.

6.23. Let

$$r_n : \mathscr{M}_n \to B\mathscr{G}$$

be the maps corresponding to the formal minimal presentation given by the U'_n. We claim that there are natural isomorphisms

$$\sigma_n : r_n^*\mathscr{E}_0 \simeq \mathscr{E}_n$$

for all n. Indeed this follows from noting that on the reduction $Z_n \to \mathscr{M}_n$ of Z there is a tautological $\mathscr{G}$-equivariant isomorphism $\mathscr{E}_{n,Z_n} \simeq E_0$ which the restricts to a $\mathscr{G}$-equivariant isomorphism over U'_n.

Summary 6.24. *Associated to the splittings*

$$\{s_i : M_{d_0} \to S_{d_i}\}_{i=1,\dots,r}, \quad t : V^\vee \to \mathrm{End}(E_0)/\mathrm{Lie}(\mathscr{G})$$

is a formal minimal presentation $\{r_n : \mathscr{M}_n \to B\mathscr{G}\}$ and a compatible collection of isomorphisms $\sigma_n : r_n^*\mathscr{E}_0 \simeq \mathscr{E}$ of vector bundles reducing to the identity over $B\mathscr{G}$.

If the characteristic of k is 0 then there are canonical choices of splittings, as discussed in section 5, and therefore also canonical choices of the formal presentation $\{r_n\}$ and the isomorphisms σ_n.

7. Further properties

We continue with the notation and assumptions of the previous section.

Proposition 7.1. *Let R be a complete local ring with field of fractions K, and let $(X_R, \mathscr{O}_{X_R}(1))$ be a polarized scheme flat over R with closed fiber $(Y, \mathscr{O}_Y(1))$ and smooth generic fiber X_K. Then X_K is a $K3$ surface and $\mathscr{O}_{X_K}(1)$ is a polarization of degree $2(v-2)$ (so the genus of $(X_K, \mathscr{O}_{X_K}(1))$ is $v-1$).*

Proof. By 6.5 and cohomology and base change $H^i(X_R, \mathscr{O}_{X_R}) = 0$ for $i \neq 0, 2$ and $H^0(X_R, \mathscr{O}_{X_R})$ and $H^2(X_R, \mathscr{O}_{X_R})$ are free modules of rank 1. It follows that X_K is geometrically connected, and that

$$h^i(X_K, \mathscr{O}_{X_K}) = \begin{cases} 1 & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

To prove the proposition it therefore suffices to show that the canonical sheaf ω_{X_K} is trivial. By Serre duality we have $h^0(X_K, \omega_{X_K}) = 1$ and therefore $\omega_{X_K} \simeq \mathscr{O}_{X_K}(E)$ for some effective divisor E (possibly 0). Since $\mathscr{O}_{X_K}(1)$ is very ample it therefore suffices to show that for some nonzero section $\theta \in H^0(X_K, \mathscr{O}_{X_K}(1))$ with associated divisor $C_\theta \subset X$ we have $C_\theta \cdot \omega_{X_K} = 0$. By Riemann-Roch, the constancy of Euler characteristics in flat families, and 6.4 we have

$$v = \frac{1}{2}(c_1(\mathscr{O}_{X_K}(1))^2 - C_\theta.\omega_{X_K}) + 2.$$

It therefore suffices to show that $c_1(\mathscr{O}_{X_K}(1))^2 = 2(v-2)$. Equivalently,

$$\deg_{C_\theta}(\mathscr{O}_{X_K}(1)|_{C_\theta}) = 2(v-2).$$

Lemma 7.2. *Let $(P, \mathscr{O}_P(1))$ be a polarized toric variety of dimension 2 with torus T associated to an integral polytope $\Delta \subset X_{\mathbf{R}}$, where X denotes the character group of T. Let $\theta \in \Gamma(P, \mathscr{O}_P(1))$ be a nonzero section with zero locus $C_\theta \subset P$.*

(i) The curve C_θ has arithmetic genus 0 and

$$\deg_{C_\theta}(\mathscr{O}_P(1)|_{C_\theta}) = \#\{\text{vertices of } \Delta\} - 2.$$

(ii) For a general choice of θ we have $C_\theta \simeq \mathbf{P}^1$.

Proof. Note that $\Gamma(P, \mathscr{O}_P(1))$ decomposes into a direct sum of rank 1 eigenspaces for the T-action with characters the vertices of Δ.

Let θ be a nonzero section of $\mathscr{O}_P(1)$ and let w_0 be a vertex with θ_{w_0} nonzero. We can then write

$$\theta = \theta_{w_0} + \sum_{w \neq w_0} \theta_w.$$

Consider the section $\theta_t \in \Gamma(P \times \mathbf{A}^1, \mathscr{O}_P(1))$ given by the sum

$$\theta_t := \theta_{w_0} + \sum_{w \neq w_0} t\theta_w,$$

where t denotes the coordinate on $\mathbf{A}^1$. For $t = 1$ we have the section θ, and for $t = 0$ we get the section θ_{w_0}. Let $C_t \subset P \times \mathbf{A}^1$ be the zero locus of θ_t. Then C_t is a flat family of curves over $\mathbf{A}^1$. Since arithmetic genus and degrees are constant in flat families, to prove (i) it therefore suffices to consider the case when $\theta = \theta_{w_0}$. In this case, C_θ is obtained by taking the boundary of Δ and removing the two edges meeting w. It follows that C_θ is a chain of smooth rational curves with number of irreducible components equal to 2 less than the number of vertices. Furthermore, the restriction of $\mathscr{O}_P(1)$ to each rational curve has degree 1. This proves statement (i).

For (ii), note that it also follows from the argument proving (i) that for general C_θ the curve has at worst nodal singularities. It therefore suffices to note that the generic curve C_θ is irreducible by Bertini's irreducibility theorem. ☐

For each i the map

$$\Gamma(Y, \mathscr{O}_Y(1)) \to \Gamma(Y_i, \mathscr{O}_{Y_i}(1))$$

is surjective. For a general section $\theta \in \Gamma(Y, \mathscr{O}_Y(1))$ with zero locus $C_\theta \subset Y$, the intersection $C_{\theta_i} := C_\theta \cap Y_i$ is isomorphic to $\mathbf{P}^1$. Furthermore, we have

$$\deg_{C_\theta}(\mathscr{O}_Y(1)|_{C_\theta}) = \sum_i \deg_{C_{\theta_i}}(\mathscr{O}_{Y_i}(1)|_{C_{\theta_i}}) = \sum_i (\#\{\text{vertices of } \Delta_i\} - 2)$$

Noting that

$$\#\{\text{vertices of } \Delta_i\} = \#\{\text{edges in } \Delta_i\}$$

we obtain the formula

$$\deg_{C_\theta}(\mathscr{O}_Y(1)|_{C_\theta}) = -2f + 2e.$$

Since we assume that $f - e + v = 2$ this in turn gives

$$\deg_{C_\theta}(\mathscr{O}_Y(1)|_{C_\theta}) = 2(v - 2).$$

From this it follows that for a general section $\theta \in \Gamma(X_R, \mathcal{O}_{X_R}(1))$ with generic fiber θ_K we have

$$\deg_{C_{\theta_K}}(\mathcal{O}_{X_K}(1)|_{C_{\theta_K}}) = 2(v-2)$$

as desired. This completes the proof of 7.1. $\square$

Remark 7.3. With notation as in 7.1 assume further that the characterstic of k is 0. Then we get as in 3.24 a canonical basis, well defined up to permutation and scaling, for $H^0(X_K, \mathcal{O}_{X_K}(1))$. In fact, we get an H-torsor $\pi : Q \to \mathrm{Spec}(K)$ such that $\pi^*\Gamma(X_K, \mathcal{O}_{X_K}(1))$ has a canonical decomposition into rank 1 submodules (the point being that depending on the size of H the basis for $\Gamma(X_K, \mathcal{O}_{X_K}(1))$ may be more precisely determined than up to the action of the full permutation group).

8. A family of examples

In this section we work out more details of the construction in the preceding sections for $K3$ surfaces for a specific series of examples. This will show, in particular, that there exist primitively polarized $K3$ surfaces in all genera with toric degenerations in the sense of the preceding discussion. Throughout this section k is a field of characteristic different from 2 or 3.

Remark 8.1. Bernd Siebert had discovered these, or very similar, examples independently (unpublished).

8.2. Let $(Y, \mathcal{O}_Y(1))$ be the polarized scheme obtained by gluing together copies of $(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(1))$, represented by triangles, and

$$(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1,1)) = \mathrm{Proj}(k[X, Y, Z, W]/(XY - ZW)),$$

represented by squares, according to the following diagrams:

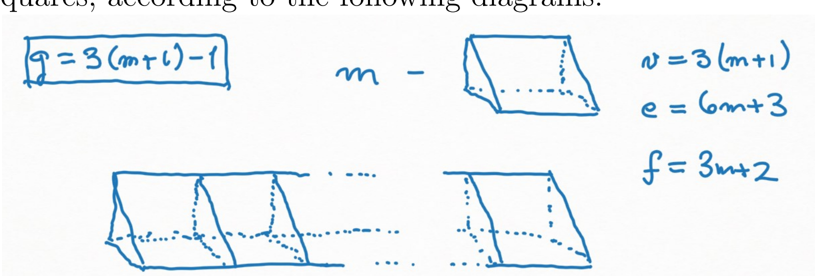

Figure 1.

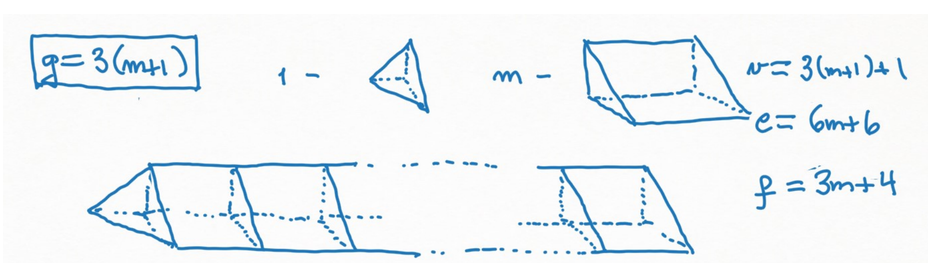

Figure 2.

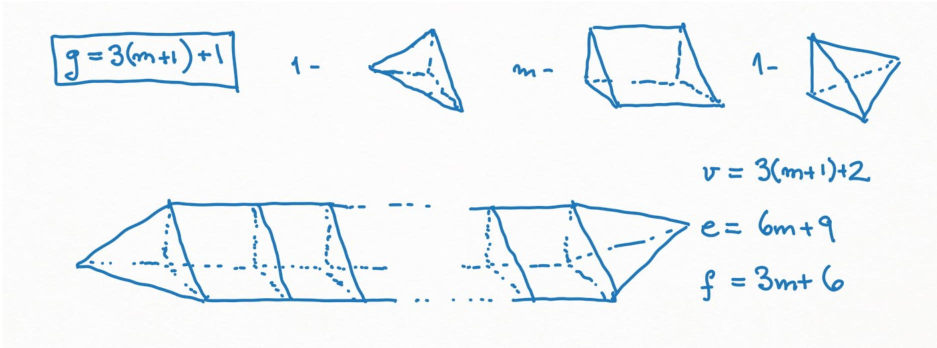

Figure 3.

Here m is assumed at least 1 in the first two cases and at least 0 in the last case.

Note that as m varies subject to these constraints g takes on all integer values ≥ 4. We have already exhibited a $K3$ surface of genus 3 with toric degeneration in the discussion of hypersurfaces. Since we assumed in the above that we had a very ample invertible sheaf, the case $g = 2$ is ruled out in our analysis.

8.3. Let us describe the automorphism group scheme $\mathscr{G}$ in each of our three cases. Let $\mathbf{T}$ denote the torus with coordinates

$$(u, w, v_0, \ldots, v_m) \quad (\text{resp. } (s, u, w, v_0, \ldots, v_m),\ (s, t, u, w, v_0, \ldots, v_m))$$

in case 1 (resp. case 2, case 3). Then $\mathbf{T}$ acts on $(Y, \mathscr{O}_Y(1))$ according to the labelling in the following figures:

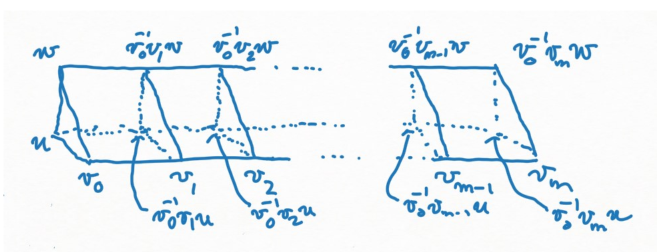

Figure 4.

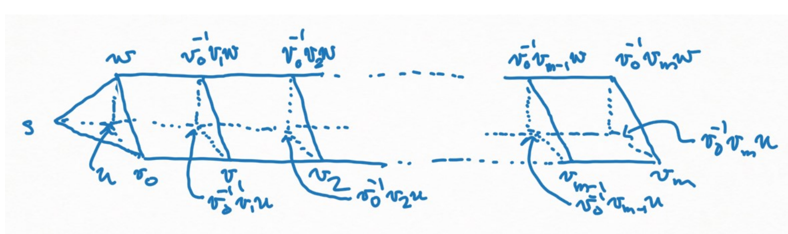

Figure 5.

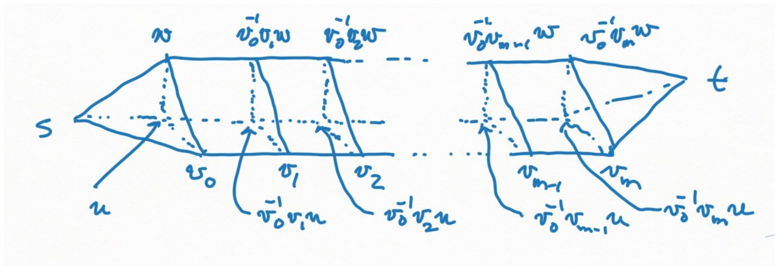

Figure 6.

Note that for each component Y_i of Y the map

$$\mathbf{T} \to \mathbf{T}_i$$

is surjective (that is, assumption (v) in section 6 holds). From this it follows that the automorphism group scheme $\mathscr{G}$ of $(Y, \mathscr{O}_Y(1))$ is an extension

$$1 \to \mathbf{T} \to \mathscr{G} \to H \to 1,$$

where H is the subgroup of the permutation group of the vertices preserving the edges and faces. An elementary calculation shows that H is equal to S_3 in cases 1 and 3 (generated by rotation and reflection across the central axis), and $\mathbf{Z}/(3)$ in case 2 (generated by rotation).

8.4. With this we can also now readily verify the assumptions (i)–(vii) in section 6. Assumptions (i)–(iv) hold by definition of $(Y, \mathscr{O}_Y(1))$, we have just verified (v), (vi) holds because we assumed that k does not have characteristic 2 or 3, and (vii) holds by inspection of the diagrams Figure 4-6.

8.5. In each of the above cases we can write down explicitly degenerations of polarized $K3$ surfaces to $(Y, \mathscr{O}_Y(1))$.

To do so let us first consider the deformation space of a genus 0 nodal curve C_0 given by a chain of m copies of $\mathbf{P}^1$. Let $P \subset \mathbf{R} \times \mathbf{R}$ be the integral points of the cone over $[0, m] \times \{1\}$. For $i = 1, \dots, m-1$ let $h_i : \{0, \dots, m\} \to \mathbf{N}$ be the function

$$h_i(j) := \max\{0, j - i\},$$

and extend this function to the piecewise linear function on $[0, m]$ which is linear on $[0, i]$ and $[i, m]$. We can then further extend h_i to a function

$$\tilde{h}_i : P \to \mathbf{N}, \quad p \mapsto \deg(p) \cdot h_i(p/\deg(p)).$$

Define

$$\mathscr{R} \subset k[[t_1, \dots, t_{m-1}]][P]$$

to be the subring generated by elements

$$t_1^{h_1(p)} t_2^{h_2(p)} \cdots t_{m-1}^{h_{m-1}(p)} p,$$

and set

$$(\mathscr{P}, \mathscr{O}_P(1)) := \operatorname{Proj}(\mathscr{R}).$$

Then $(\mathscr{P}, \mathscr{O}_{\mathscr{P}}(1))$ is a family of polarized genus 0 nodal curves over $k[[t_1, \dots, t_{m-1}]]$ with closed fiber $(C_0, \mathscr{O}_{C_0}(1))$ and smooth generic fiber. Here $\mathscr{O}_{C_0}(1)$ is the invertible sheaf given by $\mathscr{O}_{\mathbf{P}^1}(1)$ on each irreducible component.

The face in P spanned by 0 (resp. m) defines a section of $\mathscr{P} \to \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]])$ which we denote by 0 (resp. ∞). We refer to the component of C_0 which contains 0 (resp. ∞) as the first component (resp. the last component).

Observe that since $H^i(C_0, \mathscr{O}_{C_0}) = 0$ for $i > 0$, any line bundle on C_0 lifts uniquely to $\mathscr{P}$. Let $\mathscr{L}_{0,0}$ be the invertible sheaf on C_0 given by $\mathscr{O}_{\mathbf{P}^1}(1)$ on the first component on C_0 and the structure sheaf on other components, and let $\mathscr{L}_{\infty,0}$ be the invertible sheaf which is $\mathscr{O}_{\mathbf{P}^1}(1)$ on the last component and the structure sheaf on other components. Let $\bar{\theta}_0 \in \Gamma(C_0, \mathscr{L}_{0,0})$ be the (unique up to scalar) nonzero section which vanishes at the point 0, and let $\bar{\theta}_\infty \in \Gamma(C_0, \mathscr{L}_{\infty,0})$ be the section which vanishes at the point ∞.

Let $\mathscr{L}_0$ (resp. $\mathscr{L}_\infty$) be the unique lift of $\mathscr{L}_{0,0}$ (resp. $\mathscr{L}_{\infty,0}$), so we have $\mathscr{L}_0 \simeq \mathscr{O}_{\mathscr{P}}(0)$ (resp. $\mathscr{L}_\infty \simeq \mathscr{O}_{\mathscr{P}}(\infty)$), and let $\theta_0 \in \Gamma(\mathscr{P}, \mathscr{L}_0)$ (resp. $\theta_\infty \in \Gamma(\mathscr{P}, \mathscr{L}_\infty)$) be the canonical lifting of $\bar{\theta}_0$ (resp. $\bar{\theta}_\infty$).

8.6. **The family for Figure 1.** Consider the space

$$\mathbf{P}^2_k \times_k \mathscr{P}$$

with the invertible sheaf

$$\mathscr{M} := \mathscr{O}_{\mathbf{P}^2}(3) \boxtimes (\mathscr{L}_0 \otimes \mathscr{L}_\infty).$$

By cohomology and base change, the $k[[t_1, \dots, t_{m-1}]]$-module

$$M := \Gamma(\mathbf{P}^2_k \times_k \mathscr{P}, \mathscr{M})$$

is locally free of finite rank, and the formation of this space of global section sections commutes with arbitrary base change.

Let $\mathbf{M}$ be the affine space over $k[[t_1, \dots, t_{m-1}]]$ given by the symmetric algebra on $M^\vee$. If X_0, X_1, X_2 denote the coordinates on $\mathbf{P}^2_k$ then the global section

$$X_0X_1X_2 \otimes \theta_0\theta_\infty \in M$$

defines a section

$$s : \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]]) \to \mathbf{M}.$$

Let $\widehat{\mathbf{M}}$ be the spectrum of the completion of $\mathbf{M}$ along this section, and let A denote its coordinate ring.

So A is a complete local ring with residue field k, and over A we have a polarized scheme $(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(1))$ defined by the universal section of M over $\mathbf{M}$, where $\mathscr{O}_{\mathscr{X}}(1)$ is obtained by pulling back $\mathscr{O}_{\mathbf{P}^2_k}(1) \boxtimes \mathscr{O}_{\mathscr{P}}(1)$.

Let K be the field of fractions of A. By Bertini's theorem the generic fiber $\mathscr{X}_K$ is smooth, and therefore by 7.1 we have a polarized $K3$ surface $(\mathscr{X}_K, \mathscr{O}_{\mathscr{X}_K}(1))$ degenerating to $(Y, \mathscr{O}_Y(1))$.

Note also that there is a projection map

$$q : \mathscr{X} \to \mathscr{P}.$$

Consider a section $x : \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]]) \to \mathscr{P}$ of $\mathscr{P} \to \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]])$, with image in the smooth locus. Let $\mathscr{L}_x$ be the pullback of $\mathscr{O}_{\mathscr{P}}(x)$ along q, and let $e : \mathscr{O}_{\mathscr{X}} \to \mathscr{L}_x$ be the section defined by pulling back the canonical section of $\mathscr{O}_{\mathscr{P}}(x)$. The zero locus of e is then a closed subscheme $\mathscr{E} \subset \mathscr{X}$, flat over $k[[t_1, \dots, t_{m-1}]]$ whose closed fiber is isomorphic to the standard 3-gon. It follows that $\mathscr{E}$ is a family of genus 1 nodal curves.

From this it follows that the polarization $\mathcal{O}_{\mathfrak{X}_K}(1)$ is primitive. Indeed suppose $n \geq 1$ is an integer such that there exists an invertible sheaf $\mathcal{A}_K$ on $\mathfrak{X}_K$ such that $\mathcal{A}_K^{\otimes n} \simeq \mathcal{O}_{\mathfrak{X}_K}(1)$. Then n must be 1 or 3 since the degree of $\mathcal{O}_{\mathfrak{X}_K}(1)$ pulled back to $\mathcal{E}_K$ is 3. On the other hand, we know that $c_1(\mathcal{O}_{\mathfrak{X}_K}(1))^2 = 2g - 2$, which in this case is not divisible by 3.

8.7. **The family for Figure 2.** In this case consider the space

$$\mathbf{P} := \mathbf{P}^3_k \times_k \mathcal{P},$$

with coordinates X_0, X_1, X_2, X_3 on $\mathbf{P}^3_k$. Let $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) denote the line bundle $\mathcal{O}_{\mathbf{P}^3_k}(1) \boxtimes \mathcal{L}_0$ (resp. $\mathcal{O}_{\mathbf{P}^3_k}(3) \boxtimes \mathcal{L}_\infty$) on $\mathbf{P}$. Again by cohomology and base change the $k[[t_1, \dots, t_{m-1}]]$-modules $M_1 := \Gamma(\mathbf{P}, \mathcal{M}_1)$ (resp. $M_2 := \Gamma(\mathbf{P}, \mathcal{M}_2)$) are locally free of finite rank and their formation commutes with arbitrary base change. Let $\mathbf{M}_1$ (resp. $\mathbf{M}_2$) denote the affine space given by the symmetric algebra on $M_1^\vee$ (resp. $M_2^\vee$), and set

$$\mathbf{M} := \mathbf{M}_1 \times_{k[[t_1, \dots, t_{m-1}]]} \mathbf{M}_2.$$

The sections

$$X_0 \otimes \theta_0 \in M_1, \quad X_1 X_2 X_3 \otimes \theta_\infty \in M_2$$

define a section

$$s : \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]]) \to \mathbf{M}.$$

We let $\widehat{\mathbf{M}}$ be the spectrum of the completion of $\mathbf{M}$ along this section, and let A denote its coordinate ring.

The universal sections over $\mathbf{M}$ then define a flat family of polarized schemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}(1))$ over A with closed fiber $(Y, \mathcal{O}_Y(1))$ (given by the data in Figure 2). Here $\mathcal{O}_{\mathfrak{X}}(1)$ is the pullback of $\mathcal{O}_{\mathbf{P}^3_k} \boxtimes \mathcal{O}_{\mathcal{P}}(1)$.

Let K be the field of fractions of A. The generic fiber $\mathfrak{X}_K$ is smooth by Bertini's theorem, and therefore we have a polarized $K3$ surface $(\mathfrak{X}_K, \mathcal{O}_{\mathfrak{X}_K}(1))$ of genus $g = 3(m+1)$.

The primitivity of $\mathcal{O}_{\mathfrak{X}_K}(1)$ follows from a similar argument to the previous case. The projection map

$$q : \mathfrak{X} \to \mathcal{P}$$

again has general fiber a flat family of genus 1 nodal curves, and the restriction of $\mathcal{O}_{\mathfrak{X}_K}(1)$ to one of these fibers has degree 3. Therefore again if we have $\mathcal{A}$ such that $\mathcal{A}^{\otimes n} \simeq \mathcal{O}_{\mathfrak{X}_K}(1)$ then we must have $n = 1$ or 3. Furthermore, since $c_1(\mathcal{O}_{\mathfrak{X}_K})^2 = 2(g-1)$ is not divisible by 3 it follows that $n = 1$ and $\mathcal{O}_{\mathfrak{X}_K}(1)$ is primitive.

8.8. **The family for Figure 3.** Again we consider the space

$$\mathbf{P} := \mathbf{P}^3_k \times_k \mathcal{P}$$

but this time the line bundles

$$\mathcal{M}_1 := \mathcal{O}_{\mathbf{P}^3_k}(1) \boxtimes (\mathcal{L}_0 \otimes \mathcal{L}_\infty), \quad \mathcal{M}_2 := \mathcal{O}_{\mathbf{P}^3_k}(3) \boxtimes \mathcal{O}_{\mathcal{P}},$$

and the associated spaces $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M} = \mathbf{M_1} \times_{k[[t_1, \dots, t_{m-1}]]} \mathbf{M}_2$. The section s of $\mathbf{M} \to \mathrm{Spec}(k[[t_1, \dots, t_{m-1}]])$ in this case is given by

$$X_0 \otimes (\theta_0 \theta_\infty) \ \text{ and } \ X_1 X_2 X_3,$$

and we obtain a ring A with a family $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))$ over A with closed fiber $(Y, \mathcal{O}_Y(1))$ described by Figure 3. Again the generic fiber $\mathcal{X}_K$ is smooth by Bertini's theorem.

The previous argument for the primitivity of $\mathcal{O}_{\mathcal{X}_K}(1)$ does not quite work here, however, since $g-1$ is divisible by 3. However, the argument can be adjusted as follows. After possibly making an extension of k we may assume that k is algebraically closed.

Let $\sigma : \mathbf{P}^3_k \to \mathbf{P}^3_k$ be the automorphism given by

$$X_0 \mapsto X_0,\ X_1 \mapsto X_2,\ X_2 \mapsto X_3,\ X_3 \mapsto X_1.$$

There is an induced action on $\mathbf{M}$ which fixes the section s, and therefore also an induced action on A and the family $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1))$. Somewhat abusively, we write σ for all these induced automorphisms. Let $K_0 \subset K$ be the fixed field of σ, so that $(\mathcal{X}_K, \mathcal{O}_{\mathcal{X}_K}(1))$ descends to a polarized variety $(\mathcal{X}_{K_0}, \mathcal{O}_{\mathcal{X}_{K_0}}(1))$ over K_0.

Now observe that the map

$$\mathrm{Pic}(\mathcal{X}_{K_0}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is injective and identifies $\mathrm{Pic}(\mathcal{X}_{K_0})$ with the σ-invariants of $\mathrm{Pic}(\mathcal{X}_K)$. Furthermore, these groups are torsion free. It follows that since the class $[\mathcal{O}_{\mathcal{X}_K}(1)]$ is σ-invariant, so is the class of any root $\mathcal{A}$ of $\mathcal{O}_{\mathcal{X}_K}(1)$. It follows that it suffices to show that $\mathcal{O}_{\mathcal{X}_{K_0}}(1)$ is primitive in $\mathrm{Pic}(\mathcal{X}_{K_0})$.

Consider the projection

$$q : \mathcal{X} \to \mathcal{P},$$

and observe that σ acts on $\mathcal{X}$ over $\mathcal{P}$. The general fiber of q is a genus 1 curve, with induced free action of $\mathbf{Z}/(3)$ (note that the action on the general fiber is free, since this is true for a fiber over the closed point, where σ acts by rotation of the 3-gon). In particular, let $t \in \mathcal{P}(k((t_1, \dots, t_{m-1})))$ be a general section, so $\mathcal{X}_t$ is a σ-linearized genus 1 curve over K.

Let $\mathcal{X}_{t,K_0}$ be the scheme over K_0 obtained by descent from $\mathcal{X}_t$, and let $\mathcal{E}_0$ be the Jacobian of $\mathcal{X}_{t,K_0}$. The natural map

$$\mathcal{X}_{t,K_0} \to \underline{\mathrm{Pic}}^1_{\mathcal{X}_{t,K_0}}$$

to the subscheme of the Picard scheme classifying degree 1 line bundles is an isomorphism, which gives $\mathcal{X}_{t,K_0}$ the structure of a torsor under $\mathcal{E}_0$. Furthermore, the third power of this torsor is trivial, and therefore $\mathcal{X}_{t,K_0}$ is given by a Galois cohomology class in

$$H^1(K_0, \mathcal{E}_0[3]).$$

By construction, the group scheme $\mathcal{E}_0[3]$ extends to a constant finite flat group scheme over A, and from this it follows that $\mathcal{E}_0[3]$ is in fact constant over K_0. Furthermore, the torsor $\mathcal{X}_t$ is trivial so the Galois cohomology class is given by a homomorphism

$$\mathrm{Gal}(K/K_0) \to \mathcal{E}_0[3];$$

that is, by a 3-torsion point α of $\mathcal{E}_0$. Concretely, the torsor $\mathcal{X}_{t,K_0}$ is isomorphic to the torsor obtained from the descent data on $\mathcal{E}_{0,K}$ given by

$$(+\alpha, \sigma_K) : \mathcal{E}_0 \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(K) \to \mathcal{E}_0 \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(K).$$

Furthermore, the point α is not the identity since this is true when specializing to the closed fiber.

From this it follows that the composition

$$\mathrm{Pic}(\mathfrak{X}_{K_0}) \longrightarrow \mathrm{Pic}(\mathfrak{X}_t) \xrightarrow{\deg} \mathbf{Z}$$

has image in $3\mathbf{Z}$. Furthermore, the image of $\mathcal{O}_{\mathfrak{X}_{K_0}}(1)$ under this map is precisely 3, from which it follows that $\mathcal{O}_{\mathfrak{X}_K}(1)$ is primitive.

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