THETA FUNCTIONS FOR MONOMIAL DEGENERATIONS

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Abstract. In an earlier article we discussed a construction of canonical bases for line bundles on certain degenerating varieties. In this article, we make the construction more explicit in the case of a family degenerating to variety given by certain monomial ideals.

1. Introduction

1.1. In the article [4] we described a rather general construction of canonical sections of line bundles on degenerating algebraic varieties. The basic setup is the following. Let k be an algebraically closed field of characteristic 0 and let R be a complete noetherian local k-algebra with residue field k. Consider a pair $(X, \mathscr{L})$, where X is a projective flat R-scheme and $\mathscr{L}$ is a very ample invertible sheaf on X. We then showed that in certain situations, when the closed fiber $(X_0, \mathscr{L}_0)$ has a sufficiently large positive dimensional automorphism group, one can obtain an R-basis for $H^0(X, \mathscr{L})$, well-defined up to scaling and permutation. In loc. cit. we studied the case of $K3$ surfaces and hypersurfaces.

The purpose of this article is to work out explicitly the construction in the case of monomial degenerations of projective varieties. On the one hand, this case is substantially easier than the more general setup we considered in [4]. On the other hand, the added assumptions enable a very explicit description of the canonical basis in this setting.

1.2. Let $\mathscr{E}$ be a free R-module of finite rank, and let

$$i : X \hookrightarrow \mathbf{P}\mathscr{E}$$

be a closed subscheme flat over R. Assume that the closed fiber $X_0 \subset \mathbf{P}\mathscr{E}_0$ is given by a monomial ideal. That is, there exists an isomorphism $\mathscr{E}_0 \simeq k^m$, where m is the rank of $\mathscr{E}_0$, such that the ideal of X_0 in $k[x_0, \dots, x_m]$ is monomial. This implies, in particular, that the action of the torus $\mathbf{G}_m^{m+1}$, given by multiplication on the variables x_i, preserves the subscheme X_0. In section 3 we introduce a notion of a *maximally monomial* scheme, which loosely means that $\mathbf{G}_m^{m+1}$ is the connected component of the maximal subgroup of $\mathbf{GL}_{m+1}$ preserving X_0 (this is a stronger condition that what we considered in [4]). Assuming this condition we construct in section 4 an R-basis for $\mathscr{E}$, well-defined up to scaling and permutation, following the method in [4], and in section 5 we give a more concrete description of this basis.

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2. Varieties with very ample linear systems

2.1. In this article we will be interested in collections of data

$$(f : X \to S, \mathscr{L}, \beta : \mathscr{E} \to f_*\mathscr{L}) \tag{2.1.1}$$

where

(i) $f : X \to S$ is a proper finitely presented flat morphism of schemes,
(ii) $\mathscr{L}$ is an invertible sheaf on X,
(iii) $\mathscr{E}$ is a locally free sheaf on S of finite rank,
(iv) β is a morphism of sheaves of $\mathscr{O}_S$-modules such that the adjoint map $f^*\mathscr{E} \to \mathscr{L}$ is surjective and defines a closed immersion
$$i : X \hookrightarrow \mathbf{P}\mathscr{E}$$
over S.

We call such a collection (2.1.1) a *variety with a very ample linear system over* S. Note that for a morphism of schemes $S' \to S$ there is a natural notion of pullback of an object (2.1.1) to S'. We therefore get a fibered category
$$\mathscr{M} \to (\text{Schemes}), \quad (f : X \to S, \mathscr{L}, \beta : \mathscr{E} \to f_*\mathscr{L}) \mapsto S.$$
Standard descent theory [1, Exposé VIII, 1.2 and 7.8] implies that $\mathscr{M}$ is a stack for the fpqc topology.

Remark 2.2. If $f : X \to S$ is a projective morphism and $\mathscr{L}$ is a relatively very ample invertible sheaf on X such that $f_*\mathscr{L}$ is locally free on S and its formation commutes with base change, then we can take $\mathscr{E} = f_*\mathscr{L}$ and β the identity map to get a variety with very ample linear system over S.

2.3. The stack $\mathscr{M}$ can be described as follows using Hilbert schemes. This description implies, in particular, that $\mathscr{M}$ is algebraic locally of finite type over $\mathbf{Z}$.

First of all note that for an object (2.1.1) the rank of $\mathscr{E}$ is a locally constant function on S. Therefore we have
$$\mathscr{M} = \coprod_{r \geq 1} \mathscr{M}_r,$$
where for an integer $r \geq 1$ the stack $\mathscr{M}_r$ is the substack classifying objects (2.1.1) with $\mathscr{E}$ of rank r.

For $r \geq 1$ let $\mathbf{Hilb}_r$ be the Hilbert scheme of closed subschemes of $\mathbf{P}^{r-1}$. Observe that we can view $\mathbf{Hilb}_r$ as the functor sending a scheme S to the set of isomorphism classes of data
$$((f : X \to S, \mathscr{L}, \beta : \mathscr{E} \to f_*\mathscr{L}), \sigma : \mathscr{O}_S^r \to \mathscr{E}) \tag{2.3.1}$$
consisting of an object (2.1.1) of $\mathscr{M}_r(S)$ together with an isomorphism $\sigma : \mathscr{O}_S^r \to \mathscr{E}$. Indeed an S-point $x \in \mathbf{Hilb}_r(S)$ is given by a closed subscheme
$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbf{P}_S^{r-1} \\ & {}_{f}\searrow & \downarrow \\ & & S, \end{array}$$
where f is flat, and we obtain a collection of data by taking $\mathscr{L} := i^*\mathscr{O}_{\mathbf{P}_S^{r-1}}(1)$, $\mathscr{E} = \mathscr{O}_S^r$, σ the identity map, and β induced by the tautological projection
$$\mathscr{O}_{\mathbf{P}_S^{r-1}}^r \to \mathscr{O}_{\mathbf{P}_S^{r-1}}(1).$$

Conversely, for a collection (2.3.1) the map β defines a closed immersion

$$X \hookrightarrow \mathbf{P}\mathscr{E} \xrightarrow[\sigma]{\simeq} \mathbf{P}^{r-1}_S.$$

A straightforward verification shows that these two constructions define inverse morphisms of functors. Furthermore, the action of $\mathbf{GL}_r$ on $\mathbf{Hilb}_r$ given by the natural action on $(\mathbf{P}^{r-1}, \mathscr{O}_{\mathbf{P}^{r-1}}(1))$ corresponds to the action on the set of data (2.3.1) given by precomposing σ with an element of $\mathbf{GL}_r$. From this it follows that we have an isomorphism

$$\mathscr{M}_r \simeq [\mathbf{Hilb}_r/\mathbf{GL}_r].$$

Appendix A.

In many instances we are interested in the setting of 2.2 and deformations of such projective varieties.

A.1. Let k be a field and let $f : X \to S$ be a proper morphism locally of finite presentation over k. We say that a locally finitely presented quasi-coherent sheaf $\mathscr{F}$ on X is *cohomologically flat over S in dimension* 0 if the sheaf $f_*\mathscr{F}$ on S is locally free on S and if for every morphism $g : S' \to S$ the natural map

$$g^*f_*\mathscr{F} \to f'_*\mathscr{F}'$$

is an isomorphism, where $f' : X_{S'} \to S'$ is the base change of f and $\mathscr{F}'$ is the pullback of $\mathscr{F}$ to $X_{S'}$.

Lemma A.2. *Let $f : X \to S$ be a proper finitely presented morphism of k-schemes and let $\mathscr{F}$ be a locally finitely presented quasi-coherent sheaf on X flat over S. Suppose that for some point $s \in S$ the map*

$$(f_*\mathscr{F})_s \otimes \kappa(s) \to H^0(X_s, \mathscr{F}_s)$$

is surjective. Then there exists a neighborhood $s \in U \subset S$ of s such that the restriction $\mathscr{F}|_{X_U}$ of $\mathscr{F}$ to $X_U := f^{-1}(U)$ is cohomologically flat over U in dimension 0*.*

Proof. This is a consequence of cohomology and base change. If S is noetherian and f is projective, this follows immediately from [3, Chapter III, 12.11].

For the general case, note that by [5, Tag 0A1G] the object $Rf_*\mathscr{F}$ of the derived category $D(S)$ of S is a perfect object, in the sense of [5, Tag 08CM], and its formation commutes with arbitrary base change. The fact that the formation of this complex commutes with arbitary base change implies that $Rf_*\mathscr{F}$ has tor-amplitude (see [5, Tag 08CG])in $[0, d]$ for some $d \geq 0$. It therefore suffices to prove the following variant statement: Let S be a k-scheme and let $\mathscr{K} \in D(S)$ be a perfect complex with tor-amplitude in $[0, d]$ for some d. Let $s \in S$ be a point and suppose that the map

$$\mathscr{H}^0(\mathscr{K})_s \otimes \kappa(s) \to H^0(\mathscr{K} \otimes^{\mathbf{L}} \kappa(s))$$

is surjective. Then there exists a neighborhood $s \in U \subset S$ of s such that

$$\mathscr{K}|_U \simeq \mathscr{H}^0(\mathscr{K})|_U \oplus \tau_{\geq 1}\mathscr{K}|_U$$

in $D(U)$, and $\mathscr{H}^0(\mathscr{K})|_U$ is a finitely presented projective $\mathscr{O}_U$-module.

To prove this statement we may assume that $S = \mathrm{Spec}(A)$ is affine. Writing A as a filtered union of finitely generated k-algebras and using [5, Tag 0BC7] we further reduce to the case when A is noetherian. Represent $\mathscr{K}$ by a complex of projective A-modules

$$E^0 \to E^1 \to \cdots \to E^d.$$

After shrinking on A we may assume that the E^i are all free modules. Choose a basis $b_1, \dots, b_n$ for $E^0(s)$ such that $b_1, \dots, b_r$ are a basis for $H^0(E^\bullet(s))$. By our assumptions we can then choose liftings $\tilde{b}_1, \dots, \tilde{b}_n \in E^0$ of the b_i such that $\tilde{b}_1, \dots, \tilde{b}_r$ lie in $H^0(E^\bullet)$. By Nakayama's lemma we then get, after possibly shrinking on $\mathrm{Spec}(A)$, that the map

$$A^r \to H^0(E^\bullet), \quad (a_1, \dots, a_r) \mapsto \sum a_i \tilde{b}_i$$

is surjective, and the map

$$A^n \to E^0, \quad (a_1, \dots, a_n) \mapsto \sum a_i \tilde{b}_i$$

is an isomorphism. From this it follows that both maps are isomorphisms and that $H^0(E^\bullet)$ is a direct summand of E^0. □

Corollary A.3. *Let $f : X \to S$ be a proper finitely presented flat morphism of k-schemes and let $\mathscr{L}$ be an invertible sheaf on X. Let $s \in S$ be a point such that $\mathscr{L}_s$ is very ample on X_s and such that the map*

$$(f_*\mathscr{L})_s \otimes \kappa(s) \to H^0(X_s, \mathscr{L}_s)$$

is surjective. Then there exists a neighborhood $s \in U \subset S$ such that $(f_*\mathscr{L})|_U$ is locally free, its formation commutes with arbitrary base change, and

$$(X|_U, \mathscr{L}|_U, (f_*\mathscr{L})|_U, \mathrm{id})$$

is a variety with very ample linear system over U.

Proof. By A.2 we may, after shrinking on S, assume that $f_*\mathscr{L}$ is locally free of finite rank and that the formation of this sheaf commutes with arbitrary base change. The locus where the adjunction map $f^*f_*\mathscr{L} \to \mathscr{L}$ fails to be surjective is closed and does not meet X_s by assumption. Since f is proper, which implies that the image of this closed set is a closed set in S not containing s, we can after further shrinking arrange that the adjunction map is surjective in which case we get a morphism

$$X \to \mathbf{P}(f_*\mathscr{L})$$

over S. After further shrinking, and using that this is a closed immersion over s, we can arrange that this is a closed immersion as desired. □

3. Maximally monomial schemes

3.1. Let k be a field and consider a variety with very ample linear system $\mathscr{X} := (X, \mathscr{L}, \mathscr{E}, \beta)$ over k. Such a collection of data gives rise to a closed immersion

$$i : X \hookrightarrow \mathbf{P}(\mathscr{E}).$$

Let $\mathscr{G}_{\mathscr{X}}$ denote the functor on k-schemes which to any T/k associates the group of automorphisms over T of the pullback $(X_T, \mathscr{L}_T, \mathscr{E}_T, \beta_T)$. We refer to $\mathscr{G}_{\mathscr{X}}$ as the *theta group* of $\mathscr{X} = (X, \mathscr{L}, \mathscr{E}, \beta)$.

Lemma 3.2. *The functor $\mathscr{G}_{\mathfrak{X}}$ is representable by an affine group scheme, which we somewhat abusively also denote by $\mathscr{G}_{\mathfrak{X}}$, and the natural map*

$$\rho : \mathscr{G}_{\mathfrak{X}} \to \mathbf{GL}(\mathscr{E})$$

is a closed immersion.

Proof. Giving an automorphism of $(X, \mathscr{L}, \mathscr{E}, \beta)$, possibly over a base T/k, is equivalent to giving an automorphism of $\mathscr{E}$ such that the induced automorphism of $\mathbf{P}\mathscr{E}$ preserves $i : X \subset \mathbf{P}\mathscr{E}$. In other words, if $\mathbf{Hilb}_{\mathbf{P}\mathscr{E}}$ denotes the Hilbert scheme of closed subschemes of $\mathbf{P}\mathscr{E}$ then there is a map

$$\mathbf{GL}(\mathscr{E}) \to \mathbf{Hilb}_{\mathbf{P}\mathscr{E}}, \quad \alpha \mapsto \alpha^*[X],$$

where we write $[X] \in \mathbf{Hilb}_{\mathbf{P}\mathscr{E}}(k)$ for the k-point defined by i, and $\mathscr{G}_{\mathfrak{X}}$ is isomorphic to the fiber product of the diagram

$$\begin{array}{ccc} & & \mathrm{Spec}(k) \\ & & \downarrow{\scriptstyle [X]} \\ \mathbf{GL}(\mathscr{E}) & \longrightarrow & \mathbf{Hilb}_{\mathbf{P}\mathscr{E}} \end{array}$$

From this the lemma follows. □

3.3. Let $\mathscr{G}^{\circ}_{\mathfrak{X}} \subset \mathscr{G}_{\mathfrak{X}}$ denote the connected component of the identity [5, Tag 0B7R]. Note that the formation of the identity component commutes with change of field.

Definition 3.4. The variety with linear system $\mathfrak{X}$ is *maximally monomial* if $\mathscr{G}^{\circ}_{\mathfrak{X}} \subset \mathbf{GL}(\mathscr{E})$ is a maximal torus.

Remark 3.5. If $(X, \mathscr{L})$ is a projective scheme with very ample invertible sheaf $\mathscr{L}$ over k, then we can take $\mathscr{E} = \Gamma(X, \mathscr{L})$ and β the identity map to get a variety with linear system $\mathfrak{X}$. In this case we will also say that $(X, \mathscr{L})$ is maximally monomial and write $\mathscr{G}_{(X,\mathscr{L})}$ for $\mathscr{G}_{\mathfrak{X}}$.

Example 3.6. Assume that k has characteristic 0 and let $n \geq 1$ be an integer. Let $a_0, \dots, a_n \geq 1$ be integers and let

$$X \subset \mathbf{P}^n$$

be the closed subscheme defined by the equation

$$X_0^{a_0} \cdots X_n^{a_n} = 0,$$

and let $\mathscr{L}$ be the pullback of $\mathscr{O}_{\mathbf{P}^n}(1)$ to X. Then $(X, \mathscr{L})$ is maximally monomial. To see this note first that the ideal of X is preserved under the action of $\mathbf{G}_m^{n+1}$, acting by multiplication on the variables, so we have an inclusion

$$\mathbf{G}_m^{n+1} \hookrightarrow \mathscr{G}^{\circ}_{(X,\mathscr{L})}.$$

We claim that this inclusion is an isomorphism. To see this we may base change to an algebraic closure of k, and therefore can assume that $k = \bar{k}$, and since k has characteristic 0 it suffices to show that the map on k-points is bijective. So let $\alpha \in \mathscr{G}^{\circ}_{(X,\mathscr{L})}(k)$ be such a point. Since α preserves X and $\mathscr{G}^{\circ}_{(X,\mathscr{L})}$ is connected, α must fix each of the irreducible components of X with their reduced structure. We conclude that α fixes each of the standard hyperplanes of $\mathbf{P}^n$, which an elementary exercise (or see [4, 3.3 (ii)]) yields that $\alpha \in \mathbf{G}_m^{n+1}(k)$.

Example 3.7. The preceding example can be generalized as follows. Consider again k of characteristic 0 and $\mathbf{P}^n$ for some integer $n \geq 1$. For a subset $J \subset \{0, \dots, n\}$ let $H_J \subset \mathbf{P}^n$ be the linear subspace given by

$$H_j := \cap_{j \in J} V(X_j).$$

Let $X \subset \mathbf{P}^n$ be defined by a monomial ideal in $k[X_0, \dots, X_n]$ so the irreducible components X_i with their reduced structure are of the form H_{J_i} for subsets $J_i \subset \{0, \dots, n\}$. For $s = 0, \dots, n$ let $P_s \in \mathbf{P}^n$ be the point given by $X_j = 0$ for $j \neq s$, and suppose that all the P_s can be realized as intersections of components X_i of X. Equivalently, suppose that each of the sets

$$\{0, \dots, \hat{s}, \dots, n\} \subset \{0, \dots, n\}$$

can be written as unions of the J_i. Then we claim that X, together with the pullback of $\mathcal{O}_{\mathbf{P}^n}(1)$, is maximally monomial. To see this note that since X is defined by a monomial idea we have $\mathbf{G}_m^{n+1} \subset \mathcal{G}^\circ_{(X, \mathcal{L})}$ and to verify that this map is an isomorphism it suffices to consider the case when $k = \bar{k}$ and to show that the inclusion is bijective on k-points. An element $\alpha \in \mathcal{G}^\circ_{(X, \mathcal{L})}(k)$ must fix the irreducible components of X and therefore also the P_i. Representing α by a matrix $M \in \mathbf{M}_{n+1}(k)$ the condition that $\alpha(P_i) = P_i$ is equivalent to the statement that the i-th column of M is zero away from the diagonal. Since this condition holds for all i we conclude that M is diagonal and therefore $\alpha \in \mathbf{G}_m^{n+1}$.

Example 3.8. For an explicit example of the preceding construction consider $X \subset \mathbf{P}^n$ defined by two equations

$$X_0^{a_0} X_1^{a_1} = 0 \quad \text{and} \quad X_2^{a_2} \cdots X_n^{a_n} = 0$$

for integers $a_i \geq 1$. The irreducible components of X then correspond to the subsets

$$J_{ij} = \{i, j\} \subset \{0, \dots, n\}, \quad i \in \{0, 1\}, \ j \in \{2, \dots, n\}.$$

We then have

$$\{1, \dots, n\} = \cup_{j=2}^n J_{1j}, \quad \{0, 2, 3, \dots, n\} = \cup_{j=2}^n J_{0j},$$

and

$$\{0, 1, \dots, \hat{i}, \dots, n\} = \cup_{j=2, \dots, n, j \neq i}(J_{0j} \cup J_{1j}).$$

It follows that $(X, \mathcal{L})$ is maximally monomial.

4. Local moduli at maximally monomial points

4.1. Let k be a field of characteristic 0 and let $\mathfrak{X}_0 = (X_0, \mathcal{L}_0, \mathcal{E}_0, \beta_0)$ be a maximally monomial variety with very ample linear system over k, corresponding to a morphism

$$[\mathfrak{X}_0] : \mathrm{Spec}(k) \to \mathcal{M},$$

where $\mathcal{M}$ is as in 2.1. Here, and throughout the rest of this section, we view $\mathcal{M}$ as a stack over k.

If $\mathcal{G}_{\mathfrak{X}_0}$ denotes the theta group of $\mathfrak{X}_0$ then this morphism descends to a closed immersion

$$s_0 : B\mathcal{G}_{\mathfrak{X}_0} \hookrightarrow \mathcal{M}.$$

We describe the formal neighborhood of this substack following the strategy of [4].

4.2. Let P be the Hilbert polynomial of $(X_0, \mathscr{L}_0)$, and let $\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$ be the Hilbert scheme of closed subschemes of $\mathbf{P}\mathscr{E}$ with Hilbert polynomial P, and let $\mathscr{M}^P \subset \mathscr{M}$ be the open and closed substack of $\mathscr{M}$ classifying varieties with very ample linear system (2.1.1) for which the Hilbert polynomial of $X_s \subset \mathbf{P}\mathscr{E}(s)$ is P for all $s \in S$. As in 2.3 we then have a presentation

$$\mathscr{M}^P \simeq [\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}/\mathbf{GL}(\mathscr{E}_0)].$$

We now use the standard imbedding of $\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$ into a Grassmanian as in [2, Proof of 3.1]. Consider a closed immersion i

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & (\mathbf{P}\mathscr{E}_0)_S \\ & {\scriptstyle f}\searrow & \downarrow \\ & & S \end{array} \tag{4.2.1}$$

over some scheme S defining an S-point of $\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$, and let $\mathscr{L}$ denote $i^*\mathscr{O}_{(\mathbf{P}\mathscr{E}_0)_S}(1)$.

By cohomology and base change, and using the quasi-compactness of $\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$, there exists an integer n_0 such that for any diagram (4.2.1) defining an S-point of $\mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$ and $n \geq n_0$ the sheaf $f_*\mathscr{L}^{\otimes n}$ is locally free of rank $P(n)$ on S, its formation commutes with arbitrary base change on S, and the natural map

$$\mathrm{Sym}^n\mathscr{E}_0 \otimes_k \mathscr{O}_S \to f_*\mathscr{L}^{\otimes n} \tag{4.2.2}$$

is surjective.

For an integer $n \geq n_0$ let $\mathbf{G}_n$ denote the Grassmanian classifying rank $P(n)$ quotients of $\mathrm{Sym}^n\mathscr{E}_0$. We then get a morphism of schemes

$$\sigma_n : \mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0} \to \mathbf{G}_n, \quad (4.2.1) \mapsto (4.2.2).$$

By [2, 3.8] for n sufficiently big this map is a closed immersion.

Observe that there is a natural action of $\mathbf{GL}(\mathscr{E}_0)$ on $\mathbf{G}_n$ and σ_n is equivariant with respect to this action.

4.3. Fix an integer n sufficiently big such that the map σ_n is a closed immersion.

Let $x_0 \in \mathbf{Hilb}^P_{\mathbf{P}\mathscr{E}_0}$ be the k-point corresponding to the closed subscheme

$$X_0 \hookrightarrow \mathbf{P}\mathscr{E}_0$$

given by $\mathscr{X}_0$. If we write M_n for $H^0(X_0, \mathscr{L}_0^{\otimes n})$, then the point $\sigma_n(x_0) \in \mathbf{G}_n(k)$ corresponds to the quotient

$$\pi_n : \mathrm{Sym}^n\mathscr{E}_0 \twoheadrightarrow M_n$$

induced by $\beta^{\otimes n}$.

Observe that this is a surjective map of $\mathscr{G}_{\mathscr{X}_0}$-representations.

Lemma 4.4. *There exists a unique $\mathscr{G}_{\mathscr{X}_0}$-equivariant section*

$$s_n : M_n \to \mathrm{Sym}^n\mathscr{E}_0$$

of π_n.

Proof. Since k has characteristic 0 the group scheme $\mathcal{G}_{\mathfrak{X}_0}$ is an extension of a finite étale group scheme over k by a torus, and is therefore linearly reductive. Since $\mathcal{G}_{\mathfrak{X}_0}^{\circ}$ is a maximal torus in $\mathbf{GL}(\mathcal{E}_0)$ each character of $\mathrm{Sym}^n\mathcal{E}_0$ occurs with multiplicity one. It follows that the irreducible representations of $\mathcal{G}_{\mathfrak{X}_0}$ that occur in $\mathrm{Sym}^n\mathcal{E}_0$ appear with multiplicity 1. Since π_n is surjective the same is true of M_n. From this the result follows. □

4.5. The section s_n induces a $\mathcal{G}_{\mathfrak{X}_0}$-equivariant decomposition

$$\mathrm{Sym}^n\mathcal{E}_0 \simeq \Sigma_n \oplus M_n. \tag{4.5.1}$$

Let $\mathbf{A}$ denote the affine space

$$\mathrm{Hom}(\Sigma_n, M_n)$$

given by the spectrum of the symmetric algebra on $V_n := M_n^\vee \otimes \Sigma_n$. We then have an open immersion

$$j_n : \mathbf{A} \hookrightarrow \mathbf{G}_n$$

associating to a morphism $g : \Sigma_n \to M_n$ with graph

$$\gamma_g : \Sigma_n \hookrightarrow \Sigma_n \oplus M_n$$

the quotient

$$\mathrm{Sym}^n\mathcal{E}_0 \simeq \Sigma_n \oplus M_n \to \mathrm{Coker}(\gamma_g).$$

The point $\sigma_n(x_0) \in \mathbf{G}_n$ is given by the origin in $\mathbf{A}$. Furthermore, there is a natural action of $\mathcal{G}_{\mathfrak{X}_0}$ on $\mathbf{A}$ and the morphism j_n is equivariant with respect to this action.

Let

$$Z_n \subset \mathbf{Hilb}^P_{\mathbf{P}\mathcal{E}_0}$$

be the open subset $\sigma_n^{-1}(\mathbf{A})$. We then have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(k) & \xrightarrow{x_0} & Z_n & \xhookrightarrow{\sigma_n} & \mathbf{A} \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \\
B\mathcal{G}_{\mathfrak{X}_0} & \hookrightarrow & \mathcal{M}, & &
\end{array}$$

where π is smooth (note that the square is *not* cartesian).

4.6. We now proceed as in [4, 6.16]. The pullback of Z_n to $B\mathcal{G}_{\mathfrak{X}_0}$ is an open subset of the quotient

$$\mathcal{G}_{\mathfrak{X}_0}\backslash\mathbf{GL}(\mathcal{E}_0),$$

where $\mathcal{G}_{\mathfrak{X}_0}$ acts by left translation. From this we get an isomorphism

$$T_{Z_n/\mathcal{M}}(x_0) \simeq \mathrm{Lie}(\mathbf{GL}(\mathcal{E}_0))/\mathrm{Lie}(\mathcal{G}_{\mathfrak{X}_0}),$$

and a $\mathcal{G}_{\mathfrak{X}_0}$-equivariant inclusion

$$\bar{\rho} : \mathrm{Lie}(\mathbf{GL}(\mathcal{E}_0))/\mathrm{Lie}(\mathcal{G}_{\mathfrak{X}_0}) \hookrightarrow W_n,$$

where we write W_n for the dual $V_n^\vee$, the tangent space of $\mathbf{A}$ at the origin.

Looking at matrix entries as in [4, 6.17] we get a canonical equivariant splitting

$$s_n : W_n \to \mathrm{Lie}(\mathbf{GL}(\mathcal{E}_0))/\mathrm{Lie}(\mathcal{G}_{\mathfrak{X}_0}).$$

Let $W'_n \subset W_n$ be the kernel of s_n, let $V_n \to V'_n$ be the quotient obtained by taking duals, and let

$$\mathbf{A}' \hookrightarrow \mathbf{A}$$

be the induced closed immersion of affine spaces. Setting

$$Z' := Z \times_{\mathbf{A}} \mathbf{A}'$$

the induced projection

$$\pi' : Z' \to \mathscr{M}$$

is smooth in a neighborhood of x_0 by [4, 4.4]. By the technique of [4, 6.21] we then get compatible presentations

$$[U'_r/\mathscr{G}_{\mathfrak{X}_0}] \simeq \mathscr{M}_r$$

for the formal neighborhoods of $B\mathscr{G}_{\mathfrak{X}_0}$ in $\mathscr{M}$ (here $\mathscr{M}_r$ denotes the r-th infinitesimal neighborhood of $B\mathscr{G}_{\mathfrak{X}_0}$ in $\mathscr{M}$). Moreover, we get a canonical isomorphism between $\mathscr{E}_r$, the restriction of the tautological vector bundle on $\mathscr{M}$ to $\mathscr{M}_r$, and the pullback of $\mathscr{E}_0$ from $B\mathscr{G}_{\mathfrak{X}_0}$.

4.7. If k is algebraically closed of characteristic 0, this has, in particular, the following consequence. Given a complete noetherian local k-algebra R with residue field k and a morphism

$$f : \mathrm{Spec}(R) \to \mathscr{M}$$

corresponding to an object $\mathfrak{X} = (X, \mathscr{L}, \mathscr{E}, \beta)$ over R such that the reduction of f factors through $B\mathscr{G}_{\mathfrak{X}_0}$ (that is, the reduction of $\mathfrak{X}$ to the closed fiber is isomorphic to $\mathfrak{X}_0$ but we do not fix such an isomorphism), then we obtain a canonical basis for $\mathscr{E}$, well-defined up to permutation and scaling. This is constructed as in [4, 3.24]. In the next section we will describe this basis directly.

5. Matrix calculations

5.1. Let k be a field of characteristic 0 and R a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $i : X \hookrightarrow \mathbf{P}^m_R$ be a closed immersion with X/R flat, and let

$$\mathfrak{X} = (X, \mathscr{L}, \mathscr{O}_R^{m+1}, \beta)$$

be the resulting projective variety over R with very ample linear system.

Assume that the closed fiber $\mathfrak{X}_0$ is maximally monomial with associated torus in $\mathbf{GL}_{m+1}$ the standard diagonal torus, so the ideal of X_0 in $\mathbf{P}^m_k$ is a monomial ideal in $k[x_0, \dots, x_m]$.

By 4.7 we get from this setup a basis, possible different from the standard basis,

$$\gamma : \mathscr{O}_R^{m+1} \to \mathscr{O}_R^{m+1}$$

reducing to the standard basis modulo $\mathfrak{m}$. In other words a matrix with coefficients in R. We can describe this matrix as follows.

5.2. Set

$$\widehat{\mathbf{GL}}_{m+1}(R) := \mathrm{Ker}(\mathbf{GL}_{m+1}(R) \to \mathbf{GL}_{m+1}(k)).$$

Let n be an integer sufficiently large as in 4.5. We have

$$\mathrm{Sym}^n \mathscr{E}_0 \simeq k[x_0, \dots, x_m]_n,$$

the homogeneous polynomials of degree n, and the decomposition (4.5.1) is given by monomials. In fact, this decomposition has an even finer decomposition given by the monomials, which define the eigenspaces for the torus action. Set

$$\mathscr{P}_n := \{\underline{i} = (i_0, \ldots, i_m) \in \mathbf{N}^{m+1} | \sum_s i_s = n\},$$

and for $\underline{i} \in \mathscr{P}_n$ write

$$x^{\underline{i}} := x_0^{i_0} \cdots x_m^{i_n}.$$

Let $I_n \subset \mathscr{P}_n$ be the subset of those indices for which $x^{\underline{i}}$ is in the n-th graded piece of the ideal of X, and let $J_n \subset \mathscr{P}_n$ be the complement so we have $\mathscr{P}_n = I_n \coprod J_n$. We then have

$$\Sigma_n = \oplus_{\underline{i} \in I_n} k \cdot x^{\underline{i}}, \quad M_n = \oplus_{\underline{j} \in J_n} k \cdot x^{\underline{j}}.$$

The vector space $W_n = M_n \otimes \Sigma_n^\vee$ (note that this is the dual of the vector space V_n in 4.5) then breaks up as

$$W_n = \oplus_{\underline{i} \in I_n, \underline{j} \in J_n} k \cdot y_{\underline{i},\underline{j}},$$

where the torus $\mathbf{G}_m^{m+1}$ acts on $y_{\underline{i},\underline{j}}$ be the formula

$$\underline{u} * y_{\underline{i},\underline{j}} = (\prod_{s=0}^m u_s^{j_s - i_s}) \cdot y_{\underline{i},\underline{j}}.$$

Define

$$\widehat{\mathbf{A}}(R)$$

to be the elements of $\mathbf{A}(R)$ which reduce to 0 in $\mathbf{A}(k)$. Since $\mathbf{A}(R) \simeq W_n \otimes R$ we have an isomorphism

$$\widehat{\mathbf{A}}(R) \simeq W_n \otimes \mathfrak{m} \simeq \oplus_{\underline{i} \in I_n, \underline{j} \in J_n} \mathfrak{m} \cdot y_{\underline{i},\underline{j}}.$$

We then get a map

$$\rho : \widehat{\mathbf{GL}}_{m+1}(R) \to \widehat{\mathbf{A}}(R).$$

This map can be described as follows. For integers $0 \leq \alpha, \beta, \leq m$ let $\mathbf{1}_{\alpha\beta} \in \mathbf{M}_{m+1}(k)$ be the matrix with a 1 in the (α, β)-entry and zeros elsewhere. Then any element $\gamma \in \widehat{\mathbf{GL}}_{m+1}(R)$ can be written as

$$\gamma = \mathrm{id} + \sum_{\alpha,\beta} a_{\alpha\beta} \cdot \mathbf{1}_{\alpha,\beta},$$

where $a_{\alpha\beta} \in \mathfrak{m}$. Then

$$\rho(\gamma) = \sum_{\underline{i} \in I_n, \underline{j} \in J_n} b_{\underline{i},\underline{j}} y_{\underline{i},\underline{j}},$$

where $b_{\underline{i},\underline{j}}$ is the coefficent of $x^{\underline{j}}$ when

(5.2.1) $$\prod_s (x_s + \sum_\alpha a_{\alpha s} x_\alpha)^{i_s}$$

is written in the basis $x^{\underline{i}}$, $\underline{i} \in \mathscr{P}_n$. This is just making explicit the standard action of $\mathbf{GL}_{m+1}$ on the Hilbert scheme inducing by linear change of variables. Because the connected component of the automorphism group scheme of $\mathfrak{X}_0$ is the torus the induced map on tangent spaces

$$\bar{\rho} : \oplus_{\alpha \neq \beta} (\mathfrak{m}/\mathfrak{m}^2) \cdot \mathbf{1}_{\alpha\beta} \to W_n \otimes (\mathfrak{m}/\mathfrak{m}^2)$$

is injective. Note that this map is obtained from a map of vector spaces

$$\bar{\rho}_0 : \oplus_{\alpha\neq\beta} k \cdot \mathbf{1}_{\alpha\beta} \to W_n$$

by tensoring with $\mathfrak{m}/\mathfrak{m}^2$.

The map $\bar{\rho}$ is equivariant with respect to the $\mathbf{G}_m^{m+1}$-action, where the action on the source is given by

$$\underline{u} * \mathbf{1}_{\alpha\beta} = u_\alpha u_\beta^{-1}.$$

Observe here that the monomials possibly occurring in the expansion (5.2.1) modulo $\mathfrak{m}^2$ are the ones of the form $x^{\underline{j}}$, where $\underline{i} - \underline{j}$ is either 0 or has two nonzero entries, one of them +1 and the other −1.

5.3. For $\alpha \neq \beta$ let $T_{\alpha\beta} \subset I_n \times J_n$ be the set of pairs $(\underline{i}, \underline{j})$ for which $\underline{j} - \underline{i}$ is equal to the vector with 1 in the α-th component, −1 in the β-th component, and 0's in the other components, and let $N_{\alpha\beta}$ be the number of elements in $T_{\alpha\beta}$. Let $T^c \subset I_n \times J_n$ be the complement of the union of the sets $T_{\alpha\beta}$ so we have a decomposition

$$I_n \times J_n = T_n^c \coprod \left(\coprod_{\alpha\neq\beta} T_{\alpha\beta} \right).$$

Let

$$s_n : W_n \to \oplus_{\alpha\neq\beta} k \cdot \mathbf{1}_{\alpha,\beta}$$

be the map sending $y_{\underline{i},\underline{j}}$ to 0 if $(\underline{i}, \underline{j}) \in T_n^c$ and to

$$\frac{1}{N_{\alpha\beta}} \mathbf{1}_{\alpha\beta}$$

if $(\underline{i}, \underline{j}) \in T_{\alpha\beta}$. Then s_n defines a splitting of the inclusion $\bar{\rho}_0$, so if $\lambda_n : W'_n \hookrightarrow W_n$ is the kernel of s_n we get a decomposition

$$\lambda_n + \bar{\rho}_0 : W'_n \oplus W''_n \xrightarrow{\simeq} W_n,$$

where

$$W''_n := \oplus_{\alpha\neq\beta} k \cdot \mathbf{1}_{\alpha\beta}.$$

Let $\mathbf{A}''$ denote the spectrum of the symmetric algebra on the dual of W''_n and let $\widehat{\mathbf{A}}''(R)$ denote the subset of elements of $\mathbf{A}''(R)$ of elements reducing to 0. So we have

$$\widehat{\mathbf{A}}''(R) \simeq W''_n \otimes \mathfrak{m}.$$

The map s_n then induces a surjection

$$\mathbf{s}_n : \widehat{\mathbf{A}}(R) \to \widehat{\mathbf{A}}''(R).$$

Lemma 5.4. *The composition*

$$\widehat{\mathbf{GL}}_{m+1}(R) \xrightarrow{\rho} \widehat{\mathbf{A}}(R) \xrightarrow{\mathbf{s}_n} \widehat{\mathbf{A}}''(R)$$

induces a bijection of sets

$$\widehat{\mathbf{GL}}_{m+1}(R)/\widehat{\mathbf{G}}_m^{m+1}(R) \to \widehat{\mathbf{A}}''(R),$$

where

$$\widehat{\mathbf{G}}_m^{m+1}(R) := \mathrm{Ker}(\mathbf{G}_m^{m+1}(R) \to \mathbf{G}_m^{m+1}(k)).$$

Proof. Let R_n denote the quotient $R/\mathfrak{m}^n$, and let $\widehat{\mathbf{GL}}_{m+1}(R_n)$ (resp. $\widehat{\mathbf{A}}''(R_n)$, $\widehat{\mathbf{G}}_m^{m+1}(R_n)$) denote the elements of $\mathbf{GL}_{m+1}(R_n)$ (resp. $\mathbf{A}''(R_n)$, $\mathbf{G}_m^{m+1}(R_n)$) reducing to the identity (resp. 0, 1) modulo $\mathfrak{m}$. We then have

$$\widehat{\mathbf{GL}}_{m+1}(R) = \varprojlim_n \widehat{\mathbf{GL}}_{m+1}(R_n), \quad \widehat{\mathbf{A}}''(R) = \varprojlim_n \mathbf{A}''(R_n),$$

and it suffices to show that for each n the map

$$\tau_n : \widehat{\mathbf{GL}}_{m+1}(R_n) \to \widehat{\mathbf{A}}''(R_n)$$

induces a bijection

$$\bar{\tau}_n : \widehat{\mathbf{GL}}_{m+1}(R_n)/\widehat{\mathbf{G}}_m^{m+1}(R_n) \to \widehat{\mathbf{A}}''(R_n).$$

This we do by induction on n, the case of $n = 1$ being trivial.

For the inductive step note that for each $n \geq 1$ we have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathbf{GL}}_{m+1}(R_n) & \xrightarrow{\tau_n} & \widehat{\mathbf{A}}''(R_n) \\ \downarrow & & \downarrow \\ \widehat{\mathbf{GL}}_{m+1}(R_{n-1}) & \xrightarrow{\tau_n} & \widehat{\mathbf{A}}''(R_{n-1}), \end{array}$$

where the vertical maps are the reduction maps, the fiber of the left reduction map is a torsor under the matrices $\mathbf{M}_{m+1}(k) \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1}$, the fibers of the right vertical map are torsors under the group $W''_n \otimes \mathfrak{m}^n \otimes \mathfrak{m}^{n+1}$, and the map on fibers induced by τ_n is compatible with these actions in the sense that it is linear with respect to the maps

$$\mathbf{M}_{m+1}(k) \otimes (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \xrightarrow{\text{projection}} (\oplus_{\alpha \neq \beta} k \cdot \mathbf{1}_{\alpha\beta}) \otimes (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \xrightarrow{s_n} W''_n \otimes (\mathfrak{m}^n/\mathfrak{m}^{n+1}).$$

From this the result follows. □

5.5. Returning to the setup of 5.1, note that the surjection

$$R[x_0, \ldots, x_m]_n \twoheadrightarrow \Gamma(X, \mathscr{L}^{\otimes n})$$

induces an isomorphism $M_n \otimes R \simeq \Gamma(X, \mathscr{L}^{\otimes n})$, by Nakayama's lemma, and therefore also a map

$$\chi : \Sigma_n \otimes R \to M_n \otimes R$$

reducing to zero modulo $\mathfrak{m}$; that is, a point $\chi \in \widehat{\mathbf{A}}(R)$. Let $\bar{\chi} \in \widehat{\mathbf{A}}''(R)$ be the image point. Then by 5.4 we obtain an element

$$\bar{\gamma} \in \widehat{\mathbf{GL}}_{m+1}(R)/\widehat{\mathbf{G}}_m^{m+1}(R).$$

That is, a basis for R^{m+1} reducing to the standard basis modulo $\mathfrak{m}$. This is the basis provided by the construction of section 4.

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